A FRIENDLY INTRODUCTION TO LIGHT-FRONT QUANTUM FIELD THEORY

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Structure of light-front vacuum sector diagrams

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A B S T R A C T

We study the structure of scalar field light-front quantization vacuum graphs. In instant-time quantization both non-vacuum and vacuum graphs can equivalently be described by either the off-shell four-dimensional Feynman diagram approach or the on-shell three-dimensional Fock space approach, with this being the case since the relevant Feynman diagrams are given entirely by pole terms. This is also the case for light-front quantization non-vacuum graphs. However this is not the case for light-front vacuum sector diagrams, since then there are also circle at infinity contributions to Feynman diagrams. These non-pole contributions cause light-front vacuum diagrams to be nonzero and to not be given by a light-front Hamiltonian Fock space analysis. The three-dimensional approach thus fails in the light-front vacuum sector. In consequence, the closely related infinite momentum frame approach also fails in the light-front vacuum sector.

1. Introduction

Since the original work of Dirac [1], there has been continuing interest in light-front (also known as "light-cone" or "front-form") quantization of quantum field theories. Comprehensive reviews can be found in [2–5]. The light-front approach is based on 3-dimensional Hamiltonian field theory quantized at fixed light-front time $x^+ = x^0 + x^3$. The rules for calculations for Light-Front Hamiltonian QCD for both perturbative and nonperturbative applications are summarized in [6]. As is the case with the standard four-dimensional covariant Feynman Lagrangian theory, the light-front formalism is Poincaré invariant and causal. Observables in hadron physics such as form factors, structure functions, and distribution amplitudes are based on the nonperturbative light-front hadronic wave functions, the eigenfunctions of the QCD Light-Front Hamiltonian [7,8]. In the case of scattering amplitudes, the covariant Feynman and the Light-Front Hamiltonian approaches give identical results. One can also replicate the calculation rules for light-front x^+-ordered perturbation theory using standard time-ordered perturbation theory based on quantization at fixed time (also known as instant-time or "instant-form") by choosing a Lorentz frame where the observer moves at infinite momentum [9–12].

While the light-front non-vacuum (i.e., scattering) sector is well understood, in the light-front literature there has been a spirited discussion as to the status of perturbative light-front vacuum graphs (see e.g. [2,11–15]). In the light-front vacuum sector differing results have been obtained for the off-shell four-dimensional Feynman diagram approach and the on-shell three-dimensional Fock space approach, and the literature has not yet settled on which particular one might have fundamental validity, or identified what it is that causes differences between the various approaches. It is the purpose of this paper to address this issue in the scalar field theory case, and to show that because of circle at infinity contributions in four-dimensional light-front vacuum Feynman diagrams it is the Feynman approach that one must use as the light-front Fock space approach is equivalent to the pole term contribution to Feynman diagrams alone. Because of these non-pole circle at infinity contributions, light-front vacuum diagrams are not only nonzero, they are equal to instant-time vacuum diagrams, even though instant-time vacuum Feynman diagrams receive no circle at infinity contributions. Our result is initially surprising since the instant-time Fock space analysis correctly describes the instant-time vacuum sector, and in the infinite momentum frame the instant-time Fock space procedure transforms into the light-front Fock space description. However, even though circle at infinity contributions are suppressed in the instant-time case, when

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Equivalence of light-front quantization and instant-time quantization

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Commutation or anticommutation relations quantized at equal instant time and commutation or anticommutation relations quantized at equal light-front time not only cannot be transformed into each other, they take completely different forms. While they would thus appear to describe different theories, we show that this is not in fact the case. By looking not at equal times but at unequal times, we show that unequal instant-time commutation or anticommutation relations are completely equivalent to unequal light-front time commutation or anticommutation relations. Light-front quantization and instant-time quantization are thus the same and thus describe the same theory, with it being only the restriction to equal times that makes them look different. However, for fermions there is a caveat, as the light-front anticommutation relations involve projection operators acting on the fermion fields. Nonetheless, not only can one still derive fermion unequal light-front time anticommutators starting from unequal instant-time ones, one can even derive unequal instant-time fermion anticommutators starting from unequal light-front time anticommutators even though the fermion projection operators that are relevant in the light-front case are not invertible. To establish the equivalence for gauge fields we present a quantization procedure that does not involve the zero-mode singularities that are commonly encountered in light-front gauge field studies. We also study time-ordered products of fields, and again show the equivalence despite the fact that there are additional terms in the fermion light-front case. We establish our results first for free theories, and then to all orders in interacting theories though comparison of the instant-time and light-front Lehmann representations. Finally, we compare instant-time Hamiltonians and light-front Hamiltonians and show that in the instant-time rest frame they give identical results.

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I. INTRODUCTION

In quantum field theory various choices of quantization are considered. The most common choice is to take commutation relations of pairs of fields at equal instant time x^0 to be specific singular c-number functions. Thus for a free scalar field with action

$$I_S = \int dx^0 dx^1 dx^2 dx^3 \times \frac{1}{2}[(\partial_0\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2 - m^2\phi^2], \quad (1.1)$$

for instance, one identifies a canonical conjugate $\delta I_S/\delta\partial_0\phi = \partial^0\phi = \partial_0\phi$ (one can of course add on interaction terms to I_S, but as long as they contain no derivatives they do not affect the identification of the canonical conjugate), and then quantizes the theory according to the equal instant-time canonical commutation relation

$$[\phi(x^0, x^1, x^2, x^3), \partial_0\phi(x^0, y^1, y^2, y^3)] = i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3). \quad (1.2)$$

In light-front quantization (see e.g., [1] for a review) one introduces coordinates $x^\pm = x^0 \pm x^3$, a line element $g_{\mu\nu}x^\mu x^\nu = x^+x^- - (x^1)^2 - (x^2)^2$ with $(-g)^{1/2} = 1/2$, and a free scalar field action of the form

$$I_S = \frac{1}{2}\int dx^+ dx^1 dx^2 dx^- \frac{1}{2} \times [2\partial_+\phi\partial_-\phi + 2\partial_-\phi\partial_+\phi - (\partial_1\phi)^2 - (\partial_2\phi)^2 - m^2\phi^2]. \quad (1.3)$$

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Comparing light-front quantization with instant-time quantization

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ABSTRACT

In this paper we compare light-front quantization and instant-time quantization both at the level of operators and at the level of their Feynman diagram matrix elements. At the level of operators light-front quantization and instant-time quantization lead to equal light-front time commutation (or anticommutation) relations that appear to be quite different from equal instant-time commutation (or anticommutation) relations. Despite this we show that at unequal times instant-time and light-front commutation (or anticommutation) relations actually can be transformed into each other, with it only being the restriction to equal times that makes the commutation (or anticommutation) relations appear to be so different. While our results are valid for both bosons and fermions, for fermions there are subtleties associated with tip of the light cone contributions that need to be taken care of. At the level of Feynman diagrams we show for non-vacuum Feynman diagrams that the pole terms in four-dimensional light-front Feynman diagrams reproduce the widely used three-dimensional light-front on-shell Hamiltonian Fock space formulation in which the light-front energy and light-front momentum are on shell. Moreover, we show that the contributions of pole terms in non-vacuum instant-time and non-vacuum light-front Feynman diagrams are equal. However, because of circle at infinity contributions we show that this equivalence of pole terms fails for four-dimensional light-front vacuum tadpole diagrams. Then, and precisely because of these circle at infinity contributions, we show that light-front vacuum tadpole diagrams are not only nonzero, they quite remarkably are actually equal to the pure pole term instant-time vacuum tadpole diagrams. Light-front vacuum diagrams are not correctly describable by the on-shell Hamiltonian formalism, and thus not by the closely related infinite momentum frame prescription either. Thus for the light-front vacuum sector we must use the off-shell Feynman formalism as it contains information that is not accessible in the on-shell Hamiltonian Fock space approach. We show that light-front quantization is intrinsically nonlocal, and that for fermions this nonlocality is present in Ward identities. One can project fermion spinors into so-called good and bad components, and both of these components contribute in Ward identities. Central to our analysis is that the transformation from instant-time coordinates and fields to light-front coordinates and fields is a unitary, spacetime-dependent translation. Consequently, not only are instant-time quantization and light-front quantization equivalent, because of general coordinate invariance they are unitarily equivalent.

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PROCEEDINGS OF SCIENCE

Light-front quantization is the same as instant-time quantization

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Commutation or anticommutation relations quantized at equal instant time and commutation or anticommutation relations quantized at equal light-front time cannot be transformed into each other. While they would thus appear to describe different theories, we show that this is not in fact the case. In instant-time quantization unequal instant-time commutation or anticommutation relations for free scalar, fermion, or gauge boson fields are c-numbers. We show that when these unequal instant-time commutation or anticommutation relations are evaluated at equal light-front time they are identical to the equal light-front time commutation or anticommutation relations. Light-front quantization and instant-time quantization are thus the same and thus describe the same physics.

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Physics on and off the light cone

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Abstract We study light-front physics and conformal symmetry, and their interplay both on and off the light cone. The full symmetry of the light cone is conformal symmetry not just Lorentz symmetry. Spontaneously breaking conformal symmetry gives masses to particles and takes them off the light cone. Canonical quantization specifies equal-time commutators on the light cone. Equal instant-time and equal light-front-time commutators look very different, but can be shown to be equivalent by looking at unequal-time commutators. We discuss the connection of the light-front approach to the infinite momentum frame approach, and show that vacuum graphs are outside this framework. We show that there is a light-front structure to both AdS/CFT and the eikonal approximation. While mass generation involves scale-breaking mass scales, we show that such mass scales can arise via dynamical symmetry breaking in the presence of scale invariant interactions at a renormalization group fixed point.

1 Minkowski signature predates special relativity

While Minkowski signature is central to special relativity and light cone studies, it is of interest to note that the Minkowski signature predates twentieth-century special relativity having originated in differential geometry in the nineteenth century. To be specific, consider the 2-dimensional Gauss–Bolyai–Lobachevski geometry with line element

$$ds^2 = \frac{a^2 dr^2}{a^2 + r^2} + r^2 d\theta^2. \tag{1}$$

To construct it, we introduce a flat 3-dimensional space with a Minkowski-signatured line element

$$ds^2 = dx^2 + dy^2 - dt^2, \tag{2}$$

as constrained by the hyperbola

$$t^2 - x^2 - y^2 = a^2. \tag{3}$$

Eliminating t gives

$$ds^2 = dx^2 + dy^2 - \frac{(xdx + ydy)^2}{a^2 + x^2 + y^2}. \tag{4}$$

On introducing polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ we recover (1):

$$ds^2 = dr^2 + r^2 d\theta^2 - \frac{r^2 dr^2}{a^2 + r^2} = \frac{a^2 dr^2}{a^2 + r^2} + r^2 d\theta^2. \tag{5}$$

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1 OUTLINE

These lectures provide a friendly introduction to light-front quantum field theory. Topics include:

(1) Infinite momentum frame and the light front.

(2) Canonical commutation/anticommutation relations for scalars, fermions and gauge bosons at equal light-front time.

(3) Connection to commutators at equal instant time.

(4) Good and bad fermions.

(5) Feynman diagrams and Fock space.

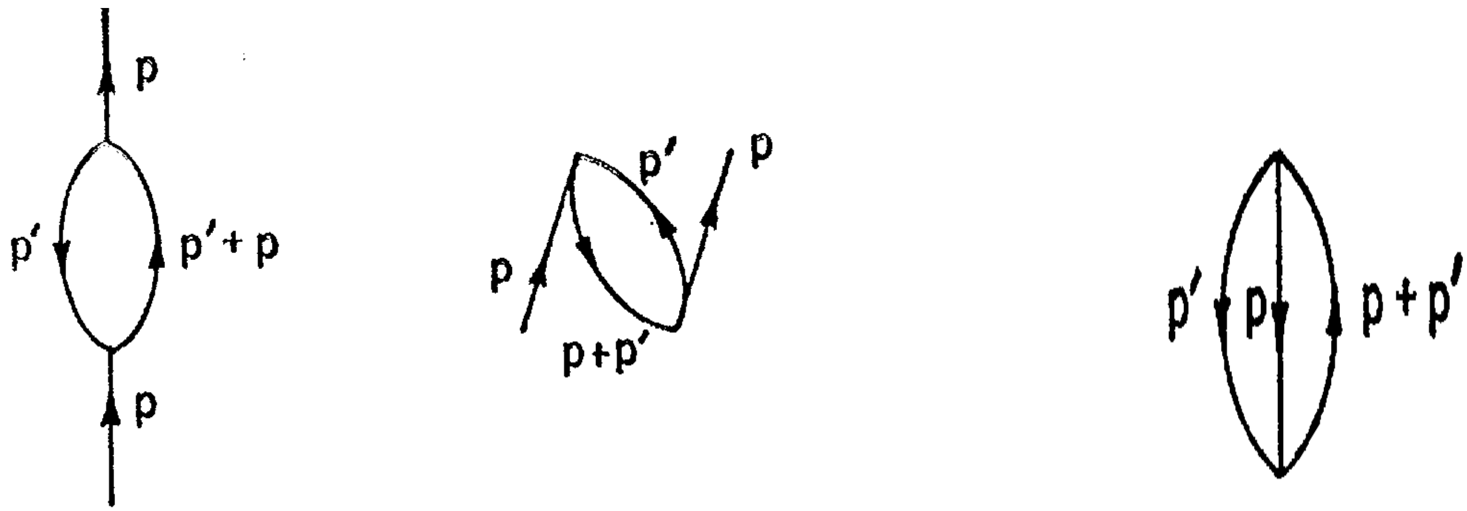
(6) Structure of light front vacuum graphs.

