A friendly introduction to the Light Front

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About these lectures

- Introductory/elementary lectures on the light front
  - Intended to be graduate-level
  - Apologies to anyone who is bored from this being too elementary
- Covers kinematics (light front coordinates) and dynamics (light front quantization)
  - More focus on kinematics & group theory—just from my personal interests
- Please interject with questions or requests for clarification

References

- Designed as lectures rather than a seminar
- Won't have citations on most slides, but the physics is a result of many researchers
- Some standard references:
  - P.A.M. Dirac,
    Forms of Relativistic Dynamics,
    Rev. Mod. Phys. 21 (1949) 392-399
  - D.E. Soper,
    Field Theories in the Infinite Momentum Frame,
    PhD thesis
  - S.J. Brodsky, H.-C. Pauli and S.S. Pinsky,
    Quantum chromodynamics and other field theories on the light cone,
    Phys. Rept. 301 (1998) 299-487
  - M. Burkardt,
    Impact parameter space interpretation for genrealized parton distributions,
    Int. J. Mod. Phys. A (2003), 173-208
- Apologies to anyone I left out; this list is not meant to be complete or comprehensive

Outline

1. **Basics**
   - Basic overview of light front coordinates
   - Conceptualization of the light front
2. **Group Theory**
   - Finite boosts and kinematics
   - The Galilei subgroup
3. **Quantum Mechanics**
   - One-particle wave functions
   - Separation of variables
4. **The harmonic oscillator** (example problem)
5. **Quantum Field Theory**
   - Light front quantization
   - Fock space

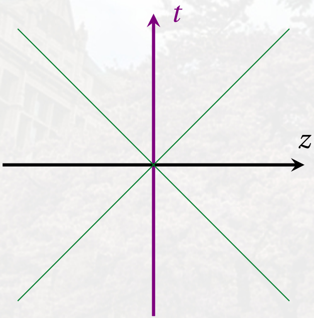
Basic Overview

Light front coordinates

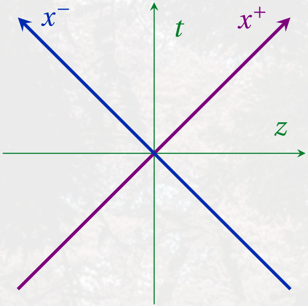
- **Light front coordinates** reparametrize spacetime.

$$x^{\pm} = \underbrace{\frac{1}{\sqrt{2}}}_{\text{conventional factor}}\big(t \pm z\big) \qquad \boldsymbol{x}_{\perp} = (x, y) \qquad x^{+} = \frac{1}{\sqrt{2}}\big(t + z\big) \equiv \text{time}$$

- **Fixed-time surface** given by light front moving in $-z$ direction.



Instant form coordinates



Light front coordinates

Coordinate conversions

- Contravariant (upper-index) four-vector:

$$(t; x, y, z) \to \left(x^0; x^1, x^2, x^3\right) \equiv x^\mu$$

- Conversion between instant form & light front coordinates:

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3) \qquad x^0 = \frac{1}{\sqrt{2}}(x^+ + x^-)$$

$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3) \qquad x^3 = \frac{1}{\sqrt{2}}(x^+ - x^-)$$

$$x^1 = x^1 \qquad x^1 = x^1$$

$$x^2 = x^2 \qquad x^2 = x^2$$

- Symmetric inversion formulas are why $\frac{1}{\sqrt{2}}$ is nice

Coordinate conversion matrix

- A matrix $\mathscr{C}$ easily converts between light front & instant form:

$$\mathscr{C} = \mathscr{C}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

- Can easily check:

$$\mathscr{C} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{bmatrix} \qquad \mathscr{C} \begin{bmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{bmatrix} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

- For any matrix M,

$$M_{\mathrm{LF}} = \mathscr{C} M_{\mathrm{IF}} \mathscr{C}$$

Four-products and the metric

- Rule for four-products:

$$\begin{aligned} x \cdot y &= x^0 y^0 - x^3 y^3 - x^1 y^1 - x^2 y^2 \\ &= \frac{1}{2}(x^+ + x^-)(y^+ + y^-) - \frac{1}{2}(x^+ - x^-)(y^+ - y^-) - x^1 y^1 - x^2 y^2 \\ &= \frac{1}{2}(x^+ y^+ + x^- y^- + x^+ y^- + x^- y^+) - \frac{1}{2}(x^+ y^+ + x^- y^- - x^+ y^- - x^- y^+) - x^1 y^1 - x^2 y^2 \\ &= x^+ y^- + x^- y^+ - \boldsymbol{x}_\perp \cdot \boldsymbol{y}_\perp \end{aligned}$$

- Defines the metric in light front coordinates:

$$x \cdot y = g_{\mu\nu} x^\mu x^\nu \qquad g_{\mu\nu}^{(\mathrm{LF})} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathscr{C} g_{\mu\nu}^{(\mathrm{IF})} \mathscr{C}$$

  - Metric is off-diagonal in x^+ and x^-!

Light front metric via $\mathscr{C}$-matrix

- Great example of how helpful the $\mathscr{C}$-matrix is!

