A friendly introduction to the Light Front

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About these lectures

- Introductory/elementary lectures on the light front
  - Intended to be graduate-level
  - Apologies to anyone who is bored from this being too elementary
- Covers kinematics (light front coordinates) and dynamics (light front quantization)
  - More focus on kinematics & group theory—just from my personal interests
- Please interject with questions or requests for clarification

References

- Designed as lectures rather than a seminar
- Won't have citations on most slides, but the physics is a result of many researchers
- Some standard references:
  - P.A.M. Dirac,
    Forms of Relativistic Dynamics,
    Rev. Mod. Phys. 21 (1949) 392-399
  - D.E. Soper,
    Field Theories in the Infinite Momentum Frame,
    PhD thesis
  - S.J. Brodsky, H.-C. Pauli and S.S. Pinsky,
    Quantum chromodynamics and other field theories on the light cone,
    Phys. Rept. 301 (1998) 299-487
  - M. Burkardt,
    Impact parameter space interpretation for genrealized parton distributions,
    Int. J. Mod. Phys. A (2003), 173-208
- Apologies to anyone I left out; this list is not meant to be complete or comprehensive

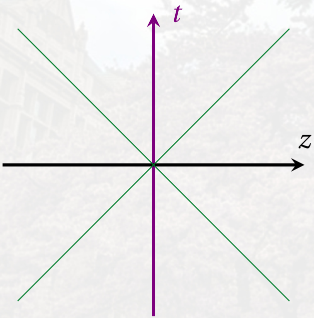
Outline


Basic Overview

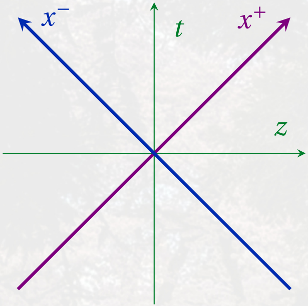
- **Light front coordinates** reparametrize spacetime.

$$x^{\pm} = \underbrace{\frac{1}{\sqrt{2}}}_{\text{conventional factor}}\big(t \pm z\big) \qquad \boldsymbol{x}_\perp = (x, y) \qquad x^+ = \frac{1}{\sqrt{2}}\big(t+z\big) \equiv \text{time}$$

- **Fixed-time surface** given by light front moving in $-z$ direction.



Instant form coordinates



Light front coordinates

Coordinate conversions

- Contravariant (upper-index) four-vector:

$$(t; x, y, z) \to \left(x^0; x^1, x^2, x^3\right) \equiv x^\mu$$

- Conversion between instant form & light front coordinates:

$$\begin{aligned} x^+ &= \frac{1}{\sqrt{2}}\left(x^0 + x^3\right) \\ x^- &= \frac{1}{\sqrt{2}}\left(x^0 - x^3\right) \\ x^1 &= x^1 \\ x^2 &= x^2 \end{aligned}$$

$$\begin{aligned} x^0 &= \frac{1}{\sqrt{2}}\left(x^+ + x^-\right) \\ x^3 &= \frac{1}{\sqrt{2}}\left(x^+ - x^-\right) \\ x^1 &= x^1 \\ x^2 &= x^2 \end{aligned}$$

- Symmetric inversion formulas are why $\frac{1}{\sqrt{2}}$ is nice

- A matrix $\mathscr{C}$ easily converts between light front & instant form:

$$\mathscr{C} = \mathscr{C}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

- Can easily check:

$$\mathscr{C} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{bmatrix} \qquad \mathscr{C} \begin{bmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{bmatrix} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

- For any matrix M,

$$M_{\mathrm{LF}} = \mathscr{C} M_{\mathrm{IF}} \mathscr{C}$$

Four-products and the metric

- Rule for four-products:

$$\begin{aligned} x \cdot y &= x^0 y^0 - x^3 y^3 - x^1 y^1 - x^2 y^2 \\ &= \frac{1}{2}(x^+ + x^-)(y^+ + y^-) - \frac{1}{2}(x^+ - x^-)(y^+ - y^-) - x^1 y^1 - x^2 y^2 \\ &= \frac{1}{2}(x^+ y^+ + x^- y^- + x^+ y^- + x^- y^+) - \frac{1}{2}(x^+ y^+ + x^- y^- - x^+ y^- - x^- y^+) - x^1 y^1 - x^2 y^2 \\ &= x^+ y^- + x^- y^+ - \boldsymbol{x}_\perp \cdot \boldsymbol{y}_\perp \end{aligned}$$

- Defines the metric in light front coordinates:

$$x \cdot y = g_{\mu\nu} x^\mu x^\nu \qquad g_{\mu\nu}^{(\mathrm{LF})} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathscr{C} g_{\mu\nu}^{(\mathrm{IF})} \mathscr{C}$$

  - Metric is off-diagonal in x^+ and x^-!

Light front metric via $\mathscr{C}$-matrix

- Great example of how helpful the $\mathscr{C}$-matrix is!

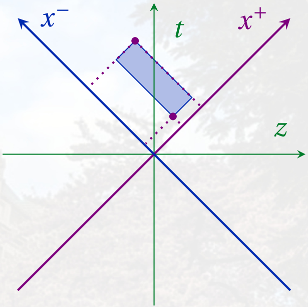
$$
g_{\mu\nu}^{(\mathrm{LF})} = \mathscr{C} g_{\mu\nu}^{(\mathrm{IF})} \mathscr{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
$$

Area law for proper time

- For events in (t, z) or (x^+, x^-) plane, proper time given by an area law:



$$\Delta\tau^2 = 2\Delta x^+ \Delta x^-$$

- Twice the area of an enclosed rectable with sides along the light cone
- Just a neat curiosity

Raising and lowering indices

- Four-vector indices are raised and lowered as usual:

$$x_\mu = g_{\mu\nu} x^\nu \qquad\qquad x^\mu = g^{\mu\nu} x_\nu \equiv \left(g_{\mu\nu}\right)^{-1} x_\nu$$

- Metric and its inverse are identical:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

  - Another perk of the $\frac{1}{\sqrt{2}}$ convention

- Raising/lowering swaps + and −:

$$x_+ = x^- \qquad\qquad x_- = x^+$$

  - We verbally refer to components by upper-index form
  - "x-plus" means $x^+ = x_-$ ..."x-minus" means $x^- = x_+$

Derivatives and four-momenta

- Light front derivatives found via chain rule:

$$\frac{\partial}{\partial x^+} = \frac{\partial x^0}{\partial x^+}\frac{\partial}{\partial x^0} + \frac{\partial x^3}{\partial x^+}\frac{\partial}{\partial x^3} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^3}\right)$$

$$\frac{\partial}{\partial x^-} = \frac{\partial x^0}{\partial x^-}\frac{\partial}{\partial x^0} + \frac{\partial x^3}{\partial x^-}\frac{\partial}{\partial x^3} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^3}\right)$$

$$\Longrightarrow \qquad \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_3)$$

- Four-momenta are spacetime translation generators—related to derivatives:

$$p_\mu = i\partial_\mu$$

$$p^- = p_+ = i\partial_+ = \text{energy} = \text{Hamiltonian}$$

$$p^+ = p_- = i\partial_- = \text{momentum component}$$

  - Energy is the time translation generator—singles out $p^- = p_+$ (usually called "p minus")

Light front energy formula

- Relativistic mass-shell condition:

$$m^2 = g_{\mu\nu} p^\mu p^\nu = 2 \underset{\substack{\uparrow \\ \text{momentum}}}{p^+} \underset{\text{energy}}{p^-} - \boldsymbol{p}_\perp^2$$

- Formula for energy:

$$H = p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$$

  - Reminiscent of non-relativistic formula:

$$H_{\mathrm{NR}} = \text{constant} + \frac{\boldsymbol{p}^2}{2m}$$

  - p^+ seems to behave like a mass
  - There are many more examples of this!

Light front momentum and velocity

- Velocities can be obtained via Hamilton's equations:

$$v^\mu = -\frac{\partial H}{\partial p_\mu}$$

$$\boldsymbol{v}_\perp = \frac{\partial H}{\partial \boldsymbol{p}_\perp} = \frac{\boldsymbol{p}_\perp}{p^+}$$

$$v^- = -\frac{\partial H}{\partial p^+} = \frac{p^-}{p^+}$$

$$H = p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$$

  - Velocity is rate of motion *with respect to* x^+

- Momentum and velocity related by:

$$(\boldsymbol{p}_\perp, p^-) = p^+(\boldsymbol{v}_\perp, v^-)$$

  - Reminiscent of $\boldsymbol{p} = m\boldsymbol{v}$
  - Again, p^+ behaves kind of like a mass

Summary so far

- Light front coordinates & conjugate momenta:

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3) \quad \text{time}$$
$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3)$$
$$\boldsymbol{x}_\perp = (x^1, x^2)$$

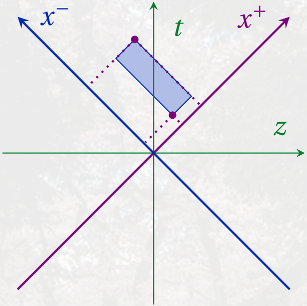
energy
$$p_+ = p^- = \frac{1}{\sqrt{2}}(p^0 - p^3)$$
$$p_- = p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3)$$
$$\boldsymbol{p}_\perp = (p^1, p^2)$$

- Invariant four-product & proper time:

$$x \cdot y = x^+ y^- + x^- y^+ - \boldsymbol{x}_\perp \cdot \boldsymbol{y}_\perp$$
$$\Delta x^2 = 2x^+ x^- - \boldsymbol{x}_\perp^2$$

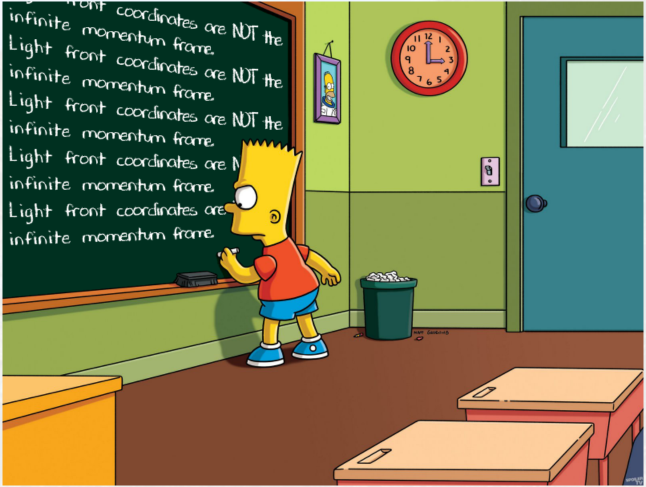
- p^+ acts somewhat like a non-relativistic mass:

$$E = p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$$
$$(p^-, \boldsymbol{p}_\perp) = p^+(v^-, \boldsymbol{v}_\perp)$$



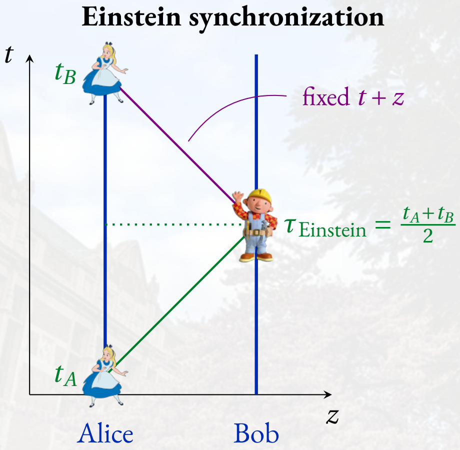

Conceptualization

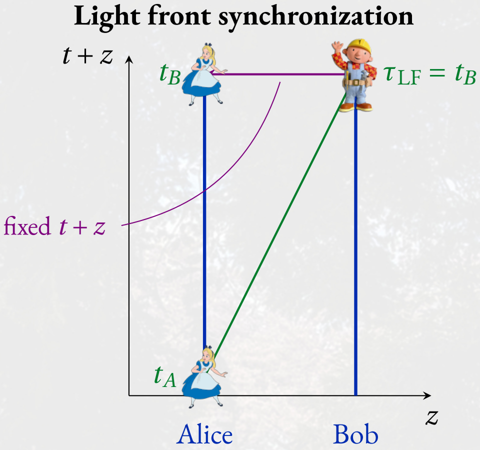
Not the infinite momentum frame!



- Light front coordinates are valid in **any** frame.
  - They're not a reference frame.
- Light front coordinates are **not** the infinite-momentum frame.
  - A common misconception.
- Light front coordinates redefine **synchronization convention**.
  - What we mean by "simultaneous."

Synchronization conventions





- **Einstein synchronization** defined to be isotropic.
- **Light front synchronization** defines hyperplanes with fixed $t+z$ to be "simultaneous."
  - Light travels instantaneously in $-z$ direction *by definition.*
  - We take what we see as literally happening now.

Equal-“time” surfaces are just a convention

- Relativity requires *round-trip* speed of light to be invariant.
- Convention that one-way speed of light be c is a *definition*, not an empirical fact.
  - Pointed out in Einstein’s original paper.
- Redefining “time” coordinate means changing this definition.
  - Light front coordinates do exactly this!