(7) Structure of the light-front Hamiltonian.

(8) Chiral symmetry breaking in the light front.

2 INFINITE MOMENTUM FRAME

In 1966 Weinberg (Phys. Rev. 150, 1313 (1966)) showed that instant-time quantization perturbation theory would be simplified in the frame in which an observer moved with an infinite three-momentum with respect to the center of mass system of a scattering process, i.e., $p^3 = \alpha P$ where P is large and α is a constant and $p^0 = [(p^3)^2 + (p^1)^2 + (p^2)^2 + m^2]^{1/2} \to \alpha P + [(p^1)^2 + (p^2)^2 + m^2]/2\alpha P$.



a b c

Specifically, Graph (b) would be suppressed with respect to Graph (a). Graph (c) was not discussed. In Weinberg's case the x^0 time axis runs up the diagram and the analysis was made using old-fashioned perturbation theory. Old-fashioned (i.e. pre-Feynman) perturbation theory is off the energy shell but on the mass shell. (The Feynman approach is off the mass shell). Graph (b) takes support outside the light cone, and is suppressed at infinite momentum. Graph (c) involves both time orderings, i.e., forward and backward in time.

3 INSTANT-TIME FEYNMAN GRAPHS AND OLD-FASHIONED PERTURBATION THEORY

In the instant-time case one can take an instant-time forward in time Green's function such as $D(x^0 > 0, \text{instant}) = -i\langle\Omega_I|\theta(x^0)\phi(x^0, x^1, x^2, x^3)\phi(0)|\Omega_I\rangle$ as evaluated in the instant-time vacuum $|\Omega_I\rangle$, and expand the field in terms of instant-time creation and annihilation operators that create and annihilate particles out of that vacuum as

$$\phi(x^0, \vec{x}) = \int \frac{d^3p}{(2\pi)^{3/2}(2E_p)^{1/2}}[a(\vec{p})\exp(-iE_pt + i\vec{p}\cdot\vec{x}) + a^\dagger(\vec{p})\exp(+iE_pt - i\vec{p}\cdot\vec{x})], \tag{3.1}$$

where $E_p = (\vec{p}^2 + m^2)^{1/2}$ and $[a(\vec{p}), a^\dagger(\vec{p}\,')] = \delta^3(\vec{p} - \vec{p}\,')$. The insertion of $\phi(\vec{x}, x^0)$ into $D(x^0 > 0, \text{instant})$ immediately leads to the on-shell three-dimensional integral

$$D(x^0 > 0, \text{instant}, \text{Fock}) = -\frac{i\theta(x^0)}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0 + i\vec{p}\cdot\vec{x}}. \tag{3.2}$$

Alternatively, one can look for solutions to $(\partial_\alpha\partial^\alpha + m^2)D(x^\mu, \text{instant}) = -\delta^4(x)$, and obtain the off-shell four-dimensional integral

$$D(x^\mu, \text{instant}) = \frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon} = \frac{1}{(2\pi)^4}\int \frac{d^4p}{2E_p}e^{-ip\cdot x}\left[\frac{1}{p_0 - E_p + i\epsilon} - \frac{1}{p_0 + E_p - i\epsilon}\right], \tag{3.3}$$

with the p_0 integration being along a contour integral in the complex p_0 plane. One can then proceed from (3.3) to (3.2) by closing the Feynman contour below the real p_0 axis, to yield a contour integral in which the lower-half p_0 plane circle at infinity makes no contribution when the instant-time x^0 is positive, while the pole term yields (3.2).

Similarly, one can proceed in reverse from (3.2) to (3.3) by writing the theta function as a contour integral in the complex ω plane:

$$\theta(x^0) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega x^0}}{\omega + i\epsilon}, \tag{3.4}$$

so that the pole contribution yields $\theta(x^0) = 1$ when $x^0 > 0$ and yields $\theta(x^0) = 0$ when $x^0 < 0$. With this representation of the theta function (3.2) takes the form

$$D(x^0 > 0, \text{instant}) = \frac{1}{(2\pi)^4}\int \frac{d^3p}{2E_p}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega x^0}}{\omega + i\epsilon} e^{-iE_p x^0 + i\vec{p}\cdot\vec{x}}. \tag{3.5}$$

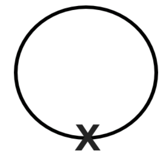
On setting $p_0 = \omega + E_p$, we can rewrite (3.5) as

$$D(x^0 > 0, \text{instant}) = \frac{1}{(2\pi)^4}\int \frac{d^4p}{2E_p}\frac{e^{-ip_0 x^0 + i\vec{p}\cdot\vec{x}}}{(p_0 - E_p + i\epsilon)}. \tag{3.6}$$

We recognize (3.6) as the forward in time, positive frequency component of (3.3), and thus establish the equivalence of the instant-time off-shell four-dimensional Feynman and on-shell three-dimensional Hamiltonian (Fock space) formalisms, and see that the equivalence occurs because the four-dimensional Feynman contour is given by **on-shell poles alone**. Pole dominance thus leads to old-fashioned perturbation theory.

And this is true in light front as well, with an analogous Fock space description.

But how do we know how to normalize the light-front creation and annihilation commutators?



However, poles is not the whole story. When $x^0 = 0$ (the vacuum bubble case) we obtain

$$\theta(0) = -\frac{1}{2\pi i}\oint_{-\infty}^{\infty} d\omega \frac{1}{\omega + i\epsilon} = -\frac{1}{2\pi i}[-2\pi i + \pi i] = \frac{1}{2}, \tag{3.7}$$

with there being a circle at infinity contribution and not just a pole term. For the instant-time case the circle contribution is suppressed because there are two powers of p_0 in the denominator of $D(x^0 = 0, \text{instant})$, viz. $1/((p^0)^2 - (p^3)^2 - (p^1)^2 - (p^2)^2 - m^2)$. However, in the light front case there is only one power of p^+ or p^-, viz $1/(p^+p^- - (p^1)^2 - (p^2)^2 - m^2)$, and the circle does contribute to $D(x^+ = 0, \text{front})$. Surprisingly (Mannheim, Lowdon and Brodsky 2019), this enables instant-form and light-front vacuum bubbles to be equal.

In the vacuum sector there is four-dimensional information that is not accessible using three-dimensional old fashioned perturbation theory or three-dimensional Fock space.

The momenta in instant-time and front-form Feynman diagrams are related by a change of variable, and thus must coincide (Yan 1973). Likewise for path integrals. However that only means that Feynman diagram poles plus circles at infinity equals poles plus circles, not that poles equal poles and circles equal circles. Also it means that with p^0 and p^3 varying between $-\infty$ and ∞, then p^+ and p^- must also vary between $-\infty$ and ∞, and not just be positive.

4 LIGHT-FRONT VARIABLES

In 1969 Chang and Ma (Phys. Rev. 180, 1506 (1969)) recovered Weinberg's infinite momentum frame result by working with the light-front variables. We introduce $x^+ = x^0 + x^3$, $x^- = x^0 - x^3$, $p^+ = p^0 + p^3$, $p^- = p^0 - p^3$. Under a Lorentz boost in the z direction with velocity u we obtain

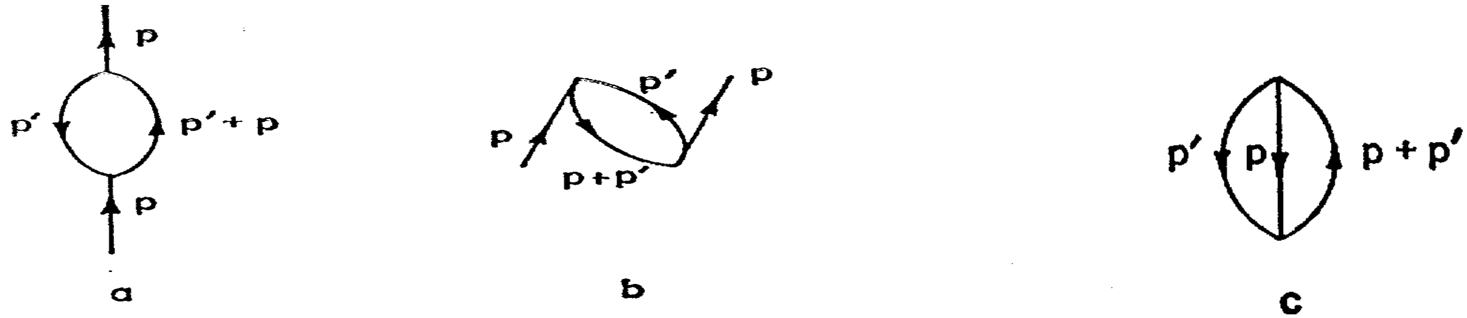
$$x^0 \to \frac{x^0 + ux^3}{(1-u^2)^{1/2}}, \qquad x^3 \to \frac{x^3 + ux^0}{(1-u^2)^{1/2}},$$
$$x^+ \to x^+ \frac{(1+u)^{1/2}}{(1-u)^{1/2}}, \qquad x^- \to x^- \frac{(1-u)^{1/2}}{(1+u)^{1/2}}, \qquad x^+x^- \to x^+x^-. \tag{4.1}$$

Thus a Lorentz boost does **NOT** take us from the instant-time frame to the light-front frame. Rather, as $u \to 1$ (infinite momentum frame) both x^0 and x^3 become infinite. What is preserved is x^+ and x^- though each gets multiplied by a factor, one of which goes to infinity and the other to zero.

So what form of transformations are they? $x^0 \to x^0 + x^3 = x^+$, $x^3 \to x^0 - x^3 = x^-$ are **TRANSLATIONS**. But these translations are spacetime dependent, i.e., they are **LOCAL** translations, i.e., they are **GENERAL COORDINATE TRANSFORMATIONS**, viz. observers move not just with constant velocity (special relativity) but also can accelerate (general relativity). Thus as long as we have general coordinate invariance (which we do) instant time quantum field theory and light-front quantum field theory must be the same theory, though they do not appear to be so. Also they must be unitarily equivalent. The objective of these lectures is to show how this comes about.

Even for a system at rest the observer can move at high velocity, so the only non-relativistic systems that can exist in nature must be the non-relativistic limit of relativistic systems. Hence $g = 2$ for an electron at rest (up to radiative corrections).

Observers can accelerate in flat space where the Riemann tensor is zero, so still need general coordinate invariance (such as with polar coordinates). If the Riemann tensor is not zero then we have gravity, with the Einstein equations writing the second order weak gravity Poisson equation in an accelerating coordinate system.



For momenta we have

$$p^0 + p^3 \to (p^0 + p^3)\left(\frac{1+u}{1-u}\right)^{1/2}, \quad p^0 - p^3 \to (p^0 - p^3)\left(\frac{1-u}{1+u}\right)^{1/2} \tag{4.2}$$

Setting $(1+u)^{1/2}/(1-u)^{1/2} = 1/2P$, $p^3 = \alpha P$, for large P and $(p^0)^2 - (p^3)^2 = p^+p^- = m^2 + (p^1)^2 + (p^2)^2$ we obtain

$$p^0 + p^3 \to \frac{2\alpha P}{2P} = \alpha, \quad p^0 - p^3 \to \frac{[m^2 + (p^1)^2 + (p^2)^2]}{2\alpha P}2P = \frac{[m^2 + (p^1)^2 + (p^2)^2]}{2\alpha}, \tag{4.3}$$

i.e., we recover the momenta used by Weinberg. With this choice a Green's function as evaluated with a complex plane p_+ contour becomes equal to Graph (a) when Graph (a) is evaluated with a complex plane p_0 contour at large p^3.