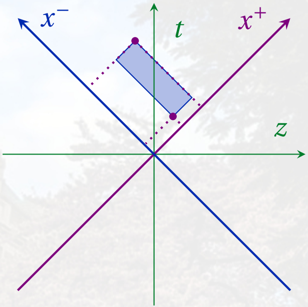
$$g^{(\mathrm{LF})}_{\mu\nu} = \mathscr{C}\, g^{(\mathrm{IF})}_{\mu\nu} \mathscr{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Area law for proper time

- For events in (t, z) or (x^+, x^-) plane, proper time given by an area law:



$$\Delta\tau^2 = 2\Delta x^+ \Delta x^-$$

- Twice the area of an enclosed rectable with sides along the light cone
- Just a neat curiosity

Raising and lowering indices

- Four-vector indices are raised and lowered as usual:

$$x_\mu = g_{\mu\nu} x^\nu \qquad x^\mu = g^{\mu\nu} x_\nu \equiv \left(g_{\mu\nu}\right)^{-1} x_\nu$$

- Metric and its inverse are identical:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

  - Another perk of the $\frac{1}{\sqrt{2}}$ convention

- Raising/lowering swaps + and −:

$$x_+ = x^- \qquad x_- = x^+$$

  - We verbally refer to components by upper-index form
  - "x-plus" means $x^+ = x_-$ ..."x-minus" means $x^- = x_+$

Derivatives and four-momenta

- Light front derivatives found via chain rule:

$$\frac{\partial}{\partial x^+} = \frac{\partial x^0}{\partial x^+}\frac{\partial}{\partial x^0} + \frac{\partial x^3}{\partial x^+}\frac{\partial}{\partial x^3} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^3}\right)$$
$$\frac{\partial}{\partial x^-} = \frac{\partial x^0}{\partial x^-}\frac{\partial}{\partial x^0} + \frac{\partial x^3}{\partial x^-}\frac{\partial}{\partial x^3} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^3}\right)$$
$$\implies \qquad \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_3)$$

- Four-momenta are spacetime translation generators—related to derivatives:

$$p_\mu = i\partial_\mu$$
$$p^- = p_+ = i\partial_+ = \text{energy} = \text{Hamiltonian}$$
$$p^+ = p_- = i\partial_- = \text{momentum component}$$

  - Energy is the time translation generator—singles out $p^- = p_+$ (usually called "p minus")

Light front energy formula

- Relativistic mass-shell condition:

$$m^2 = g_{\mu\nu} p^\mu p^\nu = 2 \underset{\text{momentum}}{\underset{\uparrow}{p^+}} \overset{\text{energy}}{p^-} - \boldsymbol{p}_\perp^2$$

- Formula for energy:

$$H = p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$$

  - Reminiscent of non-relativistic formula:

$$H_{\text{NR}} = \text{constant} + \frac{\boldsymbol{p}^2}{2m}$$

  - p^+ seems to behave like a mass
  - There are many more examples of this!

Light front momentum and velocity

- Velocities can be obtained via Hamilton's equations:

$$v^\mu = -\frac{\partial H}{\partial p_\mu} \qquad \begin{aligned} \boldsymbol{v}_\perp &= \frac{\partial H}{\partial \boldsymbol{p}_\perp} = \frac{\boldsymbol{p}_\perp}{p^+} \\ v^- &= -\frac{\partial H}{\partial p^+} = \frac{p^-}{p^+} \end{aligned} \qquad H = p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$$

  - Velocity is rate of motion *with respect to* x^+

- Momentum and velocity related by:

$$(\boldsymbol{p}_\perp, p^-) = p^+(\boldsymbol{v}_\perp, v^-)$$

  - Reminiscent of $\boldsymbol{p} = m\boldsymbol{v}$
  - Again, p^+ behaves kind of like a mass

Summary so far

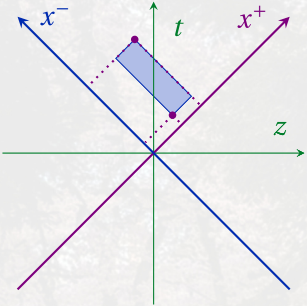
- Light front coordinates & conjugate momenta:

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3) \quad \text{time}$$
$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3)$$
$$\boldsymbol{x}_\perp = (x^1, x^2)$$

$$p_+ = p^- = \frac{1}{\sqrt{2}}(p^0 - p^3) \quad \text{energy}$$
$$p_- = p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3)$$
$$\boldsymbol{p}_\perp = (p^1, p^2)$$

- Invariant four-product & proper time:

$$x \cdot y = x^+ y^- + x^- y^+ - \boldsymbol{x}_\perp \cdot \boldsymbol{y}_\perp$$
$$\Delta x^2 = 2x^+ x^- - \boldsymbol{x}_\perp^2$$

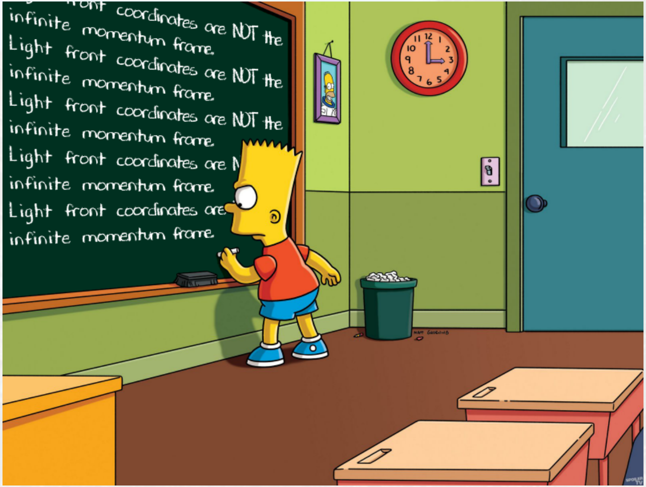


- p^+ acts somewhat like a non-relativistic mass:

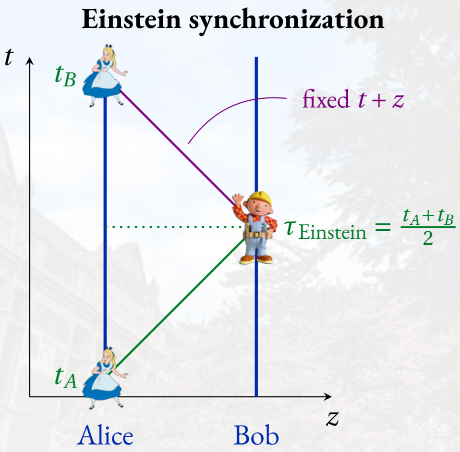
$$E = p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$$
$$(p^-, \boldsymbol{p}_\perp) = p^+(v^-, \boldsymbol{v}_\perp)$$

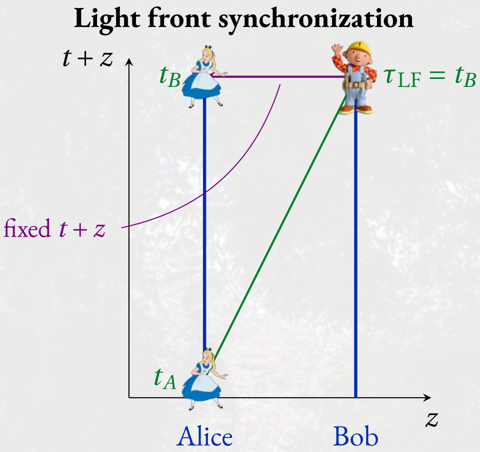

Conceptualization

Not the infinite momentum frame!



- Light front coordinates are valid in **any** frame.
  - They're not a reference frame.
- Light front coordinates are **not** the infinite-momentum frame.
  - A common misconception.
- Light front coordinates redefine **synchronization convention**.
  - What we mean by "simultaneous."

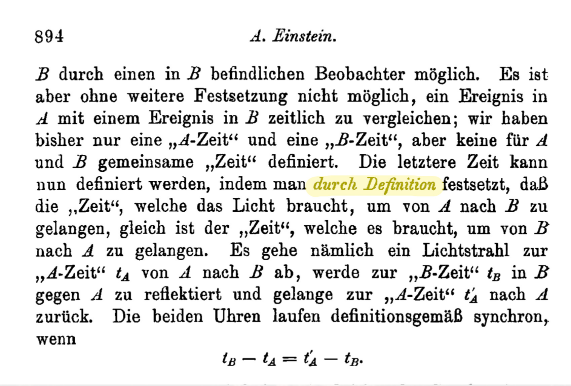




- **Einstein synchronization** defined to be isotropic.
- **Light front synchronization** defines hyperplanes with fixed $t+z$ to be "simultaneous."
  - Light travels instantaneously in $-z$ direction *by definition.*
  - We take what we see as literally happening now.

Equal-"time" surfaces are just a convention

- Relativity requires *round-trip* speed of light to be invariant.
- Convention that one-way speed of light be c is a *definition*, not an empirical fact.
  - Pointed out in Einstein's original paper.
- Redefining "time" coordinate means changing this definition.
  - Light front coordinates do exactly this!



894 *A. Einstein.*

B durch einen in *B* befindlichen Beobachter möglich. Es ist aber ohne weitere Festsetzung nicht möglich, ein Ereignis in *A* mit einem Ereignis in *B* zeitlich zu vergleichen; wir haben bisher nur eine „*A*-Zeit" und eine „*B*-Zeit", aber keine für *A* und *B* gemeinsame „Zeit" definiert. Die letztere Zeit kann nun definiert werden, indem man *durch Definition* festsetzt, daß die „Zeit", welche das Licht braucht, um von *A* nach *B* zu gelangen, gleich ist der „Zeit", welche es braucht, um von *B* nach *A* zu gelangen. Es gehe nämlich ein Lichtstrahl zur „*A*-Zeit" t_A von *A* nach *B* ab, werde zur „*B*-Zeit" t_B in *B* gegen *A* zu reflektiert und gelange zur „*A*-Zeit" t'_A nach *A* zurück. Die beiden Uhren laufen definitionsgemäß synchron, wenn

$$t_B - t_A = t'_A - t_B.$$

Einstein, Ann. Phys. 322 (1905) 891

- **Didactic overview**: Veritasium, "Why No One Has Measured The Speed of Light" (YouTube)
- **Technical review**: Anderson, Stedman & Vetharaniam, Phys. Rept. 295 (1998) 93

- Foliating spacetime (slicing it up) into fixed-x^+ surfaces is *invariant under z-direction boosts!*

$$\begin{aligned} t' &= \gamma t + \beta\gamma z \\ z' &= \gamma z + \beta\gamma t \end{aligned} \quad\Longrightarrow\quad \begin{aligned} x'^{+} &= (\gamma + \beta\gamma)x^{+} = \sqrt{\frac{1+\beta}{1-\beta}}\,x^{+} \\ x'^{-} &= (\gamma - \beta\gamma)x^{-} = \sqrt{\frac{1-\beta}{1+\beta}}\,x^{-} \end{aligned}$$

  - Foliation is invariant, but time is **redshifted** ($\beta > 0$) or **blueshifted** ($\beta < 0$).
  - Instant form always gives time dilation upon boosts
  - **Light front describes what you actually see**