894 *A. Einstein.*

B durch einen in *B* befindlichen Beobachter möglich. Es ist aber ohne weitere Festsetzung nicht möglich, ein Ereignis in *A* mit einem Ereignis in *B* zeitlich zu vergleichen; wir haben bisher nur eine „*A*-Zeit“ und eine „*B*-Zeit“, aber keine für *A* und *B* gemeinsame „Zeit“ definiert. Die letztere Zeit kann nun definiert werden, indem man *durch Definition* festsetzt, daß die „Zeit“, welche das Licht braucht, um von *A* nach *B* zu gelangen, gleich ist der „Zeit“, welche es braucht, um von *B* nach *A* zu gelangen. Es gehe nämlich ein Lichtstrahl zur „*A*-Zeit“ t_A von *A* nach *B* ab, werde zur „*B*-Zeit“ t_B in *B* gegen *A* zu reflektiert und gelange zur „*A*-Zeit“ t'_A nach *A* zurück. Die beiden Uhren laufen definitionsgemäß synchron, wenn

$$t_B - t_A = t'_A - t_B.$$

Einstein, Ann. Phys. 322 (1905) 891

- **Didactic overview**: Veritasium, “Why No One Has Measured The Speed of Light” (YouTube)
- **Technical review**: Anderson, Stedman & Vetharaniam, Phys. Rept. 295 (1998) 93

Boosts and simultaneity

- Foliating spacetime (slicing it up) into fixed-x^+ surfaces is *invariant under z-direction boosts!*

$$\begin{aligned} t' &= \gamma t + \beta\gamma z \\ z' &= \gamma z + \beta\gamma t \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x'^+ &= (\gamma + \beta\gamma)x^+ = \sqrt{\frac{1+\beta}{1-\beta}}\,x^+ \\ x'^- &= (\gamma - \beta\gamma)x^- = \sqrt{\frac{1-\beta}{1+\beta}}\,x^- \end{aligned}$$

  - Foliation is invariant, but time is **redshifted** ($\beta > 0$) or **blueshifted** ($\beta < 0$).
  - Instant form always gives time dilation upon boosts
  - **Light front describes what you actually see**

- Compact formulas if we use **rapidity**, η:

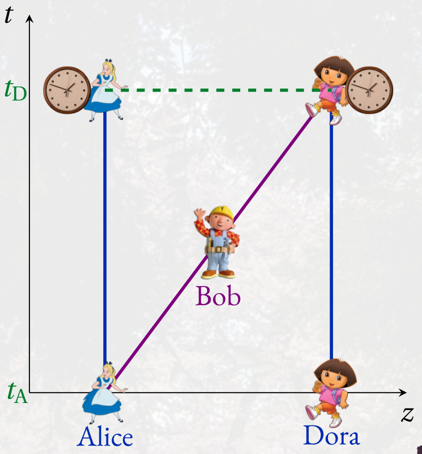
$$\begin{aligned} \gamma &= \cosh\eta \\ \beta\gamma &= \sinh\eta \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x'^+ &= e^{\eta} x^+ \\ x'^- &= e^{-\eta} x^- \end{aligned}$$

Time dilation as an artifact of instant form

- **Alice** and **Dora** synchronize their clocks by Einstein's convention.
  - The two of them together constitute a **reference frame**.
- **Bob** moves from Alice to Dora.
  - A time Δt_B passes on Bob's clock.
  - At start: Bob sees Alice's clock reads t_A.
  - At end: Bob sees Dora's clock reads t_D.
- **Time dilation** means:

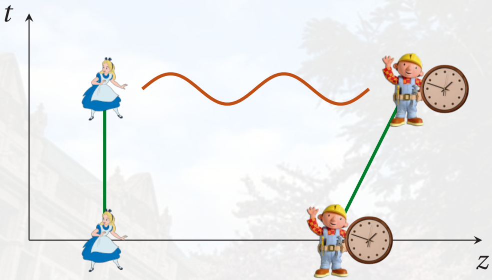
$$\Delta t_B = \sqrt{1 - \frac{v^2}{c^2}}(t_D - t_A) < (t_D - t_A)$$

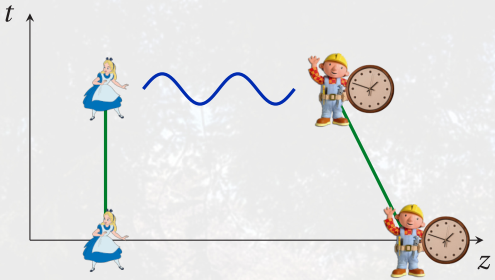
  - Bob's clock slow compared to *reference frame*.
  - Result depends on Einstein synchronization.



Redshift/blueshift as what's actually seen

- When Bob is boosted—Alice *sees* redshift or blueshift:





Redshift: Bob moves away, clock appears slower

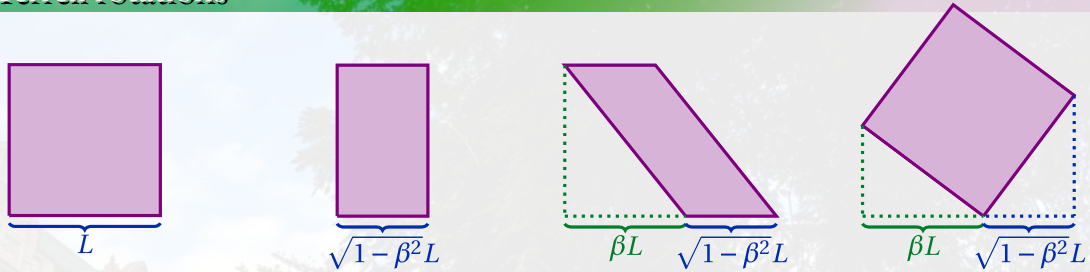
Blueshift: Bob moves towards, clock appears faster

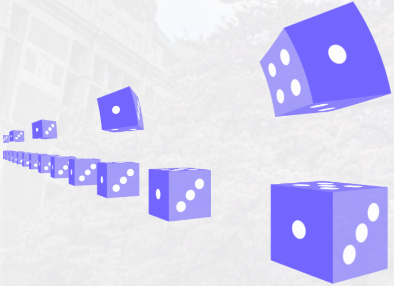
- Light front *automatically gives redshift/blueshift formula*:

$$x^+_{\text{Bob}} = \sqrt{\frac{1+\beta}{1-\beta}}\, x^+_{\text{Alice}} = e^{\eta} x^+_{\text{Alice}}$$

  - Light front prioritizes what we actually see

Terrell rotations





- Lorentz-boosted objects *appear rotated*.
  - **Terrell rotation** (PR116, 1959)
  - Optical effect: contraction + delay
- Light front takes Terrell rotations seriously!
  - Instant form: contractions are real
  - Light front: rotations are real
  - Both equally valid viewpoints

Dice image: Ute Kraus, `https://www.spacetimetravel.org/`

- Transverse boost—instant form:

$$t' = \gamma t + \beta\gamma y \approx t + \beta y + \mathcal{O}(\beta^2)$$
$$y' = \gamma y + \beta\gamma t \approx y + \beta t + \mathcal{O}(\beta^2)$$
$$x' = x$$
$$z' = z$$

- Transverse boost—light front:

$$\delta x^+ \approx \frac{\beta}{\sqrt{2}} y$$
$$\delta y \approx \frac{\beta}{\sqrt{2}} x^+ + \frac{\beta}{\sqrt{2}} x^-$$
$$\delta x^- \approx \frac{\beta}{\sqrt{2}} y$$

- Rotation around x axis—instant form:

$$t' = t$$
$$y' = \cos\theta\, y - \sin\theta\, z \approx y - \theta z + \mathcal{O}(\theta^2)$$
$$x' = x$$
$$z' = \cos\theta\, z + \sin\theta\, y \approx z + \theta y + \mathcal{O}(\theta^2)$$

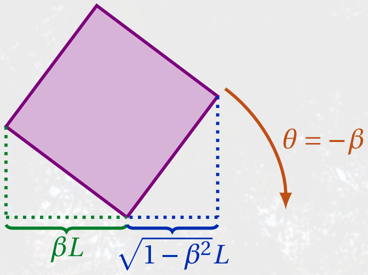
- Rotation around x axis—light front:

$$\delta x^+ \approx \frac{\theta}{\sqrt{2}} y$$
$$\delta y \approx -\frac{\theta}{\sqrt{2}} x^+ + \frac{\theta}{\sqrt{2}} x^-$$
$$\delta x^- \approx -\frac{\theta}{\sqrt{2}} y$$

- What if we boost by β and rotate by $\theta = -\beta$?

Undoing Terrell rotations

- Combined boost + counterrotation ($\theta = -\beta$):



$$\delta x^+ = \frac{\beta+\theta}{\sqrt{2}}\, y = 0$$

$$\delta y = \frac{\beta-\theta}{\sqrt{2}}\, x^+ + \frac{\beta+\theta}{\sqrt{2}}\, x^- = \sqrt{2}\beta x^+$$

  - Combined transformation leaves x^+ invariant!
  - Object apparently moving in y direction
  - Apparent velocity is $\sqrt{2}\beta$
- Suggests new generator:

$$B_y = \frac{1}{\sqrt{2}}\left(K_y - J_x\right)$$

counterrotation

ordinary boost

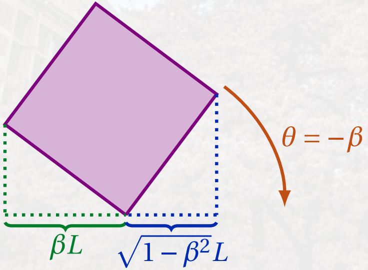
  - B_y generates a **light front transverse boost**
  - (contrasted with oridnary transverse boost)
  - Similar equation for generator of x-direction light front boost:

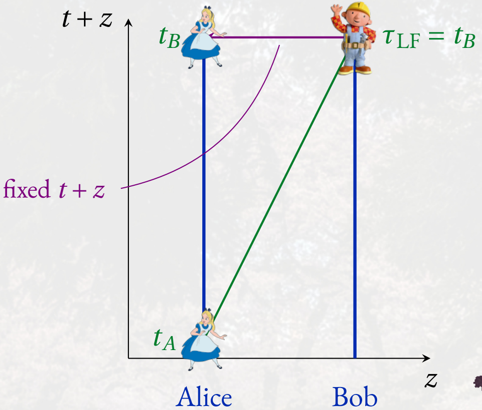
$$B_x = \frac{1}{\sqrt{2}}\left(K_x + J_y\right)$$

Summary so far

- Light front coordinates describe spacetime *as we see it*
  - Distant events (in $+z$ direction) are considered "present" instead of "past"
  - Longitudinal boosts produce *observed* redshifts/blueshifts instead of time dilation
  - Ordinary transverse boosts produce *observed* Terrell rotations instead of Fitzgerald contractions
- Can define **light front transverse boost** as transverse boost + counterrotation
  - Leaves transverse spatial structure *invariant*

$$B_y = \frac{1}{\sqrt{2}}(K_y - J_x)$$






Finite Boosts & Kinematics

Transverse boost generators: commutation rule

- Something amazing happens with the light front transverse boost generators:

$$B_x = \frac{1}{\sqrt{2}}(K_x + J_y) \qquad B_y = \frac{1}{\sqrt{2}}(K_y - J_x)$$

$$[B_x, B_y] = \frac{1}{2}\Big(\overbrace{[K_x, K_y]}^{=-iJ_z} - \underbrace{[J_y, J_x]}_{=-iJ_z} - \overbrace{[K_x, J_x]}^{=0} + \underbrace{[J_y, K_y]}_{=0}\Big) = 0$$

  - They commute!
- This is one manifestation of the **Galilei subgroup** of the Poincaré group
  - The classical-looking formulas we saw previously are other manifestations
- Let's look at *finite* boosts before fleshing out the full group

- **Generators** quantify infinitesimal transformations

$$\underbrace{\mathscr{B}_y(v)}_{\text{finite boost}} \approx 1 - i \underbrace{B_y}_{\text{generator}} v + \mathscr{O}(v^2)$$

  - The factor $-i$ is a popular convention
  - I prefer to define generators without it in my own personal notes ...
  - ...but let's use standard conventions in these lectures.
- x-boost and y-rotation generators (instant form):

$$\mathscr{K}_x^{(\mathrm{IF})}(\eta) = \begin{bmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies K_x^{(\mathrm{IF})}(\eta) = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathscr{J}_y^{(\mathrm{IF})}(\eta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \implies J_y^{(\mathrm{IF})}(\eta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

- As usual, can use $\mathscr{C}$-matrix to convert to light front:

$$K_x^{(\mathrm{LF})} = \mathscr{C} K_x^{(\mathrm{IF})} \mathscr{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

- As usual, can use $\mathscr{C}$-matrix to convert to light front:

$$J_y^{(\mathrm{LF})} = \mathscr{C} J_y^{(\mathrm{IF})} \mathscr{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Transverse boost generators and nilpotency

- Combining the traditional boost & counter-rotation...

$$K_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \qquad J_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$$

$$B_x = \frac{1}{\sqrt{2}}(K_x + J_y) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \qquad B_x^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$B_x^3 = 0\,, \quad B_x^4 = 0\,, \quad B_x^5 = 0\,, \quad \text{etc.}$$

- Light front boost generators are **nilpotent** for transverse boosts!
  - Nilpotent means there's a power of B_x that gives zero.