There is a caveat. In the infinite momentum frame case the flow of time is forward in x^0, while the flow of time in the light-front case is posited to be forward in $x^+ = x^0 + x^3$. But for timelike events

$$(x^0)^2 - (x^3)^2 = x^+x^- > (x^1)^2 + (x^2)^2 > 0$$

is positive, where $x^- = x^0 - x^3$. Thus

$$x^+x^- \text{ is positive}$$

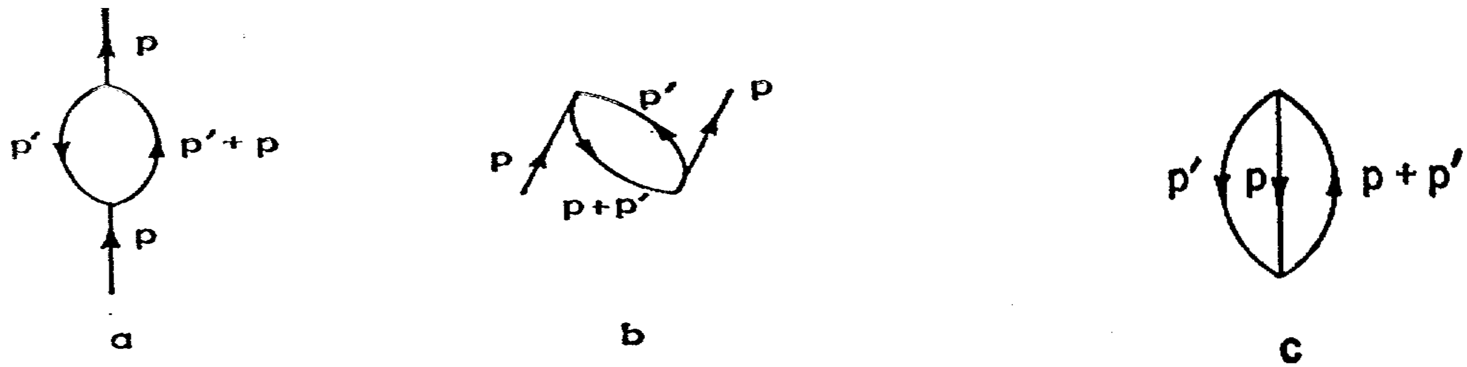
Consequently, x^+ and x^- have the same sign. And thus for

$$x^0 = (x^+ + x^-)/2 > 0$$

(the sign of x^0 is Lorentz invariant for timelike events) it follows that x^+ is positive too.

Thus for timelike events, forward in x^+ is the same as forward in x^0, with the sign of x^+ being Lorentz invariant.

5 THE TAKEAWAY



In their work Chang and Ma showed that

for Graph (a) x^+ is positive and all the p^- poles have both p^- and p^+ positive,

for Graph (b) x^+ is negative and all the p^- poles have both p^- and p^+ negative,

for Graph (c) x^+ is zero and so is p^+. But if p^+ is zero then p^- is infinite. Thus $p_+ = p^-/2$ is infinite too, just as it should be since it is the conjugate of x^+. $(\Delta x^+ \Delta p_+ > \hbar)$.

However, and this is the key point, **all of these statements are true without going to the infinite momentum frame.** They thus can define a strategy for evaluating diagrams as diagrams are segregated by the sign of the time variable x^+. And since x^+ is positive for scattering processes they only involve positive p^- and p^+, with the p^- pole contributions then corresponding to old-fashioned perturbation theory diagrams. Only needing positive p^- and p^+ provides enormous computational benefits.

The vacuum Graph (c) is expressly non-zero, something known as early as 1969. However it involves $p^+ = 0$ zero modes, whose evaluation is tricky. Resolved in Mannheim, Lowdon and Brodsky 2019.

But what about the instant-time graphs that are not at infinite momentum. Are they different from or the same as the light-front graphs. And if they are different, then which ones describe the real world. In Mannheim, Lowdon and Brodsky (2019) they were shown to be the same, though developments since 1969 would suggest that this would be far from the case.

6 LIGHT-FRONT QUANTUM FIELD THEORY

Instead of replacing instant-time momenta by light-front momenta in Feynman diagrams, we can obtain a fully-fledged light-front quantum field theory by constructing equal x^+ commutators rather than equal x^0 commutators. For a scalar field [Neville and Rohrlich, Nuovo Cimento A 1, 625 (1971)]

Scalar field light-front commutators at equal x^+

$$
\begin{aligned}
[\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)] &= -\frac{i}{4}\epsilon(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2),\\
[\phi(x^+,x^1,x^2,x^-),2\partial_-\phi(x^+,y^1,y^2,y^-)] &= i\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^- - y^-).
\end{aligned}
\tag{6.1}
$$

Scalar field instant-time commutators at equal x^0

$$
\begin{aligned}
[\phi(x^0,x^1,x^2,x^3),\partial_0\phi(x^0,y^1,y^2,y^3)] &= i\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3),\\
[\phi(x^0,x^1,x^2,x^3),\phi(x^0,y^1,y^2,y^3)] &= 0.
\end{aligned}
\tag{6.2}
$$

Gauge field instant-time commutators at equal x^0

$$
\begin{aligned}
[A_\nu(x^0,x^1,x^2,x^3),\partial_0 A_\mu(x^0,y^1,y^2,y^3)] &= -ig_{\mu\nu}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3),\\
[A_\nu(x^0,x^1,x^2,x^3),A_\mu(x^0,y^1,y^2,y^3)] &= 0.
\end{aligned}
\tag{6.3}
$$

Using gauge fixing, for light-front gauge fields we obtain (Mannheim, Lowdon and Brodsky 2021)

Gauge field light-front commutators at equal x^+

$$
\begin{aligned}
[A_\nu(x^+,x^1,x^2,x^-),2\partial_- A_\mu(x^+,y^1,y^2,y^-)] &= -ig_{\mu\nu}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^- - y^-),\\
[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)] &= \frac{i}{4}g_{\mu\nu}\epsilon(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2).
\end{aligned}
\tag{6.4}
$$

Analogous results in the non-Abelian case.

The instant-time and light-front commutators are completely different. And for the moment the light-front normalization while obvious is actually arbitrary.

7 INSTANT-TIME AND LIGHT-FRONT ANTI-COMMUTATORS

Fermion instant-time anti-commutators at equal x^0

$$\Big\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(x^0,y^1,y^2,y^3)\Big\} = \delta_{\alpha\beta}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3). \tag{7.1}$$

Fermion light-front anti-commutators at equal x^+

$$\big\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\big\} = \Lambda^+_{\alpha\beta}\delta(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{7.2}$$

[Chang, Root and Yan, Phys. Rev. D 7, 1133 (1973).]

Non-Invertible Projectors

$$\begin{aligned}
&\Lambda^\pm = \tfrac{1}{2}(1\pm\gamma^0\gamma^3),\quad \Lambda^+ + \Lambda^- = I,\quad (\Lambda^\pm)^2 = \Lambda^\pm,\quad \Lambda^+\Lambda^- = 0,\quad \gamma^\pm = \gamma^0\pm\gamma^3,\quad (\gamma^\pm)^2 = 0,\\
&\psi_{(\pm)} = \Lambda_\pm\psi.
\end{aligned} \tag{7.3}$$

$\psi_{(+)}(x)$ is a dynamical variable (it obeys a dynamical Dirac equation) and is known as a **good** fermion, $\psi_{(-)}(x)$ is a constrained variable and is known known as a **bad** fermion. It obeys the constraint

$$\psi_{(-)}(x^+,x^1,x^2,x^-) = -\frac{i}{4}\int du^-\epsilon(x^- - u^-)[-i\gamma^0(\gamma^1\partial_1+\gamma^2\partial_2)+m\gamma^0]\psi_{(+)}(x^+,x^1,x^2,u^-), \tag{7.4}$$

with anti-commutation relations of the form

$$\Big\{[\psi_{(+)}]_\nu(x),[\psi^\dagger_{(-)}]_\sigma(y)\Big\} = \tfrac{i}{8}\epsilon(x^- - y^-)[i(\gamma^-\gamma^1\partial^x_1+\gamma^-\gamma^2\partial^x_2)-m\gamma^-]_{\nu\sigma}\delta(x^1-y^1)\delta(x^2-y^2), \tag{7.5}$$

$$\begin{aligned}
&\Big\{\psi^{(-)}_\mu(x^+,x^1,x^2,x^-),[\psi^\dagger_{(-)}]_\nu(x^+,y^1,y^2,y^-)\Big\}\\
&= \frac{1}{16}\Lambda^-_{\mu\nu}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\int du^-\epsilon(x^- - u^-)\epsilon(y^- - u^-)\delta(x^1-y^1)\delta(x^2-y^2).
\end{aligned} \tag{7.6}$$

Thus light-front anti-commutators are completely different from instant-time anti-commutators, they are non-local and even not invertible.

Things get even worse. The x^+-ordered product does not always satisfy the field wave equation with a delta function source (the propagator equation). This is not a problem for scalar fields, but for fermions we obtain [Yan, Phys. Rev. D 7, 1780 (1973)]

$$-i\langle\Omega|[\theta(x^+)\psi_\beta(x^\mu)\bar{\psi}_\alpha(0)-\theta(-x^+)\bar{\psi}_\alpha(0)\psi_\beta(x^\mu)]|\Omega\rangle = \tfrac{i}{4}\gamma^+_{\beta\alpha}\delta(x^+)\epsilon(x^-)\delta(x^1)\delta(x^2)$$
$$+\frac{2}{(2\pi)^4}\int_{-\infty}^{\infty} dp_+dp_1dp_2dp_-\left[\frac{e^{-i(p_+x^++p_1x^1+p_2x^2+p_-x^-)}}{\gamma^+p_++\gamma^-p_-+\gamma^1p_1+\gamma^2p_2-m+i\epsilon}\right]_{\beta\alpha}, \tag{8.1}$$

i.e., a propagator plus a delta function term. This delta function term only contributes at $x^+=0$, and thus can only contribute in vacuum graphs.

For gauge fields quantized in the $A^+=0$ axial gauge we have [Harindranath, arXiv:hep-ph/9612244]

$$-i\langle\Omega|[\theta(x^+)A^\mu(x)A^\nu(0)+\theta(-x^+)A^\nu(0)A^\mu(x)]|\Omega\rangle$$
$$=2\int\frac{dp_+dp_-dp_1dp_2}{(2\pi)^4}\frac{e^{-ip\cdot x}}{p^2+i\epsilon}\left(g^{\mu\nu}-\frac{n^\mu p^\nu+n^\nu p^\mu}{n\cdot p}+\frac{p^2}{(n\cdot p)^2}n^\mu n^\nu\right), \tag{8.2}$$

i.e., a propagator plus an n^μ-dependent term with only non-zero element $n_+=1$. The n^μ-dependent terms are absent in the instant-time case and lead to a zero mode problem at $p^+=0$.

Fortunately, both of the fermion and gauge field problems are readily fixable. The gauge field n^μ-dependent term does not appear at all if we use gauge fixing. Rather, if one takes the action to be of the form

$$I_G = \int d^4x \left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)^2\right] = \int d^4x \left[-\tfrac{1}{2}\partial_\nu A_\mu \partial^\nu A^\mu\right], \tag{8.3}$$

the x^+-ordered product is then nicely given by $D^{\mu\nu}(p) = g^{\mu\nu}/(p^2 + i\epsilon)$ [Mannheim, Lowdon and Brodsky 2021], just as in the instant-time x^0-ordered case. So no zero mode problem.

For fermions we note that because of Lorentz invariance the vacuum graphs have no external indices, and so the α and β indices in (8.1) must be contracted with $\delta_{\alpha\beta}$. But γ^+ is traceless, and so the delta function term in (8.1) decouples [Mannheim, Lowdon and Brodsky 2021].

We thus see that the instant-time and light-front propagators (and thus Dyson-Wick expansions) are identical in form, and only differ from each other by a change of integration variable from p^0, p^3 to p^+, p^- in expressions that are Poincare invariant. Thus unlike in the infinite momentum frame study, now we can identify the two sets of propagators and Feynman diagrams at all momenta. The two theories are thus equivalent.

But what about the commutators and anti-commutators?