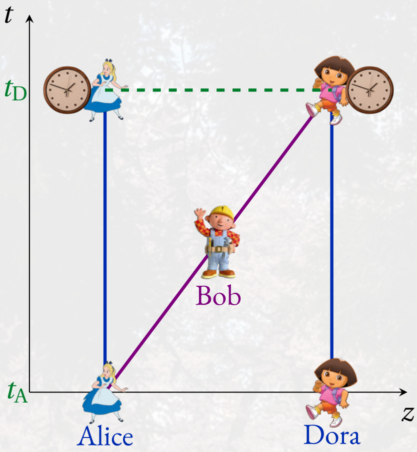
- Compact formulas if we use **rapidity**, η:

$$\begin{aligned} \gamma &= \cosh\eta \\ \beta\gamma &= \sinh\eta \end{aligned} \quad\Longrightarrow\quad \begin{aligned} x'^{+} &= e^{\eta}x^{+} \\ x'^{-} &= e^{-\eta}x^{-} \end{aligned}$$

- **Alice** and **Dora** synchronize their clocks by Einstein's convention.
  - The two of them together constitute a **reference frame**.
- **Bob** moves from Alice to Dora.
  - A time Δt_{B} passes on Bob's clock.
  - At start: Bob sees Alice's clock reads t_{A}.
  - At end: Bob sees Dora's clock reads t_{D}.
- **Time dilation** means:

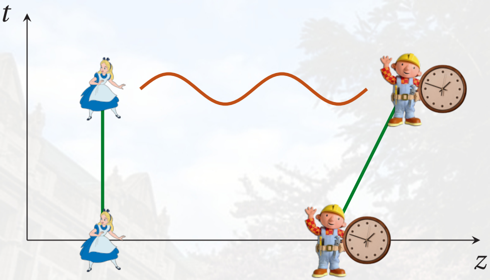
$$\Delta t_{\mathrm{B}} = \sqrt{1 - \frac{v^2}{c^2}}(t_{\mathrm{D}} - t_{\mathrm{A}}) < (t_{\mathrm{D}} - t_{\mathrm{A}})$$

  - Bob's clock slow compared to *reference frame*.
  - Result depends on Einstein synchronization.

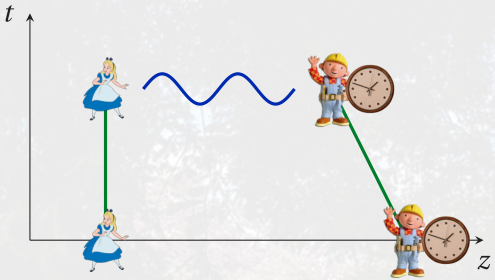


Redshift/blueshift as what's actually seen

- When Bob is boosted—Alice *sees* redshift or blueshift:



Redshift: Bob moves away, clock appears slower



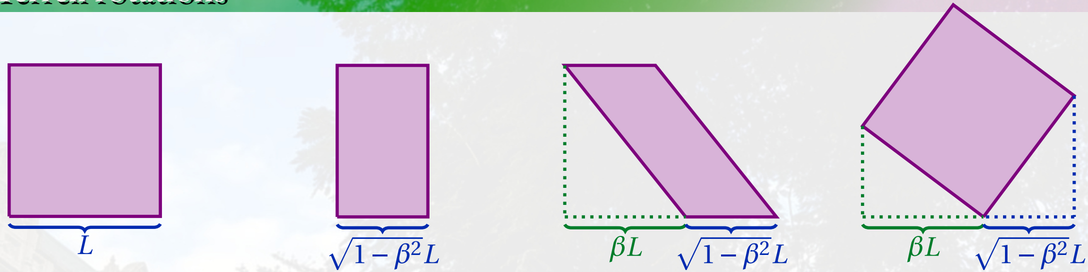
Blueshift: Bob moves towards, clock appears faster

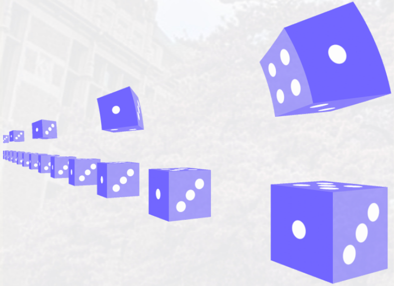
- Light front *automatically gives redshift/blueshift formula*:

$$x^+_{\mathrm{Bob}} = \sqrt{\frac{1+\beta}{1-\beta}}\, x^+_{\mathrm{Alice}} = e^{\eta} x^+_{\mathrm{Alice}}$$

  - Light front prioritizes what we actually see

Terrell rotations





- Lorentz-boosted objects *appear rotated*.
  - **Terrell rotation** (PR116, 1959)
  - Optical effect: contraction + delay
- Light front takes Terrell rotations seriously!
  - Instant form: contractions are real
  - Light front: rotations are real
  - Both equally valid viewpoints

Dice image: Ute Kraus, `https://www.spacetimetravel.org/`

Algebra of Terrell rotations

- Transverse boost—instant form:

$$t' = \gamma t + \beta\gamma y \approx t + \beta y + \mathcal{O}(\beta^2)$$
$$y' = \gamma y + \beta\gamma t \approx y + \beta t + \mathcal{O}(\beta^2)$$
$$x' = x$$
$$z' = z$$

- Transverse boost—light front:

$$\delta x^+ \approx \frac{\beta}{\sqrt{2}} y$$
$$\delta y \approx \frac{\beta}{\sqrt{2}} x^+ + \frac{\beta}{\sqrt{2}} x^-$$
$$\delta x^- \approx \frac{\beta}{\sqrt{2}} y$$

- Rotation around x axis—instant form:

$$t' = t$$
$$y' = \cos\theta\, y - \sin\theta\, z \approx y - \theta z + \mathcal{O}(\theta^2)$$
$$x' = x$$
$$z' = \cos\theta\, z + \sin\theta\, y \approx z + \theta y + \mathcal{O}(\theta^2)$$

- Rotation around x axis—light front:

$$\delta x^+ \approx \frac{\theta}{\sqrt{2}} y$$
$$\delta y \approx -\frac{\theta}{\sqrt{2}} x^+ + \frac{\theta}{\sqrt{2}} x^-$$
$$\delta x^- \approx -\frac{\theta}{\sqrt{2}} y$$

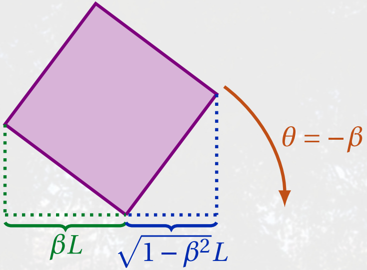
- What if we boost by β and rotate by $\theta = -\beta$?