Finite transverse boosts

$$B_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \qquad B_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{bmatrix} \qquad B_x^2 = B_y^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

- Finite boosts built from infinitesimal boosts by exponentiation:

$$\mathscr{B}_\perp(\boldsymbol{v}_\perp) = \lim_{N\to\infty}\left(1 - \frac{i}{N}\boldsymbol{v}_\perp \cdot B_\perp\right)^N = e^{-i\boldsymbol{v}_\perp\cdot\boldsymbol{B}_\perp} = \underbrace{1 - i\boldsymbol{v}_\perp \cdot \boldsymbol{B}_\perp - \frac{1}{2}\boldsymbol{v}_\perp^2 B_x^2}_{\textbf{exact}\text{ because of nilpotency}}$$

  - Basically, just continually compound infinitesimal boosts
  - Note also that $B_x B_y = B_y B_x = 0$
- Transforms a four-vector as:

$$\begin{pmatrix} x^+ \\ \boldsymbol{x}_\perp \\ x^- \end{pmatrix} \to \begin{pmatrix} x^+ \\ \boldsymbol{x}_\perp + \boldsymbol{v}_\perp x^+ \\ x^- + \boldsymbol{v}_\perp \cdot \boldsymbol{x}_\perp + \frac{1}{2}\boldsymbol{v}_\perp^2 x^+ \end{pmatrix}$$

leaves time invariant!

Boosting particle from rest

- Four-momentum at rest:

$$p_{\text{rest}}^{(\text{LF})} = \mathscr{C}\, p_{\text{rest}}^{(\text{IF})} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{m}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Can reach any momentum by longitudinal then transverse boost:

$$p = \mathscr{B}_\perp(\boldsymbol{v}_\perp)\,\mathscr{B}_\parallel(\eta)\, p_{\text{rest}}$$

  - Longitudinal boost by rapidity η
  - Transverse boost by velocity $\boldsymbol{v}_\perp$

The longitudinal boost generator

- Use $\mathscr{C}$-matrix to convert to light front:

$$K_z^{(\mathrm{LF})} = \mathscr{C} K_z^{(\mathrm{IF})} \mathscr{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

Finite longitudinal boosts

- Finite boost from exponentiation:

$$K_z^{(\mathrm{LF})} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \qquad \Longrightarrow \qquad \mathcal{K}_z^{(\mathrm{LF})} = e^{-iyK_z} = \begin{bmatrix} \mathrm{e}^{\eta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\eta} \end{bmatrix}$$

- Boosting a particle from rest:

$$\mathcal{K}_z^{(\mathrm{LF})}(\eta)\, p_{\mathrm{rest}}^{(\mathrm{LF})} = \frac{m}{\sqrt{2}} \begin{bmatrix} \mathrm{e}^{\eta} \\ 0 \\ 0 \\ \mathrm{e}^{-\eta} \end{bmatrix}$$

  - Nice scaling behavior for p^+ and p^-
- Formula for longitudinal rapidity:

$$\eta = \log\left(\frac{\sqrt{2}p^+}{m}\right)$$

  - Will remain invariant under transverse light front boosts! (The ones with Terrell counter-rotations.)

Transverse boosts of four-momenta

- Following up with a transverse boost...

$$\mathscr{B}_\perp(\boldsymbol{v}_\perp) = 1 - i\boldsymbol{v}_\perp \cdot \boldsymbol{B}_\perp - \frac{1}{2}\boldsymbol{v}_\perp^2 B_x^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_x & 1 & 0 & 0 \\ v_y & 0 & 1 & 0 \\ \frac{1}{2}\boldsymbol{v}_\perp^2 & v_x & v_y & 1 \end{bmatrix} \qquad \mathscr{K}_z^{(\mathrm{LF})}(\eta)\, p_{\mathrm{rest}}^{(\mathrm{LF})} = \frac{m}{\sqrt{2}} \begin{bmatrix} \mathrm{e}^{\eta} \\ 0 \\ 0 \\ \mathrm{e}^{-\eta} \end{bmatrix}$$

- An arbitrary four-momentum can be expressed:

$$\begin{bmatrix} p^+ \\ p^x \\ p^y \\ p^- \end{bmatrix} = \mathscr{B}_\perp(\boldsymbol{v}_\perp)\mathscr{K}_z(\eta)\, p_{\mathrm{rest}} = \frac{m}{\sqrt{2}} \begin{bmatrix} \mathrm{e}^{\eta} \\ v_x\,\mathrm{e}^{\eta} \\ v_y\,\mathrm{e}^{\eta} \\ \mathrm{e}^{-\eta} + \frac{1}{2}\boldsymbol{v}_\perp^2\,\mathrm{e}^{\eta} \end{bmatrix} = \begin{bmatrix} p^+ \\ p^+ v_x \\ p^+ v_y \\ \frac{m^2}{2p^+} + \frac{1}{2}p^+\boldsymbol{v}_\perp^2 \end{bmatrix}$$

  - Again, a classical-looking expression!
  - Transverse velocity: $\boldsymbol{v}_\perp = \boldsymbol{v}_\perp$
  - Transverse momentum: $\boldsymbol{p}_\perp = p^+\boldsymbol{v}_\perp$
  - Kinetic energy term: $\frac{1}{2}p^+\boldsymbol{v}_\perp^2$

Summary so far

- Finite boosts on the light front take simple forms:

$$\mathcal{K}_z^{(\mathrm{LF})} = \begin{bmatrix} \mathrm{e}^{\eta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\eta} \end{bmatrix} \qquad \mathcal{B}_\perp(\boldsymbol{v}_\perp) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_x & 1 & 0 & 0 \\ v_y & 0 & 1 & 0 \\ \frac{1}{2}\boldsymbol{v}_\perp^2 & v_x & v_y & 1 \end{bmatrix}$$

- The transverse boost is different from the "ordinary" transverse boost
  - Consists of an ordinary boost + counter-rotation, to cancel out Terrell rotation
  - The generator is nilpotent—series from exponentiation terminates!
- Momenta can be written in a very classical-looking form, with p^+ acting like a mass:

$$\begin{bmatrix} p^+ \\ p^x \\ p^y \\ p^- \end{bmatrix} = \begin{bmatrix} p^+ \\ p^+ v_x \\ p^+ v_y \\ \frac{m^2}{2p^+} + \frac{1}{2} p^+ \boldsymbol{v}_\perp^2 \end{bmatrix}$$


The Galilei Subgroup

Galilei subgroup

- Poincaré group has a $(2+1)$D **Galilei subgroup**.
  - x^+ is time and $\boldsymbol{x}_\perp$ is space under this subgroup
  - $p^+ = \frac{1}{\sqrt{2}}(E_{\boldsymbol{p}} + p_z)$ is the central charge—acts like a mass, commutes with rest of subgroup
  - x^+ and p^+ are invariant under this subgroup!
- Light front time gives **fully relativistic** 2D picture that looks a lot like non-relativistic physics.

$$\frac{\mathrm{d}\boldsymbol{p}_\perp}{\mathrm{d}x^+} = p^+ \frac{\mathrm{d}^2\boldsymbol{x}_\perp}{\mathrm{d}x^{+2}}$$

$$H = H_{\text{rest}} + \frac{\boldsymbol{p}_\perp^2}{2p^+} = H_{\text{rest}} + \frac{1}{2}p^+\boldsymbol{v}_\perp^2$$

$$\boldsymbol{p}_\perp = p^+\boldsymbol{v}_\perp$$

$$[B_x, B_y] = 0$$

Lorentz algebra: instant form

- Lorentz group has 6 generators $M^{\mu\nu} = -M^{\nu\mu}$

3 boosts	**3 rotations**
$K_i = M^{0i}$	$(J_1, J_2, J_3) = (M^{23}, M^{31}, M^{12})$

- Algebra defined by commutation rule:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)$$

  - Can be derived via geometric algebra, or by requiring it give the explicit forms below.
- More explicit forms:

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$
$$[J_i, J_j] = i\epsilon_{ijk} J_k$$
$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

- Let's prove the boost commutation formula!

3 boosts

$$K_i = M^{0i}$$

3 rotations

$$J_i = \tfrac{1}{2}\epsilon_{ijk}M^{jk} \implies M^{ij} = \epsilon_{ijk}J_k$$

- Fundamental commutation rule:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

- Working it out...

$$\begin{aligned}[K_i, K_j] &= [M^{0i}, M^{0j}] \\ &= i\left(\underbrace{g^{0j}}_{=0}M^{i0} + \underbrace{g^{i0}}_{=0}M^{0j} - g^{00}M^{ij} - g^{ij}\underbrace{M^{00}}_{=0}\right) = -iM^{ij} = -i\epsilon_{ijk}J_k\end{aligned}$$

Lorentz algebra: instant form, rotation commutation

- Let's prove the rotation commutation formula!

3 boosts

$$K_i = M^{0i}$$

3 rotations

$$J_i = \frac{1}{2}\epsilon_{ijk} M^{jk} \implies M^{ij} = \epsilon_{ijk} J_k$$

- Fundamental commutation rule:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)$$

- Working it out...

$$[J_i, J_j] = \frac{1}{4}\epsilon_{iab}\epsilon_{jcd}[M^{ab}, M^{cd}]$$

$$= \frac{i}{4}\epsilon_{iab}\epsilon_{jcd}\left(g^{ad} M^{bc} + g^{bc} M^{ad} - g^{ac} M^{bd} - g^{bd} M^{ac}\right)$$

$$\text{(distribute Levi-Civita)} = -\frac{i}{4}\Big(\underbrace{\epsilon_{ieb}\epsilon_{jce}}_{-(\delta_{ij}\delta_{bc}-\delta_{ic}\delta_{jb})} M^{bc} + \overbrace{\epsilon_{iae}\epsilon_{jed}}^{-(\delta_{ij}\delta_{ad}-\delta_{id}\delta_{ja})} M^{ad} - \underbrace{\epsilon_{ieb}\epsilon_{jed}}_{\delta_{ij}\delta_{bd}-\delta_{id}\delta_{jb}} M^{bd} - \overbrace{\epsilon_{iae}\epsilon_{jce}}^{\delta_{ij}\delta_{ac}-\delta_{ic}\delta_{ja}} M^{ac}\Big)$$

$$\text{(drop } M^{ii}\text{)} = -\frac{i}{4}\left(M^{ji} + M^{ji} - (-M^{ji}) - (-M^{ji})\right) = iM^{ij} = i\epsilon_{ijk}J_k$$

- Let's prove the last commutation formula!

3 boosts

$$K_i = M^{0i}$$

3 rotations

$$J_i = \tfrac{1}{2}\epsilon_{ijk}M^{jk} \implies M^{ij} = \epsilon_{ijk}J_k$$

- Fundamental commutation rule:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

- Working it out...

$$\begin{aligned}
\left[J_i, K_j\right] &= \tfrac{1}{2}\epsilon_{iab}\left[M^{ab}, M^{0j}\right] \\
&= \tfrac{i}{2}\epsilon_{iab}\Big(g^{aj}M^{b0} + \underbrace{g^{b0}}_{=0}M^{aj} - \underbrace{g^{a0}}_{=0}M^{bj} - g^{bj}M^{a0}\Big) \\
&= -\tfrac{i}{2}\left(\epsilon_{ijb}M^{b0} - \epsilon_{iaj}M^{a0}\right) = i\epsilon_{ijk}M^{0k} = i\epsilon_{ijk}K_k
\end{aligned}$$

Lorentz algebra: light front

- Lorentz group has 6 generators $M^{\mu\nu} = -M^{\nu\mu}$

3 boosts

$$(B_1, B_2, B_-) = (M^{+1}, M^{+2}, M^{+-})$$

3 rotations

$$(R_1, R_2, R_-) = (M^{2-}, M^{-1}, M^{12})$$

- x^+ is time and x^- is space!

- Algebra defined by commutation rule:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

- More explicit forms ($i, j \in \{1, 2\}$):

$$[B_i, B_j] = 0$$
$$[R_-, R_i] = i\epsilon_{3ij}R_j$$
$$[R_-, B_i] = i\epsilon_{3ij}B_j$$
$$[R_-, B_-] = 0$$

$$[B_-, B_i] = iB_i$$
$$[R_1, R_2] = 0$$
$$[B_-, R_i] = -iR_i$$
$$[R_i, B_j] = -i\delta_{ij}R_-$$

$\boldsymbol{R}_\perp$ & $\boldsymbol{B}_\perp$ transform like 2D vectors!

Lorentz algebra: light front, boost commutation rules

- Let's prove the boost formulas!