9 WHERE DID THE SCALAR COMMUTATORS COME FROM?

For a free scalar field with instant time action

$$I_S = \int dx^0 dx^1 dx^2 dx^3 \tfrac{1}{2}\left[(\partial_0\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2 - m^2\phi^2\right] \tag{9.1}$$

one identifies a canonical conjugate $\delta I_S/\delta\partial_0\phi = \partial^0\phi = \partial_0\phi$ (one can of course add on interaction terms to I_S, but as long as they contain no derivatives they do not affect the identification of the canonical conjugate), and then quantizes the theory according to the equal instant-time canonical commutation relation

$$[\phi(x^0, x^1, x^2, x^3), \partial_0\phi(x^0, y^1, y^2, y^3)] = i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3). \tag{9.2}$$

For a free scalar field with light front action

$$I_S = \tfrac{1}{2}\int dx^+ dx^1 dx^2 dx^- \tfrac{1}{2}\left[2\partial_+\phi\partial_-\phi + 2\partial_-\phi\partial_+\phi - (\partial_1\phi)^2 - (\partial_2\phi)^2 - m^2\phi^2\right]. \tag{9.3}$$

one identifies a canonical conjugate $(-g)^{-1/2}\delta I_S/\delta\partial_+\phi = \partial^+\phi = 2\partial_-\phi$, and quantizes the theory according to the equal light-front time x^+ commutation relation (Neville and Rohrlich 1971)

$$[\phi(x^+, x^1, x^2, x^-), 2\partial_-\phi(x^+, y^1, y^2, y^-)] = i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^- - y^-). \tag{9.4}$$

As written, (9.4) is already conceptually different from (9.2) since the light-front conjugate is $2\partial_-\phi$ and not $2\partial_+\phi$, i.e., not the derivative with respect to the light-front time, while the instant-time conjugate $\partial_0\phi$ is the derivative with respect to the instant time. Since $\phi(x^+, x^1, x^2, x^-)$ and $\partial_-\phi(x^+, y^1, y^2, y^-)$ are not at the same x^-, (9.4) can be integrated to

$$[\phi(x^+, x^1, x^2, x^-), \phi(x^+, y^1, y^2, y^-)] = -\frac{i}{4}\epsilon(x^- - y^-)\delta(x^1 - y^1)\delta(x^2 - y^2), \tag{9.5}$$

where $\epsilon(x) = \theta(x) - \theta(-x)$. Since the analog instant-time commutation relation is given by

$$[\phi(x^0, x^1, x^2, x^3), \phi(x^0, y^1, y^2, y^3)] = 0, \tag{9.6}$$

instant-time and light-front time quantization appear to be quite different. Similar concerns affect gauge field commutators.

But still the light-front normalization is arbitrary.

10 WHERE DID THE FERMION ANTI-COMMUTATORS COME FROM?

For a free fermion field with instant-time Dirac action

$$I_D = \int d^4x \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \tag{10.1}$$

the canonical conjugate of ψ is $i\psi^\dagger$, and the canonical anti-commutation relations are of the form

$$\begin{aligned}
&\left\{\psi_\alpha(x^0,x^1,x^2,x^3), \psi^\dagger_\beta(x^0,y^1,y^2,y^3)\right\} = \delta_{\alpha\beta}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3), \\
&\left\{\psi_\alpha(x^0,x^1,x^2,x^3), \psi_\beta(x^0,y^1,y^2,y^3)\right\} = 0.
\end{aligned} \tag{10.2}$$

For the light-front case we set

$$\partial_0 = \frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-} = \partial_+ + \partial_-, \quad \partial_3 = \frac{\partial}{\partial x^+} - \frac{\partial}{\partial x^-} = \partial_+ - \partial_-, \quad \gamma^\pm = \gamma^0 \pm \gamma^3 \tag{10.3}$$

and obtain

$$\gamma^0\partial_0 + \gamma^3\partial_3 = (\gamma^0+\gamma^3)\partial_+ + (\gamma^0-\gamma^3)\partial_- = \gamma^+\partial_+ + \gamma^-\partial_-, \tag{10.4}$$

with (10.4) serving to introduce $\gamma^\pm = \gamma^0 \pm \gamma^3$. In terms of γ^+ and γ^- the Dirac action takes the form

$$I_D = \tfrac{1}{2}\int dx^+ dx^1 dx^2 dx^- \psi^\dagger [i\gamma^0(\gamma^+\partial_+ + \gamma^-\partial_- + \gamma^1\partial_1 + \gamma^2\partial_2) - \gamma^0 m]\psi. \tag{10.5}$$

With this action the light-front time canonical conjugate of ψ is $i\psi^\dagger\gamma^0\gamma^+$. In the construction of the light-front fermion sector we find a rather sharp distinction with the instant-time fermion sector. First, unlike γ^0 and γ^3, which obey $(\gamma^0)^2 = 1$, $(\gamma^3)^2 = -1$, γ^+ and γ^- obey $(\gamma^+)^2 = 0$, $(\gamma^-)^2 = 0$, to thus both be non-invertible divisors of zero. Secondly, the quantities

$$\Lambda^+ = \tfrac{1}{2}\gamma^0\gamma^+ = \tfrac{1}{2}(1+\gamma^0\gamma^3), \quad \Lambda^- = \tfrac{1}{2}\gamma^0\gamma^- = \tfrac{1}{2}(1-\gamma^0\gamma^3) \tag{10.6}$$

obey the projector algebra and allow us to introduce good and bad fermions of the form

$$\Lambda^+ + \Lambda^- = I, \quad (\Lambda^+)^2 = \Lambda^+ = [\Lambda^+]^\dagger, \quad (\Lambda^-)^2 = \Lambda^- = [\Lambda^-]^\dagger, \quad \Lambda^+\Lambda^- = 0, \quad \psi_{(+)} = \Lambda^+\psi, \qquad \psi_{(-)} = \Lambda^-\psi. \tag{10.7}$$

We identify the conjugate of ψ as $2i\psi^{\dagger}_{(+)}$, where $\psi^{\dagger}_{(+)} = [\psi^{\dagger}]_{(+)} = \psi^{\dagger}\Lambda^{+} = [\Lambda^{+}\psi]^{\dagger} = [\psi_{(+)}]^{\dagger}$. Since the conjugate is a good fermion, in the anti-commutator of ψ with its conjugate only the good component of ψ will contribute since $\Lambda^{+}\Lambda^{-} = 0$, with the equal light-front time canonical anti-commutator being found to be of the form (Chang, Root and Yan 1973))

$$\left\{[\psi_{(+)}]_{\alpha}(x^{+},x^{1},x^{2},x^{-}),[\psi^{\dagger}_{(+)}]_{\beta}(x^{+},y^{1},y^{2},y^{-})\right\} = \Lambda^{+}_{\alpha\beta}\delta(x^{-}-y^{-})\delta(x^{1}-y^{1})\delta(x^{2}-y^{2}). \tag{10.8}$$

In this construction the bad fermion $\psi_{(-)}$ has no canonical conjugate and is thus not a dynamical variable. To understand this in more detail we manipulate the Dirac equation $(i\gamma^{+}\partial_{+} + i\gamma^{-}\partial_{-} + i\gamma^{1}\partial_{1} + i\gamma^{2}\partial_{2} - m)\psi = 0$. We first multiply on the left by γ^{0} to obtain

$$2i\partial_{+}\psi_{(+)} + 2i\partial_{-}\psi_{(-)} + i\gamma^{0}(\gamma^{1}\partial_{1} + \gamma^{2}\partial_{2})\psi - m\gamma^{0}\psi = 0. \tag{10.9}$$

Next we multiply (10.9) by Λ^{-} on the left and also separately multiply it by Λ^{+} on the left to obtain the two equations

$$2i\partial_{-}\psi_{(-)} = [-i\gamma^{0}(\gamma^{1}\partial_{1} + \gamma^{2}\partial_{2}) + m\gamma^{0}]\psi_{(+)}, \quad 2i\partial_{+}\psi_{(+)} = [-i\gamma^{0}(\gamma^{1}\partial_{1} + \gamma^{2}\partial_{2}) + m\gamma^{0}]\psi_{(-)}. \tag{10.10}$$

Since the $\partial_{-}\psi_{(-)}$ equation contains no time derivatives, $\psi_{(-)}$ is thus a constrained variable, consistent with it having no conjugate. Through the use of the inverse propagator $(\partial_{-})^{-1}(x^{-}) = \epsilon(x^{-})/2$ we can rewrite the $\partial_{-}\psi_{(-)}$ equation in (10.9) as

$$\begin{aligned}\psi_{(-)}(x^{+},x^{1},x^{2},x^{-}) &= \frac{1}{4i}\int du^{-}\epsilon(x^{-}-u^{-})[-i\gamma^{0}(\gamma^{1}\partial_{1} + \gamma^{2}\partial_{2}) + m\gamma^{0}]\psi_{(+)}(x^{+},x^{1},x^{2},u^{-}),\\ [\psi_{(-)}]^{\dagger} &= \frac{i}{4}\int du^{-}\epsilon(x^{-}-u^{-})[i\partial_{1}[\psi_{(+)}]^{\dagger}\gamma^{0}\gamma^{1} + i\partial_{2}[\psi_{(+)}]^{\dagger}\gamma^{0}\gamma^{2} + m[\psi_{(+)}]^{\dagger}\gamma^{0}],\end{aligned} \tag{10.11}$$

and recognize $\psi_{(-)}$ as obeying a constraint condition that is nonlocal. It is because $\psi_{(-)}$ obeys such a nonlocal constraint that it is known as a bad fermion. Since it is a constrained variable it does not appear in any fundamental anti-commutation relation. Nonetheless, one can still use (10.8) and (10.11) to construct a $\left\{\psi_{(-)},\psi^{\dagger}_{(-)}\right\}$ anti-commutator.

In this way we obtain (Mannheim, Lowdon and Brodsky 2019)

$$\left\{\frac{\partial}{\partial x^-}\psi_\alpha^{(-)}(x^+,x^1,x^2,x^-),\frac{\partial}{\partial y^-}[\psi^\dagger_{(-)}]_\beta(x^+,y^1,y^2,y^-)\right\}$$
$$=\frac{1}{4}\Lambda^-_{\alpha\beta}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\delta(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2), \tag{10.12}$$

$$\left\{\psi_\alpha^{(-)}(x^+,x^1,x^2,x^-),[\psi^\dagger_{(-)}]_\beta(x^+,y^1,y^2,y^-)\right\}$$
$$=\frac{1}{16}\Lambda^-_{\alpha\beta}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\int du^-\epsilon(x^- - u^-)\epsilon(y^- - u^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{10.13}$$

As we see, the equal x^+ bad fermion sector $\left\{\psi_\alpha^{(-)}(x^+,x^1,x^2,x^-),[\psi^\dagger_{(-)}]_\beta(x^+,y^1,y^2,y^-)\right\}$ anti-commutator is non-vanishing, with its nonlocal nature being apparent. However this non-locality is restricted to the light cone since with $x^+=y^+$, $x^1=y^1$, $x^2=y^2$ the quantity $(x^+-y^+)(x^--y^-)-(x^1-y^1)^2-(x^2-y^2)^2$ is zero for any value of $x^- - y^-$. As we also see, the equal light-front time fermion sector anti-commutators given in (10.8), (10.12) and (10.13) not only look different from their instant-time counterparts given in (10.2), because of the presence of the non-invertible good and bad projection operators they appear to be altogether inequivalent to their instant-time counterparts.

But still the light-front normalization is arbitrary.

To reconcile everything we will look at **UNEQUAL** time commutators and anti-commutators. But first a Dirac equation curiosity, Fock space, and the light-front Hamiltonian.