Undoing Terrell rotations

- Combined boost + counterrotation ($\theta = -\beta$):

$$\delta x^+ = \frac{\beta+\theta}{\sqrt{2}} y = 0$$

$$\delta y = \frac{\beta-\theta}{\sqrt{2}} x^+ + \frac{\beta+\theta}{\sqrt{2}} x^- = \sqrt{2}\beta x^+$$



  - Combined transformation leaves x^+ invariant!
  - Object apparently moving in y direction
  - Apparent velocity is $\sqrt{2}\beta$
- Suggests new generator:

$$B_y = \frac{1}{\sqrt{2}}(K_y - J_x)$$

counterrotation

ordinary boost

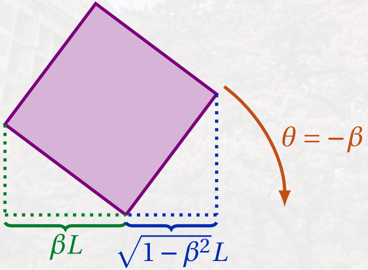
  - B_y generates a **light front transverse boost**
  - (contrasted with oridnary transverse boost)
  - Similar equation for generator of x-direction light front boost:

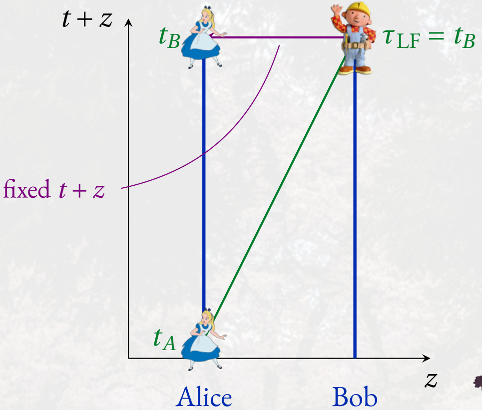
$$B_x = \frac{1}{\sqrt{2}}(K_x + J_y)$$

Summary so far

- Light front coordinates describe spacetime *as we see it*
  - Distant events (in $+z$ direction) are considered "present" instead of "past"
  - Longitudinal boosts produce *observed* redshifts/blueshifts instead of time dilation
  - Ordinary transverse boosts produce *observed* Terrell rotations instead of Fitzgerald contractions
- Can define **light front transverse boost** as transverse boost + counterrotation
  - Leaves transverse spatial structure *invariant*

$$B_y = \frac{1}{\sqrt{2}}(K_y - J_x)$$






Finite Boosts & Kinematics

Transverse boost generators: commutation rule

- Something amazing happens with the light front transverse boost generators:

$$B_x = \frac{1}{\sqrt{2}}(K_x + J_y) \qquad B_y = \frac{1}{\sqrt{2}}(K_y - J_x)$$

$$[B_x, B_y] = \frac{1}{2}\Big(\overbrace{[K_x, K_y]}^{=-iJ_z} - \underbrace{[J_y, J_x]}_{=-iJ_z} - \overbrace{[K_x, J_x]}^{=0} + \underbrace{[J_y, K_y]}_{=0}\Big) = 0$$

  - They commute!

- This is one manifestation of the **Galilei subgroup** of the Poincaré group
  - The classical-looking formulas we saw previously are other manifestations

- Let's look at *finite* boosts before fleshing out the full group

- **Generators** quantify infinitesimal transformations

$$\mathscr{B}_y(v) \approx 1 - iB_y v + \mathcal{O}(v^2)$$

finite boost generator

  - The factor $-i$ is a popular convention
  - I prefer to define generators without it in my own personal notes ...
  - ...but let's use standard conventions in these lectures.

- x-boost and y-rotation generators (instant form):

$$\mathscr{K}_x^{(\mathrm{IF})}(\eta) = \begin{bmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies K_x^{(\mathrm{IF})}(\eta) = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathscr{J}_y^{(\mathrm{IF})}(\eta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \implies J_y^{(\mathrm{IF})}(\eta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

- As usual, can use $\mathscr{C}$-matrix to convert to light front:

$$K_x^{(\mathrm{LF})} = \mathscr{C} K_x^{(\mathrm{IF})} \mathscr{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

- As usual, can use $\mathscr{C}$-matrix to convert to light front:

$$J_y^{(\mathrm{LF})} = \mathscr{C} J_y^{(\mathrm{IF})} \mathscr{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

Transverse boost generators and nilpotency

- Combining the traditional boost & counter-rotation...

$$K_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \qquad J_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$$

$$B_x = \frac{1}{\sqrt{2}}(K_x + J_y) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \qquad B_x^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$B_x^3 = 0, \quad B_x^4 = 0, \quad B_x^5 = 0, \quad \text{etc.}$$

- Light front boost generators are **nilpotent** for transverse boosts!
  - Nilpotent means there's a power of B_x that gives zero.