3 boosts

$$(B_1, B_2, B_-) = (M^{+1}, M^{+2}, M^{+-})$$

3 rotations

$$(R_1, R_2, R_-) = (M^{2-}, M^{-1}, M^{12})$$

- Fundamental commutation rule:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)$$

- Working it out, & allowing $i, j = -$ ...

$$[B_i, B_j] = [M^{+i}, M^{+j}]$$

$$= i\Big(\underbrace{g^{+j}}_{=\delta_{j-}} M^{i+} + \underbrace{g^{i+}}_{=\delta_{i-}} M^{+j} - \underbrace{g^{++}}_{=0} M^{ij} - g^{ij} \underbrace{M^{++}}_{=0}\Big) = \begin{cases} 0 & : \quad i, j \neq - \text{ or } i = j = - \\ iB_j & : \quad i = -, j \neq - \\ -iB_i & : \quad j = -, i \neq - \end{cases}$$

- Also can be written for $i, j \in \{1, 2\}$:

$$[B_i, B_j] = 0 \qquad [B_-, B_j] = iB_j$$

- Let's prove the rotation commutation rules!

3 boosts

$$(B_1, B_2, B_-) = (M^{+1}, M^{+2}, M^{+-})$$

3 rotations

$$(R_1, R_2, R_-) = (M^{2-}, M^{-1}, M^{12})$$

- Fundamental commutation rule:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)$$

- Easier to work out components explicitly:

$$[R_1, R_2] = [M^{2-}, M^{-1}] = i\big(g^{21} \underbrace{M^{--}}_{=0} + \underbrace{g^{--}}_{=0} M^{21} - \underbrace{g^{2-}}_{=0} M^{-1} - \underbrace{g^{-1}}_{=0} M^{2-}\big) = 0$$

$$[R_-, R_1] = [M^{12}, M^{2-}] = i\big(\underbrace{g^{1-}}_{=0} M^{22} + g^{22} M^{1-} - \underbrace{g^{12}}_{=0} M^{2-} - \underbrace{g^{2-}}_{=0} M^{12}\big) = iM^{-1} = iR_2$$

$$[R_-, R_2] = [M^{12}, M^{-1}] = i\big(g^{11} M^{2-} + \underbrace{g^{2-}}_{=0} M^{11} - \underbrace{g^{1-}}_{=0} M^{21} - \underbrace{g^{21}}_{=0} M^{1-}\big) = -iM^{2-} = -iR_1$$

- Let's prove remaining rules involving R_-!

3 boosts

$$(B_1, B_2, B_-) = (M^{+1}, M^{+2}, M^{+-})$$

3 rotations

$$(R_1, R_2, R_-) = (M^{2-}, M^{-1}, M^{12})$$

- Fundamental commutation rule:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)$$

- Working it out, & allowing $i = -$ ...

$$\begin{aligned}\left[R_-, B_i\right] &= \left[M^{12}, M^{+i}\right] \\ &= i\left(g^{1i} M^{2+} + \underbrace{g^{2+}}_{=0} M^{1i} - \underbrace{g^{1+}}_{=0} M^{2i} - g^{2i} M^{1+}\right) \\ &= -i\left(\delta_{1i} M^{2+} - \delta_{2i} M^{1+}\right) = \begin{cases} iB_2 & : \quad i = 1 \\ -iB_1 & : \quad i = 2 \\ 0 & : \quad i = + \end{cases}\end{aligned}$$

- I leave the other derivations as an exercise for you!

Lorentz algebra: instant form vs. light front

Instant form

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$
$$[J_i, J_j] = i\epsilon_{ijk} J_k$$
$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

Part of Galilei subgroup!

Light front

$$[B_i, B_j] = 0$$
$$[R_-, B_i] = i\epsilon_{3ij} B_j$$
$$[R_-, R_i] = i\epsilon_{3ij} R_j$$
$$[B_-, B_i] = iB_i$$
$$[B_-, R_i] = -iR_i$$
$$[R_i, B_j] = -i\delta_{ij} R_-$$
$$[R_i, R_j] = 0$$

- Can derive relationships just from index manipulation, e.g.,

$$B_i = M^{+i} = \frac{1}{\sqrt{2}}\left(M^{0i} + M^{3i}\right) = \frac{1}{\sqrt{2}}\left(K_i + \epsilon_{3ij} J_j\right) \quad \Longrightarrow \quad \begin{cases} B_x = \frac{1}{\sqrt{2}}\left(K_x + J_y\right) \\ B_y = \frac{1}{\sqrt{2}}\left(K_y - J_x\right) \end{cases}$$

  - *Exact same* result we got from conceptual arguments (Terrell rotation + counterrotation)!
- Must consider translations for rest of Galilei group

Poincaré algebra: instant form

- 10 generators: 6 Lorentz transformations $M^{\mu\nu} = -M^{\nu\mu}$ + 4 translations P^{μ}
- Algebra given by commutation rules:

$$
\begin{aligned}
\left[M^{\mu\nu}, M^{\rho\sigma}\right] &= i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right) \\
\left[M^{\mu\nu}, P^{\rho}\right] &= i\left(g^{\nu\rho} P^{\mu} - g^{\mu\rho} P^{\nu}\right) \\
\left[P^{\mu}, P^{\nu}\right] &= 0
\end{aligned}
$$

- More explicit rules (exercise for you to show!):

$$
\begin{aligned}
[J_i, P^j] &= i\epsilon_{ijk} P^k \\
[K_i, P^j] &= -i\delta_{ij} P^0 \\
[J_i, P^0] &= 0 \\
[K_i, P^0] &= -iP^i
\end{aligned}
$$

Poincaré algebra: light front

- Full Poincaré group in light front coordinates (exercise for you to show!):

$$[R_i, P^j] = -i\epsilon_{3ij}P^-$$
$$[R_i, P^+] = i\epsilon_{3ij}P^j$$
$$[R_i, P^-] = 0$$
$$[B_-, P^j] = 0$$
$$[B_-, P^+] = iP^+$$
$$[B_-, P^-] = -iP^-$$
$$[B_-, B_i] = iB_i$$
$$[R_i, B_-] = 0$$
$$[R_i, B_j] = -i\delta_{ij}R_-$$
$$[R_i, R_j] = 0$$

$$[R_-, P^i] = i\epsilon_{3ij}P^j$$
$$[R_-, B_i] = i\epsilon_{3ij}B_j$$
$$[R_-, P^-] = 0$$
$$[B_i, P^j] = -i\delta_{ij}P^+$$
$$[B_i, B_j] = 0$$
$$[B_i, P^-] = -iP^i$$
$$[P^i, P^-] = 0$$
$$[R_-, P^+] = 0$$
$$[B_i, P^+] = 0$$
$$[P^i, P^+] = 0$$
$$[P^-, P^+] = 0$$

The Galilei subgroup

Galilei subgroup: what's in and out

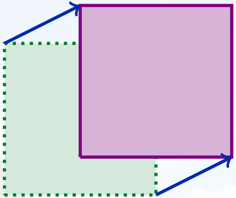
- What's in the Galilei subgroup?
  - $\boldsymbol{B}_\perp$ — transverse light front boosts, which commute
  - R^- — Rotation around z axis (aka, rotation around x^- axis with x^+ = fixed)
  - $\boldsymbol{P}_\perp$ — translations in transverse plane
  - P^- — time evolution (Hamiltonian)
  - P^+ — **central charge**; commutes with everything else, kind of like a mass

  ...that's **7 out of 10** generators!

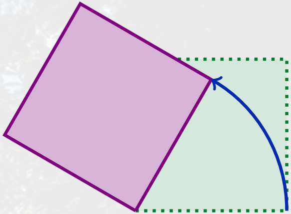
- What's *not* in the subgroup?
  - B_- — longitudinal boosts (they change P^+, so fail to commute with it)
  - $\boldsymbol{R}_\perp$ — rotations around x and y axes (they reorient the light front, so change P^+)

  ...basically, anything that fails to commute with P^+

Galilei group in 2D non-relativistic physics



2 translations — $\boldsymbol{P}$



1 rotation — R

2 boosts — $\boldsymbol{B}$

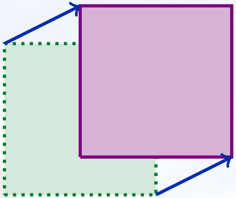
1 Hamiltonian — H

1 mass — M

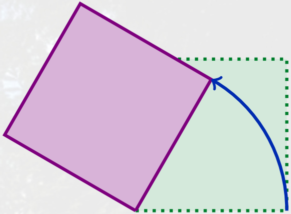
- 7 total generators
- All non-zero commutators:

$$[R, P_i] = i\epsilon_{ij3}P_j \qquad [R, B_i] = i\epsilon_{ij3}B_j \qquad [B_i, P_j] = -i\delta_{ij}M \qquad [H, B_i] = iP_i$$

Galilei group in 2D light front physics



2 translations — $\boldsymbol{P}_\perp$



1 rotation — R_-

2 boosts — $\boldsymbol{B}_\perp$

1 Hamiltonian — P^-

1 longitudinal momentum — P^+

- 7 total generators
- All non-zero commutators:

$$[R_-, P_i] = i\epsilon_{ij3}P_j \qquad [R_-, B_i] = i\epsilon_{ij3}B_j \qquad [B_i, P_j] = -i\delta_{ij}P^+ \qquad [P^-, B_i] = iP_i$$

Transverse position operator

- All non-zero commutators:

$$[R_-, P_i] = i\epsilon_{ij3}P_j \qquad [R_-, B_i] = i\epsilon_{ij3}B_j \qquad [B_i, P_j] = -i\delta_{ij}P^+ \qquad [P^-, B_i] = iP_i$$

- Position operator should obey:

$$[X_\perp^i, X_\perp^j] = 0 \qquad [X_\perp^i, P_\perp^j] = i\delta^{ij} \qquad i[P^-, X_\perp^i] = V_\perp^i$$

- Actually, the following works!:

$$X_\perp^i = -\frac{B_i}{P^+}$$

  - Transverse boosts commute
  - P^+ commutes with entire subgroup
  - $P_\perp^i = P^+ V_\perp^i$
- Exercise for you: prove the purple commutation relations

Transverse position operator and boosts

- Transverse position operator:

$$X_\perp^i = -\frac{B_i}{P^+}$$

- Recall:

$$[B_i, B_j] = 0 \qquad [B_i, P^+] = 0 \qquad [B_-, B_i] = iB_i \qquad [B_-, P^+] = iP^+$$

- Simply enough:

$$[B_i, X^j] = \overbrace{[B_i, -(P^+)^{-1}B_j]}^{[A,BC]=B[A,C]+[A,B]C} = -(P^+)^{-1}\underbrace{[B_i, B_j]}_{=0} - \underbrace{[B_i, (P^+)^{-1}]}_{=0}B_j = 0$$

- With a bit more algebra:

$$[B_-, X^i] = \overbrace{[B_-, -(P^+)^{-1}B_i]}^{[A,BC]=B[A,C]+[A,B]C} = -(P^+)^{-1}\underbrace{[B_-, B_i]}_{=iB_i} - \overbrace{[B_-, (P^+)^{-1}]}^{[A,B^{-1}]=-B^{-1}[A,B]B^{-1}}B_i$$

$$= -i(P^+)^{-1}B_i + (P^+)^{-1}\underbrace{[B_-, P^+]}_{=iP^+}(P^+)^{-1}B_i = 0$$

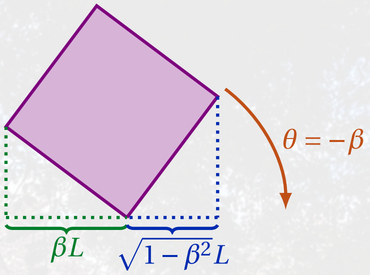
Invariance of transverse position operator

- $X_\perp^i$ commutes with transverse *and* longitudinal boosts

$$[B_i, X_\perp^j] = [B_-, X_\perp^j] = 0$$

- Can reach any momentum from rest by a sequence of boosts:

$$p = \mathscr{B}_\perp(\boldsymbol{v}_\perp)\underbrace{\mathscr{K}_z(\eta)}_{=\mathscr{B}_-(-\eta)\text{ because } B_- = M^{+-} = -M^{03} = K_z \text{ ... artifact of } x^- \sim x^0 - x^3} p_{\text{rest}}$$



- Light front provides a transverse spatial picture that ...
  - ... is quantum-mechanical (canonical commutation relations)
  - ... applies to relativistic systems
  - ... is invariant under motion
- Ultimately a result of Terrell counter-rotations (conceptually), or of the Galilei subgroup (formally)

Summary so far

- The Poincaré group has a Galilei subgroup
  - Its algebra is identical to 2D non-relativistic physics
  - Formally explains all our classical-looking formulas
- Light front time is **invariant under this subgroup**
- Transverse separations are **invariant under this subgroup**
- This will allow a spatio-temporal description of *internal properties of hadrons*


One-particle Wave Functions

One-particle Schödinger equation

- Recall the on-shell relation:

$$p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$$

- First quantization—convert to operators:

$$[X^-, P^+] = -i \qquad [X_\perp^i, P_\perp^j] = i\delta^{ij} \qquad [X^-, P_\perp^j] = [X_\perp^j, P^+] = 0$$

  - Minus sign in null coordinate—because no minus when raising/lowering index
  - This is *not* light front quantization—that'll come in a later lecture
- Momenta become derivatives in coordinate representation:

$$\boldsymbol{p}_\perp \to -i\boldsymbol{\nabla}_\perp \qquad p^- \to i\partial_+ \qquad p^+ \to i\partial_-$$

- Get a light front Schödinger equation:

$$-2\partial_+\partial_-\psi(x) = m^2\psi(x) - \boldsymbol{\nabla}_\perp^2\psi(x)$$

Longitudinal position operator

- Can define a longitudinal position operator:

$$X^- \equiv -\frac{1}{2}\left(\frac{1}{P^+}B_- + B_-\frac{1}{P^+}\right)$$

  - Defined in analogy to $X_\perp^j = -\frac{B_j}{P^+}$
  - Need average between orders make Hermitian—since $[B_-, P^+] = iP^+ \neq 0$

- Helpfully,

$$[P^-, (P^+)^{-1}B_i] = [P^-, B_i(P^+)^{-1}] = i\frac{P^-}{P^+} \qquad \Longrightarrow \qquad i[P^-, X^-] = \frac{P^-}{P^+} = V^-$$

  - Using $[B_-, P^-] = -iP^-$ from slide 47
  - Recall NR-like relation $(\boldsymbol{p}_\perp, p^-) = p^+(\boldsymbol{v}_\perp, v^-)$

- Also helpfully,

$$[X^-, X_\perp^i] = 0$$

  - Let's work it out (next slide) ...