11 A DIRAC EQUATION CURIOSITY

In writing down the Dirac equation Dirac did not start with the covariant

$$(i\gamma^0\partial_t + i\gamma^k\partial_k - m)\psi = 0, \tag{11.1}$$

but instead started with

$$i\partial_t\psi + i\alpha^k\partial_k\psi - \beta m\psi = 0. \tag{11.2}$$

These equations are equivalent since (11.1) and (11.2) can be derived from each other by multiplying through by $\beta = \gamma^0$ and setting $\gamma^k = \beta\alpha^k$. In the Dirac basis for the gamma matrices one takes $\gamma^0_{\rm D}$ to be diagonal and has

$$\gamma^0_{\rm D} = \beta_{\rm D} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^k_{\rm D} = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \gamma^k_{\rm D} = \beta_{\rm D}\alpha^k_{\rm D} = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma^5_{\rm D} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{11.3}$$

In light-front coordinates the covariant Dirac equation takes the form

$$(i\gamma^+\partial_+ + i\gamma^-\partial_- + i\gamma^1\partial_1 + i\gamma^2\partial_2 - m)\psi = 0, \tag{11.4}$$

where $\gamma^\pm = \gamma^0 \pm \gamma^3$. However there is no light-front analog of (11.2), since $(\gamma^+)^2$ and $(\gamma^-)^2$ are zero, to thus not be invertible. Thus even though the invertible γ^0 and γ^3 obey $(\gamma^0)^2 = 1$, $(\gamma^3)^2 = -1$, the non-invertible γ^+ and γ^- are divisors of zero. Consequently, one cannot multiply the light-front (11.4) by γ^+ and obtain a light-front analog of (11.2). Since γ^+ and γ^- are divisors of zero, there is no similarity transformation that can effect $S\gamma^0S^{-1} = \gamma^+$, $S\gamma^3S^{-1} = \gamma^-$, and there thus are intrinsic differences between light-front fermions and instant-time fermions. Moreover, it is because γ^+ and γ^- are divisors of zero that (11.4) breaks up into good and fermions, with $\Lambda^\pm = (1/2)\gamma^0\gamma^\pm$ being projection operators, i.e., being operators that also are not invertible.

Finally, we note that even though γ^+ and γ^- are themselves divisors of zero, the products $\gamma^+\gamma^-$ and $\gamma^-\gamma^+$ are not, with combination $\gamma^+\gamma^- + \gamma^-\gamma^+$ evaluating to 4. In consequence, $i\gamma^+\partial_+ + i\gamma^-\partial_-$ squares to $-4\partial_+\partial_-$, with the Klein-Gordon equation in the form $[4\partial_+\partial_- - (\partial_1)^2 - (\partial_2)^2 + m^2]\psi = 0$ then following from (11.4).

12 WEYL BASIS FOR THE DIRAC GAMMA MATRICES

In working with the Λ^+ and Λ^- projection operators it would be very convenient if we could find a basis for the gamma matrices in which Λ^+ and Λ^- are diagonal. It turns out that there is such a basis, the one Weyl used to diagonalize γ^5. The Weyl basis γ^μ_W is constructed from the Dirac basis γ^μ_{D} via the similarity transform

$$\gamma^\mu_{\mathrm{W}} = \tfrac{1}{\sqrt{2}}(1 - \gamma^5_{\mathrm{D}}\gamma^0_{\mathrm{D}})\gamma^\mu_{\mathrm{D}}\tfrac{1}{\sqrt{2}}(1 + \gamma^5_{\mathrm{D}}\gamma^0_{\mathrm{D}}). \tag{12.1}$$

This yields

$$\gamma^0_{\mathrm{W}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^k_{\mathrm{W}} = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \alpha^k_{\mathrm{W}} = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \quad \gamma^5_{\mathrm{W}} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{12.2}$$

$$\Lambda^\pm_{\mathrm{W}} = \tfrac{1}{\sqrt{2}}(1 - \gamma^5_{\mathrm{D}}\gamma^0_{\mathrm{D}}) \left[\tfrac{1}{2}\left(1 \pm \gamma^0_{\mathrm{D}}\gamma^3_{\mathrm{D}}\right)\right] \tfrac{1}{\sqrt{2}}(1 + \gamma^5_{\mathrm{D}}\gamma^0_{\mathrm{D}}) = \tfrac{1}{2}\left(1 \pm \gamma^0_{\mathrm{W}}\gamma^3_{\mathrm{W}}\right). \tag{12.3}$$

In this basis we find that not only is γ^5_{w} diagonal, Λ^+_{W} and Λ^-_{W} are diagonal too. They take the form

$$\Lambda^+_{\mathrm{W}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda^-_{\mathrm{W}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12.4}$$

and effect

$$\Lambda^+_{\mathrm{W}} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \\ \psi_4 \end{pmatrix}, \quad \Lambda^-_{\mathrm{W}} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_2 \\ \psi_3 \\ 0 \end{pmatrix}. \tag{12.5}$$

Hence in the Weyl basis we can treat the good and bad fermions as two-component spinors.

13 INSTANT-TIME AND LIGHT-FRONT SCALAR FOCK SPACE EXPANSIONS

The normalization of the commutation relations fixes the normalization of $[a, a^\dagger]$ commutators

$$\begin{aligned}
&[\phi(x^0, x^1, x^2, x^3), \partial_0\phi(x^0, y^1, y^2, y^3)] = i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3). \\
&[\phi(x^+, x^1, x^2, x^-), 2\partial_-\phi(x^+, y^1, y^2, y^-)] = i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^- - y^-).
\end{aligned} \tag{13.1}$$

In solutions to the field equations the instant-time scalar field Fock space expansion with $E_p^2 = p_1^2 + p_2^2 + p_3^2 + m^2$ is of form

$$\phi(x^0, x^1, x^2, x^3) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E_p)^{1/2}} [a(\vec{p})e^{-iE_pt + i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{+iE_pt - i\vec{p}\cdot\vec{x}}], \quad [a(\vec{p}), a^\dagger(\vec{p}')] = \delta^3(\vec{p} - \vec{p}') \tag{13.2}$$

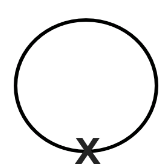
Contains $-\infty \leq p_3 \leq \infty$, well-behaved at $p_3 = 0$.

In solutions to the field equations the light-front scalar field Fock space expansion with $F_p^2 = (p_1)^2 + (p_2)^2 + m^2$ is of form

$$\begin{aligned}
&\phi(x^+, x^1, x^2, x^-) = \frac{2}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{(4p_-)^{1/2}} \\
&\times \left[e^{-i(F_p^2 x^+/4p_- + p_- x^- + p_1 x^1 + p_2 x^2)} a(p_1, p_2, p_-) + e^{i(F_p^2 x^+/4p_- + p_- x^- + p_1 x^1 + p_2 x^2)} a_p^\dagger(p_1, p_2, p_-) \right], \\
&[a(\vec{p}), a^\dagger(\vec{p}')] = \frac{1}{2}\delta(p_- - p'_-)\delta(p_1 - p'_1)\delta(p_2 - p'_2)
\end{aligned} \tag{13.3}$$

Singular at $p_- = 0$, undefined at $x^+ = 0$, $p_- = 0$, $(p_- = p^+/2,\ p_+ = p^-/2)$.
Contains $0 \leq p_- \leq \infty$ only, Light-Front Hamiltonian approach restricts to $p_- > 0$, $p_+ < \infty$.

Thus we go beyond the Light-Front Hamiltonian if have processes with $p_- = 0$. This happens in the vacuum sector where the tadpole is $-i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle$ with $x^+ = 0$. If we bring zero four-momentum into the cross in the vacuum tadpole then the only allowed momentum in loop has $p_- = 0$. If we exclude $p_- = 0$ then tadpole is zero. This is a potential solution to the cosmological constant problem. But it fails since we have to deal with the indeterminacy of x^+/p_- at $x^+ = 0$, $p_- = 0$.



14 INSTANT-TIME AND LIGHT-FRONT FERMION FOCK SPACE EXPANSIONS

The instant time anti-commutation relations are of the form

$$\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(x^0,y^1,y^2,y^3)\right\}=\delta_{\alpha\beta}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3). \tag{14.1}$$

In solutions to the field equations the instant-time fermion field Fock space expansion with $E_p^2=p_1^2+p_2^2+p_3^2+m^2$ is of form

$$\psi(\vec{x},x^0)=\sum_{s=\pm}\int\frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m}{E_p}\right)^{1/2}[b(\vec{p},s)u(\vec{p},s)e^{-ip\cdot x}+d^\dagger(\vec{p})v(\vec{p},s)e^{+ip\cdot x}], \tag{14.2}$$

where s denotes the spin projection, where the Dirac spinors $u(\vec{p},s)$ and $v(\vec{p},s)$ obey $(\not{p}-m)u(\vec{p},s)=0$, $(\not{p}+m)v(\vec{p},s)=0$, and where the non-trivial creation and annihilation operator anti-commutation relations are of the form

$$\{b(\vec{p},s),b^\dagger(\vec{p}',s')\}=\delta_{s,s'}\delta^3(\vec{p}-\vec{p}'),\quad \{d(\vec{p},s),d^\dagger(\vec{p}',s')\}=\delta_{s,s'}\delta^3(\vec{p}-\vec{p}'). \tag{14.3}$$

The light-front anti-commutation relations for the good fermions (the ones with canonical conjugates) are of the form

$$\left\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\}=\Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2) \tag{14.4}$$

and the Dirac equation is of the form

$$(i\gamma^+\partial_++i\gamma^-\partial_-+i\gamma^1\partial_1+i\gamma^2\partial_2-m)\psi=0,\quad u^\dagger_{(+)}(p,s)u_{(+)}(p,s')=2p_-\delta_{s,s'},\quad v^\dagger_{(+)}(p,s)v_{(+)}(p,s')=2p_-\delta_{s,s'} \tag{14.5}$$

In solutions to the Dirac equation the light-front fermion field Fock space expansion with $F_p^2=(p_1)^2+(p_2)^2+m^2$ is of form

$$\psi_{(+)}(x)=\sum_{s=\pm}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}dp^+\frac{1}{(2\pi)^{3/2}}\frac{1}{(p^+)^{1/2}}[b(\vec{p},s)u(\vec{p},s)e^{-ip\cdot x}+d^\dagger(\vec{p})v(\vec{p},s)e^{+ip\cdot x}], \tag{14.6}$$

where s denotes the spin projection, where the Dirac spinors $u(\vec{p},s)$ and $v(\vec{p},s)$ obey $(\not{p}-m)u(\vec{p},s)=0$, $(\not{p}+m)v(\vec{p},s)=0$, and where the non-trivial creation and annihilation operator anti-commutation relations are of the form

$$\{b(\vec{p},s),b^\dagger(\vec{p}',s')\}=\delta_{s,s'}\delta(p_--p'_-)\delta(p_1-p'_1)\delta(p_2-p'_2),\quad \{d(\vec{p},s),d^\dagger(\vec{p}',s')\}=\delta_{s,s'}\delta(p_--p'_-)\delta(p_1-p'_1)\delta(p_2-p'_2). \tag{14.7}$$

15 THE LIGHT-FRONT HAMILTONIAN

The light-front scalar field action is of the form

$$\begin{aligned} I_S &= \int d^4x(-g)^{1/2}\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi = \tfrac{1}{2}\int dx^+dx^-dx^1dx^2\tfrac{1}{2}[\partial_+\phi\partial^+\phi+\partial_-\phi\partial^-\phi+\partial_1\phi\partial^1\phi+\partial_2\phi\partial^2\phi] \\ &= \int d^4x(-g)^{1/2}\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi = \tfrac{1}{2}\int dx^+dx^-dx^1dx^2\tfrac{1}{2}[2\partial_+\phi\partial_-\phi+2\partial_-\phi\partial_+\phi-\partial_1\phi\partial_1\phi-\partial_2\phi\partial_2\phi], \end{aligned} \tag{15.1}$$

where $\partial_+=\partial\phi/\partial x^+$, $\partial_-=\partial\phi/\partial x^-$, $\partial^+\phi=2\partial_-\phi=2\partial\phi/\partial x^-$, $\partial^-\phi=2\partial_+\phi=2\partial\phi/\partial x^+$. With light-front metric $ds^2=g_{\mu\nu}dx^\mu dx^\nu=(1/2)dx^+dx^-+(1/2)dx^-dx^+-(dx^1)^2-(dx^2)^2$ we define

$$T_{\mu\nu}=\frac{1}{(-g)^{1/2}}\frac{\delta I_S}{\delta g^{\mu\nu}}=\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}\tfrac{1}{2}\partial^\alpha\phi\partial_\alpha\phi \tag{15.2}$$

and obtain

$$\begin{aligned} &T^+{}_+(\text{front})=\tfrac{1}{2}[\partial^+\phi\partial_+\phi-\partial^-\phi\partial_-\phi-\partial^1\phi\partial_1\phi-\partial^2\phi\partial_2\phi], \\ &T^+{}_-(\text{front})=\partial^+\phi\partial_-\phi,\quad T^+{}_1(\text{front})=\partial^+\phi\partial_1\phi,\quad T^+{}_2(\text{front})=\partial^+\phi\partial_2\phi. \end{aligned} \tag{15.3}$$

However, since $\partial^+\phi=g^{+\mu}\partial_\mu\phi=2\partial_-\phi$, $\partial_+\phi=g_{+\mu}\partial^\mu\phi=(1/2)\partial^-\phi$, we find that $\partial^+\phi\partial_+\phi-\partial^-\phi\partial_-\phi=0$. Thus we can replace (15.3) by

$$\begin{aligned} &T^+{}_+(\text{front})=\tfrac{1}{2}[\partial_1\phi\partial_1\phi+\partial_2\phi\partial_2\phi], \\ &T^+{}_-(\text{front})=2\partial_-\phi\partial_-\phi,\quad T^+{}_1(\text{front})=2\partial_-\phi\partial_1\phi,\quad T^+{}_2(\text{front})=2\partial_-\phi\partial_2\phi. \end{aligned} \tag{15.4}$$

With this $T_{\mu\nu}$ we fix the overall normalization of the light-front commutators as given earlier via

$$P_\mu=\tfrac{1}{2}\int dx^-dx^1dx^2T^+{}_\mu,\qquad [P_\mu,\phi]=-i\partial_\mu\phi \tag{15.5}$$

A similar calculation in the fermion case then yields the pure good fermion (Mannheim, Lowdon and Brodsky 2019)

$$P_\mu=\int dy^-dy^1dy^2i\psi^\dagger_{(+)}(y)\partial_\mu\psi_{(+)}(y),\quad [P_\mu,\psi_{(+)}(x)]=-i\partial_\mu\psi_{(+)}(x) \tag{15.6}$$

to then fix the overall normalization of the good fermion anti-commutators given earlier.