Finite transverse boosts

$$B_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \qquad B_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{bmatrix} \qquad B_x^2 = B_y^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

- Finite boosts built from infinitesimal boosts by exponentiation:

$$\mathscr{B}_\perp(\boldsymbol{v}_\perp) = \lim_{N\to\infty}\left(1 - \frac{i}{N}\boldsymbol{v}_\perp \cdot B_\perp\right)^N = e^{-i\boldsymbol{v}_\perp\cdot\boldsymbol{B}_\perp} = \underbrace{1 - i\boldsymbol{v}_\perp \cdot \boldsymbol{B}_\perp - \frac{1}{2}\boldsymbol{v}_\perp^2 B_x^2}_{\textbf{exact}\text{ because of nilpotency}}$$

  - Basically, just continually compound infinitesimal boosts
  - Note also that $B_x B_y = B_y B_x = 0$

- Transforms a four-vector as:

leaves time invariant!

$$\begin{pmatrix} x^+ \\ \boldsymbol{x}_\perp \\ x^- \end{pmatrix} \to \begin{pmatrix} x^+ \\ \boldsymbol{x}_\perp + \boldsymbol{v}_\perp x^+ \\ x^- + \boldsymbol{v}_\perp \cdot \boldsymbol{x}_\perp + \frac{1}{2}\boldsymbol{v}_\perp^2 x^+ \end{pmatrix}$$

Boosting particle from rest

- Four-momentum at rest:

$$p^{(\mathrm{LF})}_{\mathrm{rest}} = \mathscr{C}\, p^{(\mathrm{IF})}_{\mathrm{rest}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{m}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Can reach any momentum by longitudinal then transverse boost:

$$p = \mathscr{B}_{\perp}(\boldsymbol{v}_{\perp})\mathscr{B}_{\parallel}(\eta)\, p_{\mathrm{rest}}$$

  - Longitudinal boost by rapidity η
  - Transverse boost by velocity $\boldsymbol{v}_{\perp}$

The longitudinal boost generator

- Use $\mathscr{C}$-matrix to convert to light front:

$$K_z^{(\mathrm{LF})} = \mathscr{C} K_z^{(\mathrm{IF})} \mathscr{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

Finite longitudinal boosts

- Finite boost from exponentiation:

$$K_z^{(\mathrm{LF})} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \implies \mathcal{K}_z^{(\mathrm{LF})} = e^{-iyK_z} = \begin{bmatrix} \mathrm{e}^{\eta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\eta} \end{bmatrix}$$

- Boosting a particle from rest:

$$\mathcal{K}_z^{(\mathrm{LF})}(\eta)\, p_{\mathrm{rest}}^{(\mathrm{LF})} = \frac{m}{\sqrt{2}} \begin{bmatrix} \mathrm{e}^{\eta} \\ 0 \\ 0 \\ \mathrm{e}^{-\eta} \end{bmatrix}$$

  - Nice scaling behavior for p^+ and p^-
- Formula for longitudinal rapidity:

$$\eta = \log\left(\frac{\sqrt{2}p^+}{m}\right)$$

  - Will remain invariant under transverse light front boosts! (The ones with Terrell counter-rotations.)

Transverse boosts of four-momenta

- Following up with a transverse boost...

$$\mathscr{B}_\perp(\boldsymbol{v}_\perp) = 1 - i\,\boldsymbol{v}_\perp \cdot \boldsymbol{B}_\perp - \frac{1}{2}\boldsymbol{v}_\perp^2 B_x^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_x & 1 & 0 & 0 \\ v_y & 0 & 1 & 0 \\ \frac{1}{2}\boldsymbol{v}_\perp^2 & v_x & v_y & 1 \end{bmatrix} \qquad \mathscr{K}_z^{(\mathrm{LF})}(\eta)\, p_{\mathrm{rest}}^{(\mathrm{LF})} = \frac{m}{\sqrt{2}}\begin{bmatrix} \mathrm{e}^{\eta} \\ 0 \\ 0 \\ \mathrm{e}^{-\eta} \end{bmatrix}$$

- An arbitrary four-momentum can be expressed:

$$\begin{bmatrix} p^+ \\ p^x \\ p^y \\ p^- \end{bmatrix} = \mathscr{B}_\perp(\boldsymbol{v}_\perp)\mathscr{K}_z(\eta)\, p_{\mathrm{rest}} = \frac{m}{\sqrt{2}}\begin{bmatrix} \mathrm{e}^{\eta} \\ v_x\,\mathrm{e}^{\eta} \\ v_y\,\mathrm{e}^{\eta} \\ \mathrm{e}^{-\eta} + \frac{1}{2}\boldsymbol{v}_\perp^2\,\mathrm{e}^{\eta} \end{bmatrix} = \begin{bmatrix} p^+ \\ p^+ v_x \\ p^+ v_y \\ \frac{m^2}{2p^+} + \frac{1}{2}p^+\boldsymbol{v}_\perp^2 \end{bmatrix}$$

  - Again, a classical-looking expression!
  - Transverse velocity: $\boldsymbol{v}_\perp = \boldsymbol{v}_\perp$
  - Transverse momentum: $\boldsymbol{p}_\perp = p^+\boldsymbol{v}_\perp$
  - Kinetic energy term: $\frac{1}{2}p^+\boldsymbol{v}_\perp^2$

Summary so far

- Finite boosts on the light front take simple forms:

$$\mathcal{K}_z^{(\mathrm{LF})} = \begin{bmatrix} e^{\eta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\eta} \end{bmatrix} \qquad \mathcal{B}_\perp(\boldsymbol{v}_\perp) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_x & 1 & 0 & 0 \\ v_y & 0 & 1 & 0 \\ \frac{1}{2}\boldsymbol{v}_\perp^2 & v_x & v_y & 1 \end{bmatrix}$$