Longitudinal position operator (continued)

- Helpful to recall:

$$X^- = -\frac{1}{2}\left(\frac{1}{P^+}B_- + B_-\frac{1}{P^+}\right) \qquad X_\perp^i = -\frac{B_i}{P^+} \qquad \underbrace{[B_-,B_i] = iB_i \qquad [B_-,P^+] = iP^+}_{\text{slide 47}}$$

- Let's work out ...

$$[X^-,X_\perp^i] = \left[-\tfrac{1}{2}\left(\tfrac{1}{P^+}B_- + B_-\tfrac{1}{P^+}\right), -\tfrac{1}{P^+}B_i\right] = \tfrac{1}{2}\left(\tfrac{1}{P^+}\left[B_-,\tfrac{1}{P^+}B_i\right] + \left[B_-,\tfrac{1}{P^+}B_i\right]\tfrac{1}{P^+}\right)$$

  - Working out the common commutator ...

$$\overbrace{\left[B_-,\tfrac{1}{P^+}B_i\right]}^{[A,BC]=B[A,C]+[A,B]C} = \tfrac{1}{P^+}\underbrace{[B_-,B_i]}_{=iB_i} + \overbrace{\left[B_-,\tfrac{1}{P^+}\right]}^{[A,B^{-1}]=-B^{-1}[A,B]B^{-1}}B_i = i\tfrac{1}{P^+}B_i - \tfrac{1}{P^+}\underbrace{[B_-,P^+]}_{=iP^+}\tfrac{1}{P^+}B_i = i\tfrac{1}{P^+}B_i - i\tfrac{1}{P^+}B_i = 0$$

  - So the whole thing equals zero.

- Much easier to show that

$$[X^-,P^+] = -i$$

  - I leave proof as an exercise

Coordinate vs. momentum representations

- For transverse components, we can say:

Operators	Coordinate representation	Momentum representation
$\boldsymbol{X}_\perp$, $\boldsymbol{P}_\perp$	$\boldsymbol{x}_\perp$, $-i\boldsymbol{\nabla}_\perp^{(x)}$	$i\boldsymbol{\nabla}_\perp^{(p)}$, $\boldsymbol{p}_\perp$

  - Similar to all three components in non-relativistic QM!

- This doesn't work for longitudinal components—instead:

Operators	Coordinate representation	Momentum representation
X^-, P^+	x^-, $i\partial_-$	$-i\dfrac{\partial}{\partial p^+} + f(p^+)$, p^+

  - Can show this and determine $f(p^+)$ using the momentum-space integration element

Momentum-space normalization

- Lorentz-invariant one-particle completeness relation:

$$1 = \overbrace{\int \frac{\mathrm{d}^4 p}{(2\pi)^4}}^{\text{always invariant}} \underbrace{(2\pi)\delta^{(1)}(p^2 - m^2)}_{\text{invariant on-shell condition}} |p^+, \boldsymbol{p}_\perp\rangle\langle p^+, \boldsymbol{p}_\perp|$$

  - Examining the delta further:

$$\delta^{(1)}(p^2 - m^2) = \delta^{(1)}(2p^+ p^- - \boldsymbol{p}_\perp^2 - m^2) = \frac{1}{2p^+}\delta^{(1)}\left(p^- - \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}\right)$$

  - Allows the p^- integral to be eliminated:

$$1 = \underbrace{\int \frac{\mathrm{d}p^+ \,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}}_{\text{invariant momentum-space integration element}} |p^+, \boldsymbol{p}_\perp\rangle\langle p^+, \boldsymbol{p}_\perp|$$

- Inner products in momentum space:

$$\langle\phi|\psi\rangle = \int \frac{\mathrm{d}p^+ \,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3} \overbrace{\langle\phi|p^+, \boldsymbol{p}_\perp\rangle}^{\phi^*(p^+, \boldsymbol{p}_\perp)} \overbrace{\langle p^+, \boldsymbol{p}_\perp|\psi\rangle}^{\psi(p^+, \boldsymbol{p}_\perp)}$$

Longitudinal position operator (reprise)

- The longitudinal position is Hermitian, so ...

$$\langle\phi|X^-|\psi\rangle = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\langle\phi|p^+,\boldsymbol{p}_\perp\rangle\langle p^+,\boldsymbol{p}_\perp|X^-|\psi\rangle = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\langle\phi|(X^-)^\dagger|p^+,\boldsymbol{p}_\perp\rangle\langle p^+,\boldsymbol{p}_\perp|\psi\rangle$$

- Then, using:

$$\langle p^+,\boldsymbol{p}_\perp|X^-|\psi\rangle = -i\frac{\partial\psi(p^+,\boldsymbol{p}_\perp)}{\partial p^+} + f(p^+)\psi(p^+,\boldsymbol{p}_\perp)$$

$$\langle\phi|(X^-)^\dagger|p^+,\boldsymbol{p}_\perp\rangle = i\frac{\partial\phi^*(p^+,\boldsymbol{p}_\perp)}{\partial p^+} + f^*(p^+)\phi^*(p^+,\boldsymbol{p}_\perp)$$

- We find:

$$\int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\phi^*\left(-i\frac{\partial\psi}{\partial p^+} + f(p^+)\psi\right) = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\left(i\frac{\partial\phi^*}{\partial p^+} + f^*(p^+)\phi^*\right)\psi$$

Longitudinal position operator (continued)

- We found:

$$\int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\phi^*\left(-i\frac{\partial\psi}{\partial p^+}+f(p^+)\psi\right)=\int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\left(i\frac{\partial\phi^*}{\partial p^+}+f^*(p^+)\phi^*\right)\psi$$

- Rearranged:

$$\int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\frac{\partial}{\partial p^+}\left[\phi^*\psi\right]=2\int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\mathrm{Im}[f(p^+)]\phi^*\psi$$

- Using integration by parts gives us:

$$f(p^+)=\frac{i}{2p^+} \qquad \Longrightarrow \qquad X^-=-i\left(\frac{\partial}{\partial p^+}-\frac{1}{2p^+}\right)$$

  - Somewhat akin to the Newton-Wigner operator in instant form

- The position and momentum operators in the various representations are:

Operators		**Coordinate representation**		**Momentum representation**	
$\boldsymbol{X}_\perp$,	$\boldsymbol{P}_\perp$	$\boldsymbol{x}_\perp$,	$-i\boldsymbol{\nabla}_\perp^{(x)}$	$i\boldsymbol{\nabla}_\perp^{(p)}$,	$\boldsymbol{p}_\perp$
X^-,	P^+	x^-,	$i\dfrac{\partial}{\partial x^-}$	$-i\left(\dfrac{\partial}{\partial p^+} - \dfrac{1}{2p^+}\right)$,	p^+

- Often we use a **mixed representation** — transverse position + longitudinal momentum

$$i\partial_+\psi(\boldsymbol{x}_\perp, p^+) = \frac{m^2 - \boldsymbol{\nabla}_\perp^2}{2p^+}\psi(\boldsymbol{x}_\perp, p^+)$$

  - We get a nice-looking linear Schrödinger equation
  - We'll see interesting developments when we introduce a potential (next section!)

- Fourier transform utilizes the Lorentz-invariant integration element:

$$\psi(\boldsymbol{x}_\perp, x^-; x^+) = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}\,\psi(\boldsymbol{p}_\perp, p^+)\,\mathrm{e}^{i(\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp - p^+ x^-)}\,\mathrm{e}^{-ip^- x^+}$$

  - Time dependence generated by $p^- = \frac{m^2+\boldsymbol{p}_\perp^2}{2p^+}$
- Inverse transform:

$$\psi(\boldsymbol{p}_\perp, p^+) = 2p^+\,\mathrm{e}^{ix^+p^-}\int \mathrm{d}^2 x_\perp\,\mathrm{d}x^-\,\psi(\boldsymbol{x}_\perp, x^-; x^+)$$

- Can also do partial transforms — only transform longitudinal/transverse:

$$\psi(\boldsymbol{x}_\perp, x^-) = \int \frac{\mathrm{d}p^+}{4\pi p^+}\psi(\boldsymbol{x}_\perp, p^+)\,\mathrm{e}^{-ip^+x^-} \implies \psi(\boldsymbol{x}_\perp, p^+) = 2p^+\int \mathrm{d}x^-\,\psi(\boldsymbol{x}_\perp, x^-)\,\mathrm{e}^{ip^+x^-}$$

$$\psi(\boldsymbol{x}_\perp, p^+) = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2}\psi(\boldsymbol{p}_\perp, p^+)\,\mathrm{e}^{i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp} \implies \psi(\boldsymbol{p}_\perp, p^+) = \int \mathrm{d}^2 x_\perp\,\psi(\boldsymbol{x}_\perp, p^+)\,\mathrm{e}^{-i\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}$$

  - Formulas for $x^+ = 0$ for simplicity
  - Basically associate $2p^+(2\pi)$ with the $x^- \leftrightarrow p^+$ transform

Normalization

- Momentum-space normalization follows from completeness relation:

$$\langle\psi|\psi\rangle = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}\left|\psi(\boldsymbol{p}_\perp, p^+)\right|^2 \equiv 1$$

- Coordinate-space normalization derived via Fourier transform:

$$\int \mathrm{d}^2 x_\perp\,\mathrm{d}x^-\,\psi^*(\boldsymbol{x}_\perp, x^-; x^+)\, i\overleftrightarrow{\partial}_-\,\psi(\boldsymbol{x}_\perp, x^-; x^+)$$

$$= \int \frac{\mathrm{d}p^+\,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}\int \frac{\mathrm{d}k^+\,\mathrm{d}^2 k_\perp}{2k^+(2\pi)^3}(p^+ + k^+)\psi^*(\boldsymbol{k}_\perp, k^+)\psi(\boldsymbol{p}_\perp, p^+)\underbrace{\int \mathrm{d}^2 x_\perp\,\mathrm{d}x^-\,\mathrm{e}^{i\left((\boldsymbol{p}_\perp-\boldsymbol{k}_\perp)\cdot\boldsymbol{x}_\perp-(p^+-k^+)x^-\right)}}_{=(2\pi)^3\delta^{(1)}(p^+-k^+)\delta^{(2)}(\boldsymbol{p}_\perp-\boldsymbol{k}_\perp)}$$

$$= \int \frac{\mathrm{d}p^+\,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}|\psi(\boldsymbol{p}_\perp, p^+)|^2 = 1$$

- The mixed-representation normalization rule (try proving as an exercise):

$$\int \frac{\mathrm{d}p^+\,\mathrm{d}^2 x_\perp}{4\pi p^+}\left|\psi(\boldsymbol{x}_\perp, p^+)\right|^2 = 1$$

Summary so far

- Quantum mechanics looks pretty normal in transverse components
- A longitudinal position operator exists, but exhibits weird behavior

Operators		**Coordinate representation**		**Momentum representation**	
$\boldsymbol{X}_\perp$,	$\boldsymbol{P}_\perp$	$\boldsymbol{x}_\perp$,	$-i\boldsymbol{\nabla}_\perp^{(x)}$	$i\boldsymbol{\nabla}_\perp^{(p)}$,	$\boldsymbol{p}_\perp$
X^-,	P^+	x^-,	$i\dfrac{\partial}{\partial x^-}$	$-i\left(\dfrac{\partial}{\partial p^+} - \dfrac{1}{2p^+}\right)$,	p^+

- It's common to use a **mixed representation** when doing light front QM — $p+$ and $\boldsymbol{x}_\perp$
  - Fully momentum-space representation is also common
  - Basically nobody works in the x^- coordinate representation

$$\int \frac{\mathrm{d}p^+\,\mathrm{d}^2x_\perp}{4\pi p^+}\left|\psi(\boldsymbol{x}_\perp, p^+)\right|^2 = 1$$


Separation of Variables

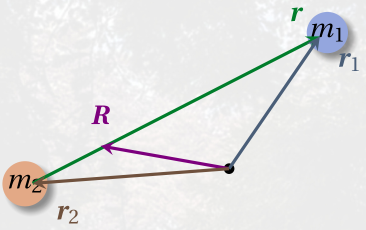
Non-relativistic separation of variables