16 UNEQUAL TIME COMMUTATORS AND ANTI-COMMUTATORS

Following Mannheim (2020):
UNEQUAL TIME Scalar instant-time commutator

$$\begin{aligned} i\Delta(x-y) &= [\phi(x^0,x^1,x^2,x^3),\phi(y^0,y^1,y^2,y^3)] \\ &= \int \frac{d^3p d^3q}{(2\pi)^3(2p)^{1/2}(2q)^{1/2}}\Big([a(\vec{p}),a^\dagger(\vec{q})]e^{-ip\cdot x+iq\cdot y}+[a^\dagger(\vec{p}),a(\vec{q})]e^{ip\cdot x-iq\cdot y}\Big) \\ &= \int \frac{d^3p}{(2\pi)^3 2p}\big(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\big) \\ &= -\frac{i}{2\pi}\frac{\delta(x^0-y^0-|\vec{x}-\vec{y}|)-\delta(x^0-y^0+|\vec{x}-\vec{y}|)}{2|\vec{x}-\vec{y}|} \\ &= -\frac{i}{2\pi}\epsilon(x^0-y^0)\delta[(x^0-y^0)^2-(x^1-y^1)^2-(x^2-y^2)^2-(x^3-y^3)^2]. \end{aligned} \tag{16.1}$$

Since it holds at ALL times, it also holds at EQUAL light front time.

Substitute $x^0=(x^++x^-)/2$**,** $x^3=(x^+-x^-)/2$**,** $y^0=(y^++y^-)/2$**,** $y^3=(y^+-y^-)/2$**:**

$$i\Delta(x-y)=-\frac{i}{2\pi}\epsilon[\tfrac{1}{2}(x^++x^--y^+-y^-)]\delta[(x^+-y^+)(x^--y^-)-(x^1-y^1)^2-(x^2-y^2)^2]. \tag{16.2}$$

$$i\Delta(x-y)\big|_{x^+=y^+} = [\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)] = -\frac{i}{4}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{16.3}$$

At $x^+=y^+$ **UNEQUAL instant-time commutator is EQUAL light-front time commutator**

Light-front quantization is instant-time quantization, and does not need to be independently postulated.

UNEQUAL TIME Abelian gauge field instant-time commutator

$$[A_\nu(x^0,x^1,x^2,x^3),A_\mu(y^0,y^1,y^2,y^3)] = ig_{\mu\nu}\Delta(x-y)$$
$$= -\frac{i}{2\pi}g_{\mu\nu}\epsilon(x^0-y^0)\delta[(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2]. \tag{16.4}$$

Leads to

$$[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)] = \frac{i}{4}g_{\mu\nu}\epsilon(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{16.5}$$

At $x^+ = y^+$ UNEQUAL instant-time commutator is EQUAL light-front time commutator Similar result holds for non-Abelian gauge field.

17 FERMION UNEQUAL INSTANT-TIME ANTI-COMMUTATOR

$$\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(y^0,y^1,y^2,y^3)\right\}=\left[(i\gamma^\mu\gamma^0\partial_\mu\right]_{\alpha\beta}i\Delta(x-y). \tag{17.1}$$

Apply projector and set $x^+=y^+$

$$\begin{aligned}&\Lambda^+_{\alpha\gamma}\left\{\psi_\gamma(x^+,x^1,x^2,x^-),\psi_\delta(x^+,y^1,y^2,y^-)\right\}\Lambda^+_{\delta\beta}\\ &=\left\{[\psi_{(+)}(x^+,x^1,x^2,x^-)]_\alpha,[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\}=\Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2).\end{aligned} \tag{17.2}$$

At $x^+=y^+$ UNEQUAL instant-time anti-commutator is EQUAL light-front time anti-commutator. Can also derive anti-commutators involving bad fermions in the same way.

All cases discussed in Mannheim (2020).

Light-front quantization is instant-time quantization, and does not need to be independently postulated. The seemingly different structure between EQUAL instant-time and EQUAL light-front time commutators is actually a consequence of the structure of UNEQUAL instant-time time commutators and anti-commutators as restricted to equal x^0 or equal x^+.

Now the transformation $x^+ = x^0 + x^3$, $x^- = x^0 - x^3$ is not a Lorentz transformation but a translation, i.e., a general coordinate transformation. But for theories that are Poincare invariant this is a symmetry. Thus:

GENERAL RULE: ANY TWO DIRECTIONS OF QUANTIZATION THAT CAN BE CONNECTED BY A GENERAL COORDINATE TRANSFORMATION DESCRIBE THE SAME THEORY.

BUT IN THE QUANTUM THEORY TRANSLATIONS ARE UNITARY TRANSFORMATIONS. THUS INSTANT-TIME AND LIGHT-FRONT THEORIES ARE UNITARILY EQUIVALENT, AND ARE THUS ONE AND THE SAME THEORY.

19 UNITARY EQUIVALENCE VIA TRANSLATION INVARIANCE

So far the discussion has only dealt with free theory commutators, and they just happen to be c-numbers. However, for interacting theories we can only discuss matrix elements. With

$$[\hat{P}_\mu, \phi] = -i\partial_\mu \phi, \quad [\hat{P}_\mu, \hat{P}_\nu] = 0 \tag{19.1}$$

to all orders in perturbation theory because of Poincare invariance, we introduce

$$U(\hat{P}_0, \hat{P}_3) = \exp(ix^3\hat{P}_0)\exp(ix^0\hat{P}_3). \tag{19.2}$$

It effects

$$U\phi(IT; x^0, x^1, x^2, -x^3)U^{-1} = \phi(IT; x^0 + x^3, x^1, x^2, x^0 - x^3) = \phi(LF; x^+, x^1, x^2, x^-) \tag{19.3}$$

Then with a light-front vacuum of the form $|\Omega_F\rangle = U|\Omega_I\rangle$ we obtain

$$\begin{aligned} -i\langle\Omega_I|[\phi(IT; x^0, x^1, x^2, -x^3), \phi(0)]|\Omega_I\rangle &= -i\langle\Omega_I|U^\dagger U[\phi(IT; x^0, x^1, x^2, -x^3), \phi(0)]U^\dagger U|\Omega_I\rangle \\ &= -i\langle\Omega_F|[\phi(LF; x^+, x^1, x^2, x^-), \phi(0)]|\Omega_F\rangle, \end{aligned} \tag{19.4}$$

to all orders in perturbation theory. We thus establish the unitary equivalence of matrix elements of instant-time and light-front commutators to all orders.

The same equivalence holds for the all-order Lehmann representations. For the instant-time case we have

$$\begin{aligned}\langle\Omega|[\phi(IT;x),\phi(IT;y)]|\Omega\rangle &= \frac{1}{(2\pi)^3}\int_0^\infty d\sigma^2\rho(\sigma^2,IT)\int d^4q\epsilon(q_0)\delta(q^2-\sigma^2)e^{-iq\cdot(x-y)} \\ &= \int_0^\infty d\sigma^2\rho(\sigma^2,IT)i\Delta(IT,FREE;x-y,\sigma^2),\end{aligned}\tag{19.5}$$

where

$$\rho(q^2,IT)\theta(q_0)=(2\pi)^3\sum_n\delta^4(p_\mu^n-q_\mu)|\langle\Omega|\phi(0)|p_\mu^n\rangle|^2,\quad \hat{P}_\mu|p_\mu^n\rangle=p_\mu^n|p_\mu^n\rangle,\tag{19.6}$$

as written in instant-time momentum eigenstates.

For the light-front case we have

$$\begin{aligned}\langle\Omega|[\phi(LF;x),\phi(LF;y)]|\Omega\rangle &= \frac{2}{(2\pi)^3}\int_0^\infty d\sigma^2\rho(\sigma^2,LF)\int d^4q\epsilon(q_+)\delta(q^2-\sigma^2)e^{-iq\cdot(x-y)}. \\ &= \int_0^\infty d\sigma^2\rho(\sigma^2,LF)i\Delta(LF,FREE;x-y,\sigma^2),\end{aligned}\tag{19.7}$$

where

$$\rho(q_\mu,LF)=\frac{(2\pi)^3}{2}\sum_n\delta^4(p_\mu^n-q_\mu)|\langle\Omega|\phi(0)|p_\mu^n\rangle|^2=\rho(q^2,LF)\theta(q_+),\tag{19.8}$$

as written in light-front momentum eigenstates. Then with

$$U|p_0^n\rangle=|p_+^n\rangle,\quad U|p_3^n\rangle=|p_-^n\rangle,\quad U|p_1^n\rangle=|p_1^n\rangle,\quad U|p_2^n\rangle=|p_2^n\rangle\tag{19.9}$$

we obtain the all-order

$$\langle\Omega|[\phi(IT;x),\phi(IT;y)]|\Omega\rangle=\langle\Omega|[\phi(LF;x),\phi(LF;y)]|\Omega\rangle.\tag{19.10}$$

With the all-order momentum operators having real and complete eigenspectra we have the all-order

$$\hat{P}_\mu(IT) = \sum |p^n(IT)\rangle p^n_\mu(IT)\langle p^n(IT)|, \quad \hat{P}_\mu(LF) = \sum |p^n(LF)\rangle p^n_\mu(LF)\langle p^n(LF)|. \tag{19.11}$$

With eigenvalues not changing under a unitary transformation, we obtain

$$\begin{aligned}\hat{P}_0(IT) = U\hat{P}_0(IT)U^{-1} = U\sum |p^n(IT)\rangle p^n_0\langle p^n(IT)|U^\dagger \\ = \sum |p^n(LF)\rangle(p^n_+ + p^n_-)\langle p^n(LF)| = \hat{P}_+(LF) + \hat{P}_-(LF).\end{aligned} \tag{19.12}$$

Given (19.11) and (19.12), there initially appears to be a mismatch between the eigenstates of $\hat{P}_0(IT)$ and $\hat{P}_+(LF)$. However, for any timelike set of instant-time momentum eigenvalues we can Lorentz boost p_1, p_2 and p_3 to zero, to yield

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 0, \quad p_0 = m. \tag{19.13}$$

If we impose this same $p_1 = 0$, $p_2 = 0$, $p_3 = 0$ condition on the light-front momentum eigenvalues we would set $p_+ = p_-$, $p^2 = 4p_+^2 = m^2$, and thus obtain

$$p_1 = 0, \quad p_2 = 0, \quad p_+ = p_-, \quad p_0 = 2p_+ = m \tag{19.14}$$

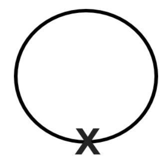
When written in terms of contravariant vectors with $p^\mu = g^{\mu\nu}p_\nu$ this condition takes the form

$$p^0 = p^- = m. \tag{19.15}$$

Thus in **the instant-time rest frame** the eigenvalues of the contravariant $\hat{P}^0(IT)$ and $\hat{P}^-(LF)$ coincide. In this sense then instant-time and light-front Hamiltonians are equivalent. And non-relativistic in the light-front case still means $p_3 = 0$, i.e., $p_+ = p_-$, and not $p_- = p^+/2 = 0$.

Having now established the equivalence of commutators and the equivalence of Hamiltonian operators, we now proceed to establish the same equivalence for both free and interacting instant-time and light-front Green's functions.