- The transverse boost is different from the "ordinary" transverse boost
  - Consists of an ordinary boost + counter-rotation, to cancel out Terrell rotation
  - The generator is nilpotent—series from exponentiation terminates!
- Momenta can be written in a very classical-looking form, with p^+ acting like a mass:

$$\begin{bmatrix} p^+ \\ p^x \\ p^y \\ p^- \end{bmatrix} = \begin{bmatrix} p^+ \\ p^+ v_x \\ p^+ v_y \\ \frac{m^2}{2p^+} + \frac{1}{2}p^+\boldsymbol{v}_\perp^2 \end{bmatrix}$$


The Galilei Subgroup

Galilei subgroup

- Poincaré group has a $(2+1)$D **Galilei subgroup**.
  - x^+ is time and $\boldsymbol{x}_\perp$ is space under this subgroup
  - $p^+ = \frac{1}{\sqrt{2}}(E_{\boldsymbol{p}} + p_z)$ is the central charge—acts like a mass, commutes with rest of subgroup
  - x^+ and p^+ are invariant under this subgroup!
- Light front time gives **fully relativistic** 2D picture that looks a lot like non-relativistic physics.

$$\frac{\mathrm{d}\boldsymbol{p}_\perp}{\mathrm{d}x^+} = p^+ \frac{\mathrm{d}^2\boldsymbol{x}_\perp}{\mathrm{d}x^{+2}}$$

$$H = H_{\text{rest}} + \frac{\boldsymbol{p}_\perp^2}{2p^+} = H_{\text{rest}} + \frac{1}{2}p^+\boldsymbol{v}_\perp^2$$

$$\boldsymbol{p}_\perp = p^+\boldsymbol{v}_\perp$$

$$[B_x, B_y] = 0$$

Lorentz algebra: instant form

- Lorentz group has 6 generators $M^{\mu\nu} = -M^{\nu\mu}$

3 boosts

$$K_i = M^{0i}$$

3 rotations

$$(J_1, J_2, J_3) = (M^{23}, M^{31}, M^{12})$$

- Algebra defined by commutation rule:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

  - Can be derived via geometric algebra, or by requiring it give the explicit forms below.

- More explicit forms:

$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$
$$[J_i, J_j] = i\epsilon_{ijk}J_k$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k$$

- Let's prove the boost commutation formula!

3 boosts

$$K_i = M^{0i}$$

3 rotations

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \implies M^{ij} = \epsilon_{ijk}J_k$$

- Fundamental commutation rule:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

- Working it out...

$$\begin{aligned}\left[K_i, K_j\right] &= \left[M^{0i}, M^{0j}\right] \\ &= i\big(\underbrace{g^{0j}}_{=0}M^{i0} + \underbrace{g^{i0}}_{=0}M^{0j} - g^{00}M^{ij} - g^{ij}\underbrace{M^{00}}_{=0}\big) = -iM^{ij} = -i\epsilon_{ijk}J_k\end{aligned}$$

- Let's prove the rotation commutation formula!

3 boosts	**3 rotations**
$K_i = M^{0i}$	$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \implies M^{ij} = \epsilon_{ijk}J_k$

- Fundamental commutation rule:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

- Working it out...

$$\begin{aligned}
[J_i, J_j] &= \tfrac{1}{4}\epsilon_{iab}\epsilon_{jcd}[M^{ab}, M^{cd}] \\
&= \tfrac{i}{4}\epsilon_{iab}\epsilon_{jcd}\left(g^{ad}M^{bc} + g^{bc}M^{ad} - g^{ac}M^{bd} - g^{bd}M^{ac}\right) \\
\text{(distribute Levi-Civita)} &= -\tfrac{i}{4}\Big(\underbrace{\epsilon_{ieb}\epsilon_{jce}}_{-(\delta_{ij}\delta_{bc}-\delta_{ic}\delta_{jb})}M^{bc} + \overbrace{\epsilon_{iae}\epsilon_{jed}}^{-(\delta_{ij}\delta_{ad}-\delta_{id}\delta_{ja})}M^{ad} - \underbrace{\epsilon_{ieb}\epsilon_{jed}}_{\delta_{ij}\delta_{bd}-\delta_{id}\delta_{jb}}M^{bd} - \overbrace{\epsilon_{iae}\epsilon_{jce}}^{\delta_{ij}\delta_{ac}-\delta_{ic}\delta_{ja}}M^{ac}\Big) \\
(\text{drop } M^{ii}) &= -\tfrac{i}{4}\left(M^{ji} + M^{ji} - (-M^{ji}) - (-M^{ji})\right) = iM^{ij} = i\epsilon_{ijk}J_k
\end{aligned}$$

- Let's prove the last commutation formula!

3 boosts

$$K_i = M^{0i}$$

3 rotations

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \implies M^{ij} = \epsilon_{ijk}J_k$$

- Fundamental commutation rule:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

- Working it out...

$$\begin{aligned}
[J_i, K_j] &= \frac{1}{2}\epsilon_{iab}[M^{ab}, M^{0j}] \\
&= \frac{i}{2}\epsilon_{iab}\left(g^{aj}M^{b0} + \underbrace{g^{b0}}_{=0}M^{aj} - \underbrace{g^{a0}}_{=0}M^{bj} - g^{bj}M^{a0}\right) \\
&= -\frac{i}{2}\left(\epsilon_{ijb}M^{b0} - \epsilon_{iaj}M^{a0}\right) = i\epsilon_{ijk}M^{0k} = i\epsilon_{ijk}K_k
\end{aligned}$$

Lorentz algebra: light front

- Lorentz group has 6 generators $M^{\mu\nu} = -M^{\nu\mu}$

3 boosts

$$(B_1, B_2, B_-) = (M^{+1}, M^{+2}, M^{+-})$$

3 rotations

$$(R_1, R_2, R_-) = (M^{2-}, M^{-1}, M^{12})$$

  - x^+ is time and x^- is space!