- Let's begin with a review—**separation of variables** for non-relativistic two-body systems:

$$\underbrace{\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}}_{\text{barycenter (center of mass)}} \qquad \underbrace{\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2}_{\text{relative separation}}$$

- Canonically conjugate momenta:

$$\underbrace{\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2}_{\text{total momentum}} \qquad \underbrace{\boldsymbol{k} = \frac{m_2 \boldsymbol{p}_1 - m_1 \boldsymbol{p}_2}{m_1 + m_2}}_{\text{relative momentum}}$$



  - Can be proved by requiring:

$$[r^i, k^j] = i\delta^{ij} \qquad [r^i, P^j] = 0$$

$$[R^i, P^j] = i\delta^{ij} \qquad [R^i, k^j] = 0$$

- Transformation formulas:

$$\underbrace{\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}}_{\text{barycenter (center of mass)}} \qquad \underbrace{\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2}_{\text{relative separation}} \qquad \underbrace{\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2}_{\text{total momentum}} \qquad \underbrace{\boldsymbol{k} = \frac{m_2 \boldsymbol{p}_1 - m_1 \boldsymbol{p}_2}{m_1 + m_2}}_{\text{relative momentum}}$$

- Inversion formulas:

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\boldsymbol{r} \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\boldsymbol{r} \qquad \boldsymbol{p}_1 = \frac{m_1}{m_1 + m_2}\boldsymbol{P} + \boldsymbol{k} \qquad \boldsymbol{p}_2 = \frac{m_2}{m_1 + m_2}\boldsymbol{P} - \boldsymbol{k}$$

- Main perk — clean separation of the Hamiltonian:

$$H = \frac{\boldsymbol{p}_1^2}{2m_1} + \frac{\boldsymbol{p}_2^2}{2m_2} + V(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \overbrace{\frac{\boldsymbol{P}^2}{2M}}^{\text{free Hamiltonian for barycentric motion}} + \underbrace{\frac{\boldsymbol{k}^2}{2\mu} + V(\boldsymbol{r})}_{\text{internal structure independent of } \boldsymbol{P}}$$

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

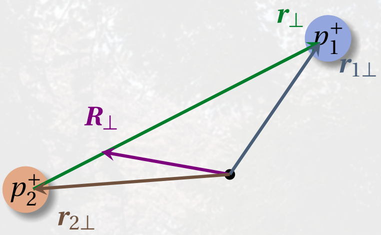
Light front separation of variables

- **Separation of variables** proceeds similarly for transverse light front coordinates:

$$\underbrace{\boldsymbol{X}_\perp = \frac{p_1^+ \boldsymbol{x}_{1\perp} + p_2^+ \boldsymbol{x}_{2\perp}}{p_1^+ + p_2^+}}_{\text{barycenter (center of } p^+)} \qquad \underbrace{\boldsymbol{r}_\perp = \boldsymbol{x}_{1\perp} - \boldsymbol{x}_{2\perp}}_{\text{relative separation}}$$

- But we've got p^+ in place of the mass
- Canonically conjugate momenta:

$$\underbrace{\boldsymbol{P}_\perp = \boldsymbol{p}_{1\perp} + \boldsymbol{p}_{2\perp}}_{\text{total momentum}} \qquad \underbrace{\boldsymbol{k} = \frac{p_2^+ \boldsymbol{p}_{1\perp} - p_1^+ \boldsymbol{p}_{2\perp}}{p_1^+ + p_2^+}}_{\text{relative momentum}}$$



  - Can be proved by requiring:

$$[r_\perp^i, k_\perp^j] = i\delta^{ij} \qquad [r_\perp^i, P_\perp^j] = 0$$

$$[R_\perp^i, P_\perp^j] = i\delta^{ij} \qquad [R_\perp^i, k_\perp^j] = 0$$

Separation of variables (light front, continued)

- Transformation formulas:

$$\underbrace{\boldsymbol{X}_\perp = \frac{p_1^+ \boldsymbol{x}_{1\perp} + p_2^+ \boldsymbol{x}_{2\perp}}{p_1^+ + p_2^+}}_{\text{barycenter (center of mass)}} \qquad \underbrace{\boldsymbol{r}_\perp = \boldsymbol{x}_{1\perp} - \boldsymbol{x}_{2\perp}}_{\text{relative separation}} \qquad \underbrace{\boldsymbol{P}_\perp = \boldsymbol{p}_{1\perp} + \boldsymbol{p}_{2\perp}}_{\text{total momentum}} \qquad \underbrace{\boldsymbol{k}_\perp = \frac{p_2^+ \boldsymbol{p}_{1\perp} - p_1^+ \boldsymbol{p}_{2\perp}}{p_1^+ + p_2^+}}_{\text{relative momentum}}$$

- Inversion formulas:

$$\boldsymbol{x}_{1\perp} = \boldsymbol{X}_\perp + \frac{p_2^+}{p_1^+ + p_2^+}\boldsymbol{r}_\perp \qquad \boldsymbol{p}_{1\perp} = \frac{p_1^+}{p_1^+ + p_2^+}\boldsymbol{P}_\perp + \boldsymbol{k}_\perp$$

$$\boldsymbol{x}_{2\perp} = \boldsymbol{X}_\perp - \frac{p_1^+}{p_1^+ + p_2^+}\boldsymbol{r}_\perp \qquad \boldsymbol{p}_{2\perp} = \frac{p_2^+}{p_1^+ + p_2^+}\boldsymbol{P}_\perp - \boldsymbol{k}_\perp$$

- What does this do to the Hamiltonian?

Separation of variables and light front Hamiltonian

- Start working it out ...

$$P^- = \frac{m_1^2 + \boldsymbol{p}_{1\perp}^2}{2p_1^+} + \frac{m_2^2 + \boldsymbol{p}_{2\perp}^2}{2p_2^+} + \underbrace{P^-_{\text{int}}(\ldots)}_{\text{interaction Hamiltonian}} = \frac{p_2^+ m_1^2 + p_1^+ m_2^2}{2p_1^+ p_2^+} + \frac{\boldsymbol{P}_\perp^2}{2(p_1^+ + p_2^+)} + \frac{\boldsymbol{k}_\perp^2}{2\frac{p_1^+ p_2^+}{p_1^+ + p_2^+}} + P^-_{\text{int}}(\ldots)$$

  - In contrast to a reduced mass, we define **momentum fractions**:

$$\underbrace{P^+ = p_1^+ + p_2^+}_{\text{total } P^+} \qquad \underbrace{\xi_1 = \frac{p_1^+}{P^+} \qquad \xi_2 = \frac{p_2^+}{P^+}}_{\text{momentum fractions}}$$

- Gives, with minor rearrangement:

$$\underbrace{M^2 = 2P^+P^- - \boldsymbol{P}_\perp^2}_{\text{free Hamiltonian for barycentric motion}} = \overbrace{\frac{m_1^2}{\xi_1} + \frac{m_2^2}{\xi_2} + \frac{\boldsymbol{k}_\perp^2}{\xi_1\xi_2} + \underbrace{2P^+P^-_{\text{int}}(\ldots)}_{\equiv V(\ldots)}}^{\text{Internal structure — must be boost-invariant, independent of } (P^+;\boldsymbol{P}_\perp).}$$

  - The left-hand side is a Lorentz-invariant
  - Can get to $(P^+;\boldsymbol{P}_\perp) = (M;\boldsymbol{0}_\perp)$ via Galilean boosts
  - Right-hand side *must be* independent of $(P^+;\boldsymbol{P}_\perp)$

Free Hamiltonian under transverse boosts

- Light front Hamiltonian equation (two bodies):

$$M^2 = \underbrace{\frac{m_1^2}{\xi_1} + \frac{m_2^2}{\xi_2} + \frac{\boldsymbol{k}_\perp^2}{\xi_1 \xi_2}}_{\text{free Hamiltonian}} + \underbrace{V(\dots)}_{\text{interaction}} = M_{\text{free}}^2 + M_{\text{int}}^2$$

  - Brodsky and others call M^2 the Hamiltonian
  - Left-hand side commutes with entire Poincaré group!
  - Whatever free part commutes with must commute with interaction part.

- Helpful to recall:

$$X_\perp^i = -\frac{B_\perp^i}{P^+} \qquad \Longrightarrow \qquad B_\perp^i = -P^+ X_\perp^i$$

  - A boost transforms all objects — so it's *barycentric* $X_\perp^i$ that's used

$$[B_\perp^i, k_\perp^j] = \overbrace{[-P^+ X_\perp^i, k_\perp^j]}^{[AB,C]=A[B,C]+[A,C]B} = -\underbrace{[P^+, k_\perp^j]}_{=0 \text{ because they're all momenta}} X_\perp^i - P^+ \overbrace{[X_\perp^i, k_\perp^j]}^{=0 \text{ from variable separation}} = 0$$

- Also have $[B_\perp^i, \xi_{1,2}] = 0$ basically by definition

(transverse boosts leave plus components invariant)

Free Hamiltonian under transverse boosts

- Light front Hamiltonian equation (two bodies):

$$M^2 = \underbrace{\frac{m_1^2}{\xi_1} + \frac{m_2^2}{\xi_2} + \frac{\boldsymbol{k}_\perp^2}{\xi_1 \xi_2}}_{\text{free Hamiltonian}} + \underbrace{V(\dots)}_{\text{interaction}} = M_{\text{free}}^2 + M_{\text{int}}^2$$

  - Brodsky and others call M^2 the Hamiltonian
  - Left-hand side commutes with entire Poincaré group!
  - Whatever free part commutes with must commute with interaction part.
- Helpful to recall:

$$\mathcal{K}_z(\eta) p_1^+ = e^\eta p_1^+ \qquad \mathcal{K}_z(\eta) p_2^+ = e^\eta p_2^+ \qquad \Longrightarrow \qquad \mathcal{K}_z(\eta)\xi_1 = \xi_1 \qquad \mathcal{K}_z(\eta)\xi_2 = \xi_2$$

- Also have $[B_-, \boldsymbol{k}_\perp] = 0$, basically by definition

  (longitudinal boosts leave transverse components invariant)
- **Free Hamiltonian commutes with boosts!**
  - ***Interaction Hamiltonian must also commute with boosts.***

What commutes with boosts?

- We've already seen the following are "good" variables:

$$\xi_1, \quad \xi_2, \quad \boldsymbol{k}_\perp$$

- Transverse separation $\boldsymbol{r}_\perp$ also works!

$$[B^i_\perp, r^j_\perp] = \overbrace{[-P^+X^i_\perp, r^j_\perp]}^{[AB,C]=A[B,C]+[A,C]B} = -[P^+, r^j_\perp]X^i_\perp - P^+\overbrace{[X^i_\perp, r^j_\perp]}^{=0 \text{ from variable separation}} = -\underbrace{[p^+_1, r^j_\perp]}_{=0}X^i_\perp - \underbrace{[p^+_2, r^j_\perp]}_{=0}X^i_\perp = 0$$

- What about longitudinal separation?

$$z_1 - z_2 = \left(\frac{x^+_1 - x^-_1}{\sqrt{2}}\right) - \left(\frac{x^+_2 - x^-_2}{\sqrt{2}}\right) \xrightarrow[\text{fixed } x^+]{} \left(\frac{x^-_2 - x^-_1}{\sqrt{2}}\right)$$

  - Commutes with $X^i_\perp$, and ...

$$[P^+, x^-_2 - x^-_1] = \underbrace{[p^+_2, x^-_2]}_{= i \text{ by CCR's}} - \underbrace{[p^+_1, x^-_1]}_{=i} = 0$$

  - So this commutes with transverse boosts—*but*:

$$\mathcal{K}_z(\eta)\left(x^-_2 - x^-_1\right) = e^{-\eta}\left(x^-_2 - x^-_1\right) \neq \left(x^-_2 - x^-_1\right)$$

More on longitudinal separation

- So $(x_2^- - x_1^-)$ is not boost-invariant, but the **Miller-Brodsky variable** $\tilde{z}$ is:

$$\tilde{z} \equiv P^+ \left(x_2^- - x_1^- \right)$$

  - Works out because P^+ scales as $\mathrm{e}^{\eta} P^+$ while x^- scales with $\mathrm{e}^{-\eta} x^-$.
  - Can put P^+ on either side since $[P^+, x_2^- - x_1 -] = 0$ (previous slide)
- So $\tilde{z}$ is a good, boost-invariant variable that can appear in the potential.
- Our list of good variables:

$$\begin{array}{ll} \boldsymbol{k}_\perp\,, & \xi_1 = 1 - \xi_2 \equiv \xi \\ \boldsymbol{r}_\perp\,, & \tilde{z} \end{array}$$

  - We have $\boldsymbol{k}_\perp$ and $\boldsymbol{r}_\perp$ as canonically conjugate
  - Can helpfully write $\boldsymbol{k}_\perp \to -i\boldsymbol{\nabla}_\perp$ in coordinate representation
  - Is there a similar relationship between ξ and $\tilde{z}$?