20 EQUIVALENCE OF INSTANT AND FRONT PROPAGATORS AND TADPOLES



Construct tadpole as $x^\mu \to 0$ limit of propagator (not two-point function), i.e., use x^μ as a regulator.

$$D(x^\mu) = -i\langle\Omega|[\theta(\sigma)\phi(x)\phi(0) + \theta(-\sigma)\phi(0)\phi(x)]|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon}, \quad \sigma = \mathrm{x}^0 \text{ or } \sigma = \mathrm{x}^+ \tag{20.1}$$

$$D(x^\mu = 0) = -i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p\frac{1}{p^2 - m^2 + i\epsilon}. \tag{20.2}$$

$$\begin{aligned}
D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0dp_1dp_2dp_3\frac{e^{-i(p_0x^0+p_1x^1+p_2x^2+p_3x^3)}}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon},\\
D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+dp_1dp_2dp_-\frac{e^{-i(p_+x^++p_1x^1+p_2x^2+p_-x^-)}}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon},\\
D(x^\mu = 0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0dp_1dp_2dp_3\frac{1}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon},\\
D(x^\mu = 0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+dp_1dp_2dp_-\frac{1}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon}.
\end{aligned} \tag{20.3}$$

For all of these Feynman contours there are only poles, except $D(x^\mu = 0, \text{front})$, for which the circle at infinity in the complex p_+ plane is not suppressed.

21 THE NON-VACUUM INSTANT-TIME CASE

In the instant-time case the Feynman integral is readily performed since it is just pole terms and for the **forward** $D(x^0 > 0, \text{instant}) = -i\langle \Omega_I | \theta(x^0)\phi(x^0, x^1, x^2, x^3)\phi(0)|\Omega_I\rangle$ we obtain

$$\begin{aligned} D(x^0 > 0, \text{instant}) &= D(x^0 > 0, \text{instant}, \text{pole}) \\ &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}} = \frac{1}{8\pi}\left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}). \end{aligned} \tag{21.1}$$

Insertion of the Fock space expansion for $\phi(x^0, x^1, x^2, x^3)$ yields

$$D(x^0 > 0, \text{instant}, \text{ Fock}) = -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}}. \tag{21.2}$$

We recognize (21.2) as (21.1), to thus establish the equivalence of the instant-time Feynman and Fock space prescriptions.

22 THE NON-VACUUM LIGHT-FRONT CASE

In the light-front case poles in the complex p_+ plane occur at

$$p_+ = E'_p - \frac{i\epsilon}{4p_-}, \quad E'_p = \frac{(p_1)^2 + (p_2)^2 + m^2}{4p_-}. \tag{22.1}$$

Poles with $p_- \geq 0^+$ thus all lie below the real p_+ axis and have positive E'_p, while poles with $p_- \leq 0^-$ all lie above the real p_+ axis and have negative E'_p. For $x^+ > 0$, closing the p_+ contour below the real axis (which for $x^+ > 0$ suppresses the circle at infinity contribution) then restricts to poles with $E'_p > 0$, $p_- \geq 0^+$. However, in order to evaluate the pole terms one has to deal with the fact that the pole at $p_- = 0^+$ has $E'_p = \infty$. Momentarily exclude the region around $p_- = 0$, and thus only consider poles below the real p_+ axis that have $p_- \geq \delta$. Evaluating the contour integral in the lower half of the complex p_+ plane thus gives

$$\begin{aligned}
D(x^+ > 0, \text{front}, \text{pole}) &= -\frac{2i}{(2\pi)^3}\int_\delta^\infty \frac{dp_-}{4p_-}\int_{-\infty}^\infty dp_1 \int_{-\infty}^\infty dp_2 e^{-i(E'_p x^+ + p_- x^- + p_1 x^1 + p_2 x^2) - \epsilon x^+/4p_-} \\
&= -\frac{1}{4\pi^2 x^+}\int_\delta^\infty dp_- e^{-ip_- x^- + i[(x^1)^2 + (x^2)^2]p_-/x^+ - im^2 x^+/4p_- - \epsilon x^+/4p_-} \\
&= -\frac{1}{4\pi^2 x^+}\int_\delta^\infty dp_- e^{-ip_- x^2/x^+ - im^2 x^+/4p_- - \epsilon x^+/4p_-}.
\end{aligned} \tag{22.2}$$

If we now set $\alpha = x^+/4p_-$, we obtain

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{1}{16\pi^2}\int_0^{x^+/4\delta} \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon}. \tag{22.3}$$

In (22.3) we can now take the limit $\delta \to 0$, $x^+/4\delta \to \infty$ without encountering any ambiguity **AS LONG AS x^+ IS NONZERO**, and with $x^+ > 0$ thus obtain

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon} = \frac{1}{8\pi}\left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}). \tag{22.4}$$

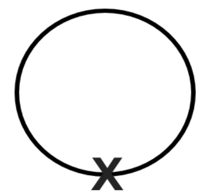
Comparing with (21.1) we see that $D(x^+ > 0, \text{instant})$ and $D(x^+ > 0, \text{front})$ are equal.

Inserting the Fock space expansion for $\phi(x^+, x^1, x^2, x^-)$ gives precisely the same result, and thus we obtain

$$\begin{aligned} D(x^0 > 0, \text{instant}) &= D(x^0 > 0, \text{instant}, \text{pole}) = D(x^0 > 0, \text{instant}, \text{Fock}) \\ &= D(x^+ > 0, \text{front}) = D(x^+ > 0, \text{front}, \text{pole}) = D(x^+ > 0, \text{front}, \text{Fock}). \end{aligned} \tag{22.5}$$

General rule: the Feynman and Fock space prescriptions will coincide whenever the only contribution to Feynman contours is poles. Thus for $x^+ > 0$ the Feynman and Light-Front Hamiltonian approaches coincide. But what about $x^+ = 0$?

23 THE INSTANT-TIME VACUUM CASE



In the instant-time case one can readily set x^μ to zero, and obtain

$$\begin{aligned} D(x^\mu=0,\text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0dp_1dp_2dp_3\frac{1}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon} \\ &= D(x^\mu=0,\text{instant},\text{pole}) = D(x^\mu=0,\text{instant},\text{Fock}) \\ &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p} = -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}. \end{aligned} \tag{23.1}$$

24 THE LIGHT-FRONT VACUUM CASE - POLE AND FOCK SPACE CONTRIBUTIONS

In the light-front case we set x^μ to zero and evaluate

$$D(x^\mu = 0, \text{front}) = \frac{2}{(2\pi)^4} \int dp_+ dp_1 dp_2 \frac{dp_-}{4p_-} \frac{1}{[p_+ - [(p_1)^2 + (p_2)^2 + m^2]/4p_- + i\epsilon]}. \tag{24.1}$$

Again we need to take care of the $p_- = 0$ region, so we again introduce the δ cutoff at small p_-. On closing below the real p_+ axis the only poles are those with $p_- > 0$, and for them we obtain a pole contribution of the form

$$D(x^\mu = 0, \text{front}, \text{pole}) = -\frac{2i}{(2\pi)^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{\delta}^{\infty} \frac{dp_-}{4p_-}, \tag{24.2}$$

with the residue at the pole being a constant, i.e., not depending on $[(p_1)^2 + (p_2)^2 + m^2]/4p_-$. Then on setting $p_- = 1/\alpha$, we are able to let p_- go to zero, to obtain

$$D(x^\mu = 0, \text{front}, \text{pole}) = -\frac{i}{16\pi^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{1/\delta} \frac{d\alpha}{\alpha} = -\frac{i}{16\pi^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{\infty} \frac{d\alpha}{\alpha}. \tag{24.3}$$

For the Fock space prescription we set

$$\phi(0) = \frac{2}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{\infty} \frac{dp_-}{(4p_-)^{1/2}} [a_p + a_p^\dagger], \tag{24.4}$$

and on inserting $\phi(0)$ into $-i\langle\Omega|\phi(0)\phi(0)]|\Omega\rangle$ obtain

$$D(x^\mu = 0, \text{front}, \text{Fock}) = -\frac{2i}{(2\pi)^3} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{\infty} \frac{dp_-}{4p_-} = D(x^\mu = 0, \text{front}, \text{pole}). \tag{24.5}$$

Comparing with (24.2) we again see the equivalence of the pole and Fock space prescriptions.

However, something is wrong. We are evaluating the m-dependent $D(x^\mu = 0, \text{front})$ as given in (24.1), and yet we obtain an answer that does not depend on m at all. **What went wrong is that we left out the circle at infinity.**

25 THE LIGHT-FRONT VACUUM CASE - CIRCLE AT INFINITY CONTRIBUTION

To evaluate the circle at infinity contribution we introduce the regulator

$$\frac{1}{(A+i\epsilon)} = -i\int_0^\infty d\alpha e^{i\alpha(A+i\epsilon)}. \tag{25.1}$$

For $p_- > 0$ the regulator converges on the **UPPER** half circle, and there are no poles at all. We obtain

$$\begin{aligned}
&D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) \\
&= \frac{2i}{(2\pi)^4}\int_0^\infty dp_- \int_{-\infty}^\infty dp_1 \int_{-\infty}^\infty dp_2 \int_0^\pi iRe^{i\theta}d\theta \int_0^\infty d\alpha e^{i\alpha(4p_-Re^{i\theta}-(p_1)^2-(p_2)^2-m^2+i\epsilon)} \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\int_0^\pi iRe^{i\theta}d\theta e^{4i\alpha p_-Re^{i\theta}} \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{(e^{4i\alpha p_-Re^{i\theta}} - e^{-4i\alpha p_-Re^{i\theta}})}{4i\alpha p_-}\Bigg|_0^\pi \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{(e^{-4i\alpha p_-R} - e^{4i\alpha p_-R})}{4i\alpha p_-} \\
&= -\frac{1}{4\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{\sin(4\alpha p_-R)}{4\alpha p_-}.
\end{aligned} \tag{25.2}$$

Then, on letting R go to infinity we obtain

$$\begin{aligned}
&D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) = -\frac{1}{4\pi^2}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-) \\
&= -\frac{1}{8\pi^2}\int_{-\infty}^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-) = -\frac{1}{32\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}.
\end{aligned} \tag{25.3}$$

We thus establish the centrality of $p_- = 0$ modes.

Similarly, for $p_- < 0$ close on the **LOWER** half circle, and again there are no poles. We obtain

$$D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) = D(x^\mu = 0, p_- < 0, \text{front}, \text{lower circle}), \tag{25.4}$$

and thus

$$
\begin{aligned}
D(x^\mu = 0, \text{front}) &= D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) + D(x^\mu = 0, p_- < 0, \text{front}, \text{lower circle}) \\
&= -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}.
\end{aligned}
\tag{25.5}
$$

Now not only is there now an m dependence, we obtain

$$
D(x^\mu = 0, \text{front}) = D(x^\mu = 0, \text{instant}). \tag{25.6}
$$

So again, light-front quantization is instant-time quantization. And even though there is only a circle at infinity contribution in the light front case, it is this circle at infinity that enables the light-front and instant-time vacuum graphs to be the same.

26 RECONCILING THE FOCK SPACE AND FEYNMAN CALCULATIONS

To avoid $p_- = 0$ difficulties we use the regulator on the real p_+ axis, and set

$$
\begin{aligned}
&D(x^\mu, \text{front}, \text{regulator}) \\
&= -\frac{2i}{(2\pi)^4}\int_{-\infty}^{\infty} dp_+ \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_- e^{-i(p_+x^+ + p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon)} \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} dp_- e^{-i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(-(p_1)^2 - (p_2)^2 - m^2 + i\epsilon)}\delta(4\alpha p_- - x^+) \\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{0} dp_- e^{-i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(-(p_1)^2 - (p_2)^2 - m^2 + i\epsilon)}\delta(4\alpha p_- - x^+).
\end{aligned}
\tag{26.1}
$$

On changing the signs of p_-, p_1 and p_2 in the last integral and setting F_p^2 equal to the positive $(p_1)^2 + (p_2)^2 + m^2$ we obtain

$$
\begin{aligned}
&D(x^\mu, \text{front}, \text{regulator}) \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} e^{-i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{ix^+(-F_p^2 + i\epsilon)/4p_-}\delta(\alpha - x^+/4p_-) \\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} e^{i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{ix^+(F_p^2 - i\epsilon)/4p_-}\delta(\alpha + x^+/4p_-) \\
&= -\frac{2i\theta(x^+)}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} e^{-i(F_p^2x^+/4p_- + p_-x^- + p_1x^1 + p_2x^2 + ix^+\epsilon/4p_-)} \\
&-\frac{2i\theta(-x^+)}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} e^{i(F_p^2x^+/4p_- + p_-x^- + p_1x^1 + p_2x^2 - ix^+\epsilon/4p_-)},
\end{aligned}
\tag{26.2}
$$

and note that the structure of (26.2) is such that for $x^+ > 0$ (forward in time) one only has positive energy propagation, while for $x^+ < 0$ (backward in time) one only has negative energy propagation. With the insertion into $D(x^\mu) = -i\langle\Omega|[\theta(x^+)\phi(x)\phi(0) + \theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle$ of the Fock space expansion for $\phi(x^\mu)$ precisely leading to (26.2), we recognize (26.2) as the $x^\mu \neq 0$ $D(x^\mu, \text{front}, \text{Fock})$.