Longitudinal separation in the momentum fraction representation

- Recall:

$$P^+ = p_1^+ + p_2^+ \qquad \xi = \frac{p_1^+}{p_1^+ + p_2^+}$$

- The derivation will be dry, but important ...

$$x_1^- = -i\left(\frac{\partial}{\partial p_1^+} - \frac{1}{2p_1^+}\right) = -i\left(\frac{\partial P^+}{\partial p_1^+}\frac{\partial}{\partial P^+} + \frac{\partial \xi}{\partial p_1^+}\frac{\partial}{\partial \xi} - \frac{1}{2\xi P^+}\right) = -i\left(\frac{\partial}{\partial P^+} + \frac{1-\xi}{P^+}\frac{\partial}{\partial \xi} - \frac{1}{2\xi P^+}\right)$$

$$x_2^- = -i\left(\frac{\partial}{\partial p_2^+} - \frac{1}{2p_2^+}\right) = -i\left(\frac{\partial P^+}{\partial p_2^+}\frac{\partial}{\partial P^+} + \frac{\partial \xi}{\partial p_2^+}\frac{\partial}{\partial \xi} - \frac{1}{2(1-\xi)P^+}\right) = -i\left(\frac{\partial}{\partial P^+} - \frac{\xi}{P^+}\frac{\partial}{\partial \xi} - \frac{1}{2(1-\xi)P^+}\right)$$

$$x_2^- - x_1^- = \frac{i}{P^+}\left(\frac{\partial}{\partial \xi} - \frac{1-2\xi}{2\xi(1-\xi)}\right)$$

- This can be written in a nice, compact form:

$$\tilde{z}\psi(\xi) = P^+(x_2^- - x_1^-)\psi(\xi) = i\sqrt{\xi(1-\xi)}\frac{\partial}{\partial \xi}\left[\frac{1}{\sqrt{\xi(1-\xi)}}\psi(\xi)\right]$$

Normalization of relative wave function

- The ξ-dependent wave function has a funky normalization
- In terms of longitudinal momentum (suppressing transverse variables):

$$\int \frac{\mathrm{d}p_1^+}{4\pi p_1^+} \int \frac{\mathrm{d}p_2^+}{4\pi p_2^+} \left|\psi(p_1^+, p_2^+)\right|^2 = 1$$

- Jacobian for transformation to (P^+, ξ):

$$\frac{\partial(P^+, \xi)}{\partial(p_1^+, p_2^+)} = \begin{vmatrix} 1 & 1 \\ \frac{1-\xi}{P^+} & -\frac{\xi}{P^+} \end{vmatrix} = \frac{1}{P^+}$$

- Normalization for factorized wave function:

$$\underbrace{\int \frac{\mathrm{d}P^+}{4\pi P^+} \left|\Psi(P^+)\right|^2}_{\equiv 1} \underbrace{\int \frac{\mathrm{d}\xi}{4\pi \xi(1-\xi)} \left|\psi(\xi)\right|^2}_{\equiv 1} = 1$$

- $M^2 = 2P^+P^- - \boldsymbol{P}_\perp^2$ is typically used as the Hamiltonian
  - Capital P signifies total four-momentum of a composite system
- Kinetic and potential energy parts are separately boost-invariant:

$$M^2 = M^2_{\text{kin}} + M^2_{\text{pot}} = \sum_{\text{particles}} \frac{m_n^2 - \boldsymbol{\nabla}^2_{n,\perp}}{\xi_n} + V_{\text{LF}}$$

- Typically parametrize wave functions in terms of **momentum fractions**:

$$\xi_n = \frac{p_n^+}{P^+}$$

  - **Caution:** most authors use x instead of ξ
  - I'm using ξ here to avoid confusion with spacetime coordinates
- Wave function normalization (two body systems):

$$\int \frac{\mathrm{d}\xi\, \mathrm{d}^2 r_\perp}{4\pi\xi(1-\xi)} \left|\psi(\boldsymbol{r}_\perp, \xi)\right|^2 = 1$$

Light Front Harmonic Oscillator

Harmonic oscillator potential — *classical* case

- Non-relativistically, we have:

reduced mass, $= \frac{m_1 m_2}{m_1+m_2}$

$\tilde{z} = \sqrt{2}P^+(z_1-z_2) \xrightarrow{\text{rest}} (m_1+m_2)(z_1-z_2)$

$$V_{\rm NR}(\boldsymbol{r}) = \frac{1}{2}\mu\omega^2\boldsymbol{r}^2 = \frac{1}{2}\mu\omega^2\left(\boldsymbol{r}_\perp^2 + (z_1-z_2)^2\right) = \frac{1}{2}\mu\omega^2\left(\boldsymbol{r}_\perp^2 + \overbrace{\frac{\tilde{z}^2}{(m_1+m_2)^2}}\right)$$

  - NR formula assumed to work *when both bodies at rest*

- Use mass-squared element in non-relativistic limit to match potentials *at rest*:

$$2P^+_{\rm rest}P^-_{\rm int} = \left(E_{\rm free} + V_{\rm NR}\right)^2 - E^2_{\rm free} \approx 2(m_1+m_2)V_{\rm NR} = m_1 m_2\omega^2\left(\boldsymbol{r}_\perp^2 + \frac{\tilde{z}^2}{(m_1+m_2)^2}\right)$$

  - Substitute masses → plus momenta to incorporate relativistic motion effects.
  - Note that frequency scales like $(P^+)^{-1}$ because of redshift/blueshift.

$$m_1 m_2 \to 2(P^+)^2\xi(1-\xi) \qquad\qquad \omega^2 \to \frac{(m_1+m_2)^2}{2(P^+)^2}\omega^2$$

- Get light front harmonic oscillator potential:

$$V_{\rm LF}(\boldsymbol{r}_\perp,\tilde{z},\xi) = \underbrace{(m_1+m_2)^2\omega^2}_{\equiv\kappa^4}\xi(1-\xi)\left(\boldsymbol{r}_\perp^2 + \frac{\tilde{z}^2}{(m_1+m_2)^2}\right)$$

  - Similar forms by Brodsky, de Teramond, Vary, etc., as confining potential!

Quantizing the potential

- The classical potential is not Hermitian — ξ and $\tilde{z}$ don't commute!

$$V_{\text{classical}} = \kappa^4 \xi(1-\xi)\left(\boldsymbol{r}_\perp^2 + \frac{\tilde{z}^2}{(m_1+m_2)^2}\right)$$

- First quantization is ambiguous; which do we use?

$$\xi(1-\xi)\tilde{z}^2 \xrightarrow{?} \frac{1}{2}\Big(\xi(1-\xi)\tilde{z}^2 + \tilde{z}^2\xi(1-\xi)\Big)?$$

$$\xi(1-\xi)\tilde{z}^2 \xrightarrow{?} \frac{1}{4}\Big(\xi(1-\xi)\tilde{z}^2 + 2\tilde{z}\xi(1-\xi)\tilde{z} + \tilde{z}^2\xi(1-\xi)\Big)?$$

- I'm going to make an arbitrary ad hoc choice:

$$\xi(1-\xi)\tilde{z}^2 \to \sqrt{\xi(1-\xi)}\,\tilde{z}^2\sqrt{\xi(1-\xi)}$$

  - It's a pedagocial choice — makes the math easier!

$$\tilde{z}\psi(\xi) = i\sqrt{\xi(1-\xi)}\frac{\partial}{\partial\xi}\left[\frac{1}{\sqrt{\xi(1-\xi)}}\psi(\xi)\right]$$

$$\implies \quad \sqrt{\xi(1-\xi)}\,\tilde{z}^2\sqrt{\xi(1-\xi)}\,\psi(\xi) = \xi(1-\xi)\frac{\partial^2\psi}{\partial\xi^2}$$

Harmonic oscillator potential in the mixed representation

- The quantum light front potential:

$$V_{\mathrm{LF}}(\boldsymbol{r}_\perp, \tilde{z}) = \kappa^4 \sqrt{\xi(1-\xi)} \left(\boldsymbol{r}_\perp^2 + \frac{\tilde{z}^2}{(m_1+m_2)^2} \right) \sqrt{\xi(1-\xi)}$$

- The action of the potential on a mixed-representation wave function is:

$$V_{\mathrm{LF}}(\boldsymbol{r}_\perp, \tilde{z}) \psi(\boldsymbol{r}_\perp, \xi) = \kappa^4 \xi(1-\xi) \left(\boldsymbol{r}_\perp^2 \psi(\boldsymbol{r}_\perp, \xi) - \frac{\xi(1-\xi)}{(m_1+m_2)^2} \frac{\partial^2 \psi(\boldsymbol{r}_\perp, \xi)}{\partial \xi^2} \right)$$

- Recall that our Schödinger equation is:

$$-\frac{\nabla_\perp^2 \psi}{\xi(1-\xi)} + \left(\frac{m_1^2}{\xi} + \frac{m_2^2}{1-\xi}\right)\psi + V_{\mathrm{LF}}\psi = M^2\psi$$

- Explicitly, for the harmonic oscillator:

$$-\frac{1}{\xi(1-\xi)}\left(\frac{\partial^2\psi}{\partial r_\perp^2} + \frac{1}{r_\perp}\frac{\partial\psi}{\partial r_\perp} + \frac{1}{r_\perp^2}\frac{\partial^2\psi}{\partial\phi^2}\right) + \left(\frac{m_1^2}{\xi} + \frac{m_2^2}{1-\xi} + \kappa^4\xi(1-\xi)r_\perp^2 - M^2\right)\psi$$
$$-\frac{\kappa^4\xi(1-\xi)}{(m_1+m_2)^2}\frac{\partial^2\psi}{\partial\xi^2} = 0$$

- Several $\xi(1-\xi)$ can be absorbed through the holographic variable:

$$\zeta = \sqrt{\xi(1-\xi)}\,r_\perp$$

  - Naturally appears in holography (see Brodsky & de Teramond)
  - Here it's just a convenient change of variables

Harmonic oscillator — separation of variables

- The (somewhat) simplified equation:

$$-\left(\frac{\partial^2\psi}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial\psi}{\partial\zeta}+\frac{1}{\zeta^2}\frac{\partial^2\psi}{\partial\phi^2}\right)+\left(\frac{m_1^2}{\xi}+\frac{m_2^2}{1-\xi}+\kappa^4\zeta^2-M^2\right)\psi-\frac{\kappa^4\xi(1-\xi)}{(m_1+m_2)^2}\frac{\partial^2\psi}{\partial\xi^2}=0$$

- This can be solved by a standard separation of variables:

$$\psi(\zeta,\phi,\xi)=Z(\zeta)F(\phi)X(\xi)$$

  - Leads to three ordinary differential equations:

$$-\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2}=m_l^2F$$

$$-\frac{\mathrm{d}^2Z}{\mathrm{d}\zeta^2}-\frac{1}{\zeta}\frac{\mathrm{d}Z}{\mathrm{d}\zeta}+\left(\frac{m_l^2}{\zeta^2}+\kappa^4\zeta^2\right)Z=\lambda Z$$

$$\frac{\kappa^4\xi(1-\xi)}{(m_1+m_2)^2}\frac{\mathrm{d}^2X}{\mathrm{d}\xi^2}+\left(M^2-\frac{m_1^2}{\xi}-\frac{m_2^2}{1-\xi}\right)X=\lambda X$$

  - We have three coupled eigenvalue problems in $\{M^2,\lambda,m_l\}$

Harmonic oscillator — azimuth dependence

- The first eigenvalue problem is trivial:

$$-\frac{\mathrm{d}^2 F}{\mathrm{d}\phi^2} = m_l^2 F$$

- Solutions:

$$F(\phi) = \mathrm{e}^{\pm i m_l \phi}$$

- Continuity requires m_l to be an integer
- m_l is just the z-axis orbital angular momentum.

- Let's look at the second eigenvalue problem:

$$-\frac{\mathrm{d}^2 Z}{\mathrm{d}\zeta^2} - \frac{1}{\zeta}\frac{\mathrm{d}Z}{\mathrm{d}\zeta} + \left(\frac{m_l^2}{\zeta^2} + \kappa^4\zeta^2\right) Z = \lambda Z$$

  - This is basically Exercise 12.3.7 in Shankar!

- Define unitless variables:

$$\tau = \kappa\zeta \qquad \Lambda = \frac{\lambda}{\kappa^2} \qquad \Longrightarrow \qquad Z''(\tau) + \frac{1}{\tau} Z'(\tau) + \left(\Lambda - \frac{m_l^2}{\tau^2} - \tau^2\right) Z(\tau) = 0$$

- Rework by factoring out $\mathrm{e}^{-\frac{1}{2}\tau^2}$ ...

$$Z(\tau) = Y(\tau)\,\mathrm{e}^{-\frac{1}{2}\tau^2} \qquad \Longrightarrow \qquad Z' = \left(Y' - \tau Y\right)\mathrm{e}^{-\frac{1}{2}\tau^2} \qquad Z'' = \left(Y'' - 2\tau Y' + (\tau^2 - 1) Y\right)\mathrm{e}^{-\frac{1}{2}\tau^2}$$

$$\Longrightarrow \quad Y''(\tau) + \left(\frac{1}{\tau} - 2\tau\right) Y'(\tau) + \left(\Lambda - 2 - \frac{m_l^2}{\tau^2}\right) Y(\tau) = 0$$

Harmonic oscillator — transverse position dependence (continued)

- So far, with $Z(\tau) = Y(\tau)\,\mathrm{e}^{-\frac{1}{2}\tau^2}$ ...

$$Y''(\tau) + \left(\frac{1}{\tau} - 2\tau\right) Y'(\tau) + \left(\Lambda - 2 - \frac{m_l^2}{\tau^2}\right) Y(\tau) = 0$$

- Can pull out another factor τ^{m_l} ...

$$Y(\tau) = \tau^{m_l} f(\tau) \quad \Longrightarrow \quad Y' = \tau^{m_l}\left(f' + \frac{m_l}{\tau} f\right) \qquad Y'' = \tau^{m_l}\left(f'' + \frac{2m_l}{\tau} f' + \frac{m_l(m_l-1)}{\tau^2} f\right)$$

$$\Longrightarrow \quad f''(\tau) + \left(\frac{2m_l+1}{\tau} - 2\tau\right) f'(\tau) + \left(\Lambda - 2m_l - 2\right) f(\tau) = 0$$

- I'm not aware of explicit tabulated solutions, but we can use standard methods (see Shankar 12.3.7):

$$f(\tau) = \sum_{n=0}^{\infty} C_n \tau^n \quad \Longrightarrow \quad C_{n+2} = \frac{2(1+n+m_l) - \Lambda}{(n+2)(2m_l+n+2)} C_n \quad \text{and} \quad C_1 = 0$$

  - Series terminates iff $\Lambda = 2(1 + N_\perp + m_l)$ for some $N_\perp \geq 0$
  - And $N_\perp \equiv 2n_\perp$ must be even (since $C_1 = 0$)

$$\lambda(n_\perp, m_l) = 2\kappa^2(1 + 2n_\perp + m_l)$$

- Let's look at the third eigenvalue problem:

$$\frac{\kappa^4 \xi(1-\xi)}{(m_1+m_2)^2}\frac{\mathrm{d}^2 X}{\mathrm{d}\xi^2} + \left(M^2 - \lambda - \frac{m_1^2}{\xi} - \frac{m_2^2}{1-\xi}\right) X = 0$$

- Define some new dimensionless constants:

$$\mu_0 = \frac{(m_1+m_2)^2(M^2-\lambda)}{\kappa^4} \qquad \mu_1 = \frac{m_1^2(m_1+m_2)^2}{\kappa^4} \qquad \mu_2 = \frac{m_2^2(m_1+m_2)^2}{\kappa^4}$$

$$\implies \quad \xi(1-\xi)X''(\xi) + \left(\mu_0 - \frac{\mu_1}{\xi} - \frac{\mu_2}{1-\xi}\right) X(\xi) = 0$$

- Expect endpoint behavior ξ^a at $\xi \sim 0$ and $(1-\xi)^b$ at $\xi \sim 1$ for some $a, b > 0$ ...
  - We'll deal with this in two steps:
    1. $X(\xi) = \xi^a Y(\xi)$
    2. $Y(\xi) = (1-\xi)^b f(\xi)$
  - We can figure out a and b by canceling out $\frac{1}{\xi}$ and $\frac{1}{1-\xi}$ in the differential equation

- Starting point:

$$\xi(1-\xi)X''(\xi) + \left(\mu_0 - \frac{\mu_1}{\xi} - \frac{\mu_2}{1-\xi}\right)X(\xi) = 0$$

- Substitution:

$$X(\xi) = \xi^a Y(\xi) \qquad \Longrightarrow \qquad X''(\xi) = \xi^a\left(Y''(\xi) + \tfrac{2a}{\xi}Y'(\xi) + \tfrac{a(a-1)}{\xi^2}Y(\xi)\right)$$

$$\Longrightarrow \quad \xi(1-\xi)Y''(\xi) + 2a(1-\xi)Y'(\xi) + \Bigg(\mu_0 - a(a-1) + \underbrace{\frac{a(a-1)-\mu_1}{\xi}}_{a(a-1)=\mu_1 \implies a=\frac{1}{2}\left(1+\sqrt{1+4\mu_1}\right)} - \frac{\mu_2}{1-\xi}\Bigg)Y(\xi) = 0$$

  - Technically $a = \frac{1}{2}\left(1 \pm \sqrt{1+4\mu_1}\right)$ are both solutions
  - But we need $a > 0$ in the large mass limit

- So far:

$$\xi(1-\xi)Y''(\xi) + 2a(1-\xi)Y'(\xi) + \left(\mu_0 - a(a-1) - \frac{\mu_2}{1-\xi}\right)Y(\xi) = 0$$

- Starting point:

$$\xi(1-\xi)Y''(\xi)+2a(1-\xi)Y'(\xi)+\left(\mu_0-a(a-1)-\frac{\mu_2}{1-\xi}\right)Y(\xi)=0$$

- Substitution:

$$Y(\xi)=(1-\xi)^b f(\xi) \quad\Longrightarrow\quad Y'=(1-\xi)^b\left(f'-\tfrac{b}{1-\xi}f\right),\quad Y''=(1-\xi)^b\left(f''-\tfrac{2b}{1-\xi}f'+\tfrac{b(b-1)}{(1-\xi)^2}f\right)$$

$$\Longrightarrow\quad \xi(1-\xi)f''(\xi)+2\big(a(1-\xi)-b\xi\big)f'(\xi)+\left(\mu_0-a(a-1)+\underbrace{\frac{\xi b(b-1)-\mu_2}{1-\xi}}_{b(b-1)=\mu_2\implies b=\frac{1}{2}\left(1+\sqrt{1+4\mu_2}\right)}-2ab\right)f(\xi)=0$$

- So far:

$$\xi(1-\xi)f''(\xi)+2\big(a(1-\xi)-b\xi\big)f'(\xi)+\big(\mu_0+(a+b)(1-a-b)\big)f(\xi)=0$$

  - From here, can use series expansion method

- With $X(\xi) = \xi^a (1-\xi)^b f(\xi)$, we need to solve:

$$\xi(1-\xi) f''(\xi) + 2\big(a(1-\xi) - b\xi\big) f'(\xi) + \Big(\mu_0 + (a+b)(1-a-b)\Big) f(\xi) = 0$$

- Use series expansion:

$$f(\xi) = \sum_{n=0}^{\infty} C_n \xi^n \qquad \Longrightarrow \qquad C_{n+1} = -\left(\frac{\mu - (a+b)\big(a+b+(2n-1)\big) - n(n-1)}{(n+1)(n+2a)}\right) C_n$$

  - Good exercise to show this
  - Series terminates iff there's some $n_\parallel \geq 0$ for which $\mu_0 = n_\parallel(n_\parallel - 1) + (a+b)\big(a+b+(2n_\parallel - 1)\big)$

$$\boxed{M^2(n_\parallel, n_\perp, m_l) = \kappa^2 \left\{ 2(1 + 2n_\perp + m_l) + \frac{\kappa^2}{(m_1+m_2)^2}\Big[n_\parallel(n_\parallel - 1) + (a+b)\big(a+b+(2n_\parallel - 1)\big)\Big]\right\}}$$

  - We got the mass spectrum for the relativistic harmonic oscillator!
  - Recall $a, b = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4m_{1,2}^2 (m_1+m_2)^2}{\kappa^4}}\right)$

Harmonic oscillator — spectrum

- Does the spectrum we got make sense?

$$M^2(n_\parallel, n_\perp, m_l) = \kappa^2 \left\{ 2(1+2n_\perp+m_l) + \frac{\kappa^2}{(m_1+m_2)^2}\left[n_\parallel(n_\parallel-1)+(a+b)\big(a+b+(2n_\parallel-1)\big)\right]\right\}$$

- Take non-relativistic limit: $m_1, m_2 \gg \kappa$:

$$a,b = \frac{1}{2}\left(1+\sqrt{1+\frac{4m_{1,2}^2(m_1+m_2)^2}{\kappa^4}}\right) \approx \frac{1}{2}+\frac{m_{1,2}(m_1+m_2)}{\kappa^2} \quad \Longrightarrow \quad a+b \approx 1+\frac{(m_1+m_2)^2}{\kappa^2}$$

$$\Longrightarrow \quad M^2 \approx (m_1+m_2)^2 + \kappa^2\Big(3+2(2n_\perp+n_\parallel+m_l)\Big) + \mathscr{O}(\kappa^4)$$

- Note that:

$$M^2 = (m_1+m_2+E_{\mathrm{NR}})^2 \approx (m_1+m_2)^2 + 2(m_1+m_2)E_{\mathrm{NR}} + \mathscr{O}(E_{\mathrm{NR}}^2) \qquad \underbrace{\kappa^2 = (m_1+m_2)\omega}_{\text{first slide in section}}$$

$$\Longrightarrow \quad E_{\mathrm{NR}} = \omega\left(\frac{3}{2}+(2n_\perp+n_\parallel+m_l)\right)$$

- This is *exactly* the usual non-relativistic spectrum if $n \equiv 2n_\perp + n_\parallel + m_l$!

Harmonic oscillator — ground state wave function

- Ground state when $(n_\parallel, n_\perp, m_l) = (0,0,0)$

$$F(\phi) = 1 \qquad Z(\zeta) = \mathrm{e}^{-\frac{1}{2}(\kappa\zeta)^2} \qquad X(\xi) = \xi^a (1-\xi)^b$$

- Also recall:

$$\zeta = \sqrt{\xi(1-\xi)}\, r_\perp \qquad \psi(\boldsymbol{r}_\perp, \xi) = Z(\zeta) F(\phi) X(\xi)$$

- So the ground state wave function is (up to a normalization factor):

$$\boxed{\psi(\boldsymbol{r}_\perp, \xi) = \xi^a (1-\xi)^b \, \mathrm{e}^{-\frac{1}{2}\xi(1-\xi)\kappa^2 r_\perp^2}}$$

  - You can get excited states by using recursive formulas for polynomial coefficients

- Also interesting to look at density in ξ:

$$f(\xi) \equiv \int d^2 r_\perp \frac{|\psi(\boldsymbol{r}_\perp,\xi)|^2}{4\pi\xi(1-\xi)} \qquad \int \frac{d\xi\, d^2 r_\perp}{4\pi\xi(1-\xi)} \left|\psi(\boldsymbol{r}_\perp,\xi)\right|^2 = 1$$

  - When the constituents are quarks or gluons, this is called a **parton distribution function**
  - A factor $\frac{1}{4\pi\xi(1-\xi)}$ is pulled from wave function normalization rule so $\int d\xi\, f(\xi) = 1$
- Applying to harmonic oscillator ground state (and ignoring normalization):

$$f(\xi) \sim \int_0^\infty dr_\perp r_\perp \xi^{2a-1}(1-\xi)^{2b-1} e^{-\xi(1-\xi)\kappa^2 r_\perp^2} \sim \xi^{2a-2}(1-\xi)^{2b-2}$$

- For intuition, consider $m_1 = m_2 \equiv m$ and look at small/large mass limits ...

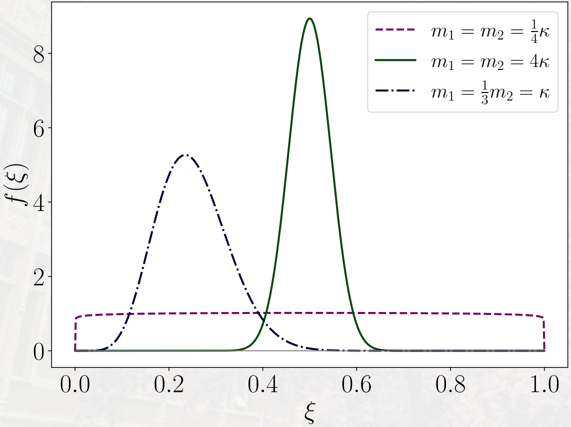
$$a = b = \frac{1}{2}\left(1 + \sqrt{1 + \frac{16m^4}{\kappa^4}}\right) \approx \begin{cases} 1 + \frac{4m^4}{\kappa^4} & : \quad m \ll \kappa \\ \frac{2m^2}{\kappa^2} & : \quad m \gg \kappa \end{cases}$$

  - For small masses, $f(\xi) \sim \left(\xi(1-\xi)\right)^{4m^4/\kappa^4}$
  - Distribution becomes flat (power $\to 0$) in massless limit
  - Super-massive limit gives very high powers

Harmonic oscillator — light front momentum density (continued)

- For more intuition, let's look at numerical examples
- Light front momentum density (with normalization restored):

$$f(\xi) = \frac{1}{\mathrm{B}(2a-1, 2b-1)} \xi^{2a-2}(1-\xi)^{2b-2}$$



- More mass $\Longrightarrow$ more concentrated density
  - Harder for one partner to have all the p^+ if they're both massive
- Mass imbalance $\Longrightarrow$ heavier partner likely to have more p^+
  - Recall $p^+ = \frac{E+p_z}{\sqrt{2}}$ and $p_{\text{rest}} = \frac{m}{\sqrt{2}}$
  - More mass $\Longrightarrow$ more p^+ at rest