Now if we set $x^\mu = 0$ in (26.2) we would appear to obtain the m-independent $D(x^\mu = 0, \text{front}, \text{Fock})$ given in (24.5). However, we cannot take the $x^+ \to 0$ limit since the quantity $x^+/4p_-$ is undefined if p_- is zero, and $p_- = 0$ is included in the integration range. Hence, just as discussed in regard to (22.3), the limit is singular.

To obtain a limit that is not singular we note that we can set x^μ to zero in (26.1) as there the limit is well-defined, and this leads to

$$
\begin{aligned}
&D(x^\mu = 0, \text{front}, \text{regulator}) \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} dp_- \int_0^{\infty} d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-) \\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{0} dp_- \int_0^{\infty} d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-) \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_- \int_0^{\infty} \frac{d\alpha}{4\alpha} e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(p_-),
\end{aligned}
\tag{26.3}
$$

and again see the **centrality of $p_- = 0$ modes**. If we do the momentum integrations we obtain the m-dependent

$$
D(x^\mu = 0, \text{front}, \text{regulator}) = -\frac{1}{16\pi^2}\int_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \tag{26.4}
$$

We recognize (26.4) as being of the same form as the m-dependent $D(x^\mu = 0, \text{front})$ given in (25.5). We thus have to conclude that the limit $x^\mu \to 0$ of (26.2) is not (24.5) but is (26.4) instead, and that

$$
D(x^\mu = 0, \text{front}) = D(x^\mu = 0, \text{instant}) = -\frac{1}{16\pi^2}\int_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \tag{26.5}
$$

Setting $p_- = 0$ and then $x^+ = 0$ is not the same as setting $x^+ = 0$ and then $p_- = 0$.

Thus because of singularities we first have to point split, and when we do so we find that it is the m-dependent (26.4) that is the correct value for the light-front vacuum graph. And it is equal to the instant-time vacuum graph.

27 RELATIVISTIC EIKONALIZATION AND THE LIGHT FRONT

For eikonalization of a light wave one defines $A_\mu = \epsilon_\mu e^{iT}$ and takes the eikonal phase to obey

$$\partial_\mu T = \frac{dx_\mu}{dq} = k_\mu, \quad k_\mu k^\mu = 0, \tag{27.1}$$

where q is an affine parameter that measures distance along the light ray (the normal to the propagating wavefront). But if we set $T = \int^x k_\mu dx^\mu$, we would have $T = 0$. If momentarily we nonetheless do set $T = \int^x k_\mu dx^\mu$, then for $k_\mu = (k, 0, 0, k)$ we would have

$$(\partial_0 + \partial_3)T = 0, \tag{27.2}$$

which we recognize as a light-front constraint. Now in light-front coordinates we have

$$k_\mu k^\mu = 4k_+k_- - k_1^2 - k_2^2, \quad \partial_+ = \frac{1}{2}(\partial_0 + \partial_3), \qquad \partial_- = \frac{1}{2}(\partial_0 - \partial_3) \tag{27.3}$$

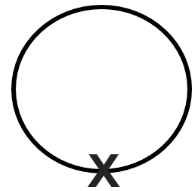
Now we can be on the light cone if $k_+ = k_1 = k_2 = 0$, with k_- unconstrained. Thus we can now set

$$T = \int^x k_- dx^-, \tag{27.4}$$

a quantity that is non-zero on the light cone. Since T does not depend on x^+ it still obeys $\partial_+ T = 0$.

The eikonalized ray thus travels on a light-front trajectory and not on an instant-time one (Mannheim, Class. Quant. Grav **39**, 245001 (2022), arXiv:2105.08556 [gr-qc]), with the trajectory being the normal to the wave front of the propagating light wave.

28 Light-front axial-vector Ward identity



Since the standard discussion of dynamical symmetry breaking only involves Feynman diagrams (non-trivial solutions to the Schwinger-Dyson, Bethe-Salpeter and vacuum energy equations), the outcome is the same in both instant-time quantization and light-front quantization, though in the light-front case we need to include the circle at infinity contributions to the tadpole graph. While the same spontaneously broken symmetry outcome must also occur in the axial vector Ward identity, the way that it does so in the light-front case is somewhat different from the way it does so in the instant-time case. This is because of the role played by the light-front bad fermions.

To see the issues involved we analyze the components of the axial-vector current $A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. In light-front components $A^+ = 2\psi^\dagger_{(+)}\gamma^5\psi_{(+)}$ is written in terms of good fermions alone, $A^- = 2\psi^\dagger_{(-)}\gamma^5\psi_{(-)}$ is written in terms of bad fermions alone, and $A^1 = \psi^\dagger\gamma^0\gamma^1\gamma^5\psi$ and $A^2 = \psi^\dagger\gamma^0\gamma^2\gamma^5\psi$ contain both good and bad fermions. We take the axial-vector current to be conserved so that $\partial_+A^+ + \partial_-A^- + \partial_1A^1 + \partial_2A^2 = 0$. While the axial charge $Q^5 = (1/2)\int dx^-dx^1dx^2A^+ = \int dx^-dx^1dx^2\psi^\dagger_{(+)}\gamma^5\psi_{(+)}$ only contains good fermions, its light-front time derivative $\partial_+Q^5 = -(1/2)\int dx^-dx^1dx^2(\partial_-A^- + \partial_1A^1 + \partial_2A^2)$ involves both good and bad fermions. Since $\psi_{(-)}$ obeys the nonlocal $\psi_{(-)} = -(i/2)(\partial_-)^{-1}[-i\gamma^0(\gamma^1\partial_1 + \gamma^2\partial_2) + m\gamma^0]\psi_{(+)}$ given in (7.4), to secure the light-front time independence of Q^5 requires that the fermion fields be more convergent asymptotically than in the instant-time case. In addition, the scalar and pseudoscalar fermion bilinears are of the form

$$\bar{\psi}\psi = \psi^\dagger_{(+)}\gamma^0\psi_{(-)} + \psi^\dagger_{(-)}\gamma^0\psi_{(+)}, \quad \bar{\psi}i\gamma^5\psi = \psi^\dagger_{(+)}i\gamma^0\gamma^5\psi_{(-)} + \psi^\dagger_{(-)}i\gamma^0\gamma^5\psi_{(+)}, \tag{28.1}$$

and thus they both contain both good and bad fermions.

Noting that generically we have

$$A^\dagger BC^\dagger D - C^\dagger DA^\dagger B \ = A^\dagger(BC^\dagger + C^\dagger B)D - C^\dagger(DA^\dagger + A^\dagger D)B - (A^\dagger C^\dagger + C^\dagger A^\dagger)BD + C^\dagger A^\dagger(BD + DB), \tag{28.2}$$

on using the equal light-front time anticommutators given earlier we obtain

$$\begin{aligned}[Q^5, \bar{\psi}(x)i\gamma^5\psi(x)] &= \int dy^- dy^1 dy^2 [\psi^\dagger_{(+)}(y)\gamma^5\psi_{(+)}(y), i\psi^\dagger_{(+)}(x)\gamma^0\gamma^5\psi_{(-)}(x) + i\psi^\dagger_{(-)}(x)\gamma^0\gamma^5\psi_{(+)}(x)] \\ &= i\psi^\dagger_{(+)}(x)\gamma^0\psi_{(-)}(x) + i\psi^\dagger_{(-)}(x)\gamma^0\psi_{(+)}(x) = i\bar{\psi}(x)\psi(x).\end{aligned} \tag{28.3}$$

Thus despite the presence of both good and bad fermions, they organize themselves to give $[Q^5, \bar{\psi}(x)i\gamma^5\psi(x)] = i\bar{\psi}(x)\psi(x)$, i.e., to give precisely the same form as in the instant-time case.

We introduce the vacuum matrix element of the light-front time-ordered product

$$\langle\Omega[|\theta(x^+)A^\mu(x)\bar{\psi}(0)i\gamma^5\psi(0) + \theta(-x^+)\bar{\psi}(0)i\gamma^5\psi(0)A^\mu(x)]|\Omega\rangle$$

Since there is only one associated momentum vector in Fourier space, we can set

$$\langle\Omega|[\theta(x^+)A^\mu(x)\bar{\psi}(0)i\gamma^5\psi(0) + \theta(-x^+)\bar{\psi}(0)i\gamma^5\psi(0)A^\mu(x)]|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p e^{ip\cdot x}p^\mu F(p^2), \tag{28.4}$$

where $F(p^2)$ is a scalar function. With $\partial_\mu A^\mu = 0$ we apply ∂_μ and then $\int d^4x$ to (28.4) to obtain

$$\delta(x^+)\langle\Omega|[A^\mu(x), \bar{\psi}(0)i\gamma^5\psi(0)]|\Omega\rangle = \frac{i}{(2\pi)^4}\int d^4p e^{ip\cdot x}p^2 F(p^2), \tag{28.5}$$

$$i\int d^4p\delta^4(p)p^2 F(p) = \langle\Omega|[Q^5(x^+ = 0), \bar{\psi}(0)i\gamma^5\psi(0)]|\Omega\rangle = i\langle\Omega|\bar{\psi}(0)\psi(0)|\Omega\rangle. \tag{28.6}$$

Thus if $\partial_\mu A^\mu = 0$ and $|\Omega\rangle$ is such that $i\langle\Omega|\bar{\psi}(0)\psi(0)|\Omega\rangle \neq 0$, Q^5 must not annihilate the vacuum, and $F(p)$ must contain a pole at $p^2 = 0$. This then is how the Goldstone theorem is satisfied in the light-front case, with the bad fermions playing a central role.

29 THE MORAL OF THE STORY

When we let $p_- \to 0$ we are letting $p_+ = [(p_1)^2 + (p_2)^2 + m^2]/4p_- \to \infty$.

However x^+ is the conjugate of p_+, and thus as $p_+ \to \infty$, $x^+ \to 0$.

The $p_- \to 0$ and the $x^+ \to 0$ limits are thus intertwined.

If we stay away from $x^+ = 0$ and restrict to $x^+ > 0$ and thus $p_- > 0$ as in the Light-Front Hamiltonian approach, there is no difficulty as there are only poles and nothing is singular, with the forward scattering on-shell Light-Front Hamiltonian approach thus being validated.

However this does become a concern for tadpole graphs as they have $x^+ = 0$, since we need both $\theta(x^+)$ and $\theta(-x^+)$ time orderings in the limit, with $\langle\Omega|[\theta(x^+)\phi(x)\phi(0) + \theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle \to \langle\Omega|[\theta(0^+)\phi(0)\phi(0) + \theta(0^-)\phi(0)\phi(0)]|\Omega\rangle = \langle\Omega|\phi(0)\phi(0)|\Omega\rangle$.

If we compare

$$
\begin{aligned}
D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{e^{-i(p_0x^0+p_1x^1+p_2x^2+p_3x^3)}}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon}, \\
D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{e^{-i(p_+x^++p_1x^1+p_2x^2+p_-x^-)}}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon},
\end{aligned}
\tag{29.1}
$$

$$
\begin{aligned}
D(x^\mu=0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon}, \\
D(x^\mu=0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon},
\end{aligned}
\tag{29.2}
$$

we can transform each instant-time graph into each corresponding light-front graph by a change of variable. Thus they must be equal. However, that does not mean that pole equals pole or that circle equals circle, only that pole plus circle equals pole plus circle, as it is only on the full closed contour that the integrals are equal.

The transformation $x^0 \to x^0 + x^3$, $x^3 \to x^0 - x^3$ is a spacetime-dependent general coordinate transformation (not a Lorentz transformation), and thus by the general coordinate invariance of the fundamental interactions it must be the case that

LIGHT-FRONT QUANTIZATION IS INSTANT-TIME QUANTIZATION, JUST ONE THEORY.