- Algebra defined by commutation rule:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}\right)$$

- More explicit forms ($i, j \in \{1,2\}$):

$$[B_i, B_j] = 0$$

$$[R_-, R_i] = i\epsilon_{3ij}R_j$$

$$[R_-, B_i] = i\epsilon_{3ij}B_j$$

$$[R_-, B_-] = 0$$

$$[B_-, B_i] = iB_i$$

$$[R_1, R_2] = 0$$

$$[B_-, R_i] = -iR_i$$

$$[R_i, B_j] = -i\delta_{ij}R_-$$

$\boldsymbol{R}_\perp$ & $\boldsymbol{B}_\perp$ transform like 2D vectors!

Lorentz algebra: light front, boost commutation rules

- Let's prove the boost formulas!

3 boosts

$$(B_1, B_2, B_-) = (M^{+1}, M^{+2}, M^{+-})$$

3 rotations

$$(R_1, R_2, R_-) = (M^{2-}, M^{-1}, M^{12})$$

- Fundamental commutation rule:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)$$

- Working it out, & allowing $i, j = -$ ...

$$[B_i, B_j] = [M^{+i}, M^{+j}]$$

$$= i\Big(\underbrace{g^{+j}}_{=\delta_{j-}} M^{i+} + \underbrace{g^{i+}}_{=\delta_{i-}} M^{+j} - \underbrace{g^{++}}_{=0} M^{ij} - g^{ij} \underbrace{M^{++}}_{=0}\Big) = \begin{cases} 0 & : \quad i, j \neq - \text{ or } i = j = - \\ iB_j & : \quad i = -, j \neq - \\ -iB_i & : \quad j = -, i \neq - \end{cases}$$

- Also can be written for $i, j \in \{1, 2\}$:

$$[B_i, B_j] = 0 \qquad [B_-, B_j] = iB_j$$

- Let's prove the rotation commutation rules!

3 boosts

$$(B_1, B_2, B_-) = (M^{+1}, M^{+2}, M^{+-})$$

3 rotations

$$(R_1, R_2, R_-) = (M^{2-}, M^{-1}, M^{12})$$

- Fundamental commutation rule:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)$$

- Easier to work out components explicitly:

$$[R_1, R_2] = [M^{2-}, M^{-1}] = i\left(g^{21} \underbrace{M^{--}}_{=0} + \underbrace{g^{--}}_{=0} M^{21} - \underbrace{g^{2-}}_{=0} M^{-1} - \underbrace{g^{-1}}_{=0} M^{2-}\right) = 0$$

$$[R_-, R_1] = [M^{12}, M^{2-}] = i\left(\underbrace{g^{1-}}_{=0} M^{22} + g^{22} M^{1-} - \underbrace{g^{12}}_{=0} M^{2-} - \underbrace{g^{2-}}_{=0} M^{12}\right) = iM^{-1} = iR_2$$

$$[R_-, R_2] = [M^{12}, M^{-1}] = i\left(g^{11} M^{2-} + \underbrace{g^{2-}}_{=0} M^{11} - \underbrace{g^{1-}}_{=0} M^{21} - \underbrace{g^{21}}_{=0} M^{1-}\right) = -iM^{2-} = -iR_1$$

Lorentz algebra: light front, Galilean rotation

- Let's prove remaining rules involving R_-!

3 boosts

$$(B_1, B_2, B_-) = (M^{+1}, M^{+2}, M^{+-})$$

3 rotations

$$(R_1, R_2, R_-) = (M^{2-}, M^{-1}, M^{12})$$

- Fundamental commutation rule:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)$$

- Working it out, & allowing $i = -$ ...

$$\begin{aligned}
\left[R_-, B_i\right] &= \left[M^{12}, M^{+i}\right] \\
&= i\left(g^{1i} M^{2+} + \underbrace{g^{2+}}_{=0} M^{1i} - \underbrace{g^{1+}}_{=0} M^{2i} - g^{2i} M^{1+}\right) \\
&= -i\left(\delta_{1i} M^{2+} - \delta_{2i} M^{1+}\right) = \begin{cases} iB_2 & : \quad i = 1 \\ -iB_1 & : \quad i = 2 \\ 0 & : \quad i = + \end{cases}
\end{aligned}$$

- I leave the other derivations as an exercise for you!

Instant form

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$
$$[J_i, J_j] = i\epsilon_{ijk} J_k$$
$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

Part of Galilei subgroup!

Light front

$$[B_i, B_j] = 0$$
$$[R_-, B_i] = i\epsilon_{3ij} B_j$$
$$[R_-, R_i] = i\epsilon_{3ij} R_j$$
$$[B_-, B_i] = iB_i$$
$$[B_-, R_i] = -iR_i$$
$$[R_i, B_j] = -i\delta_{ij} R_-$$
$$[R_i, R_j] = 0$$

- Can derive relationships just from index manipulation, e.g.,

$$B_i = M^{+i} = \frac{1}{\sqrt{2}}\left(M^{0i} + M^{3i}\right) = \frac{1}{\sqrt{2}}\left(K_i + \epsilon_{3ij} J_j\right) \implies \begin{cases} B_x = \frac{1}{\sqrt{2}}\left(K_x + J_y\right) \\ B_y = \frac{1}{\sqrt{2}}\left(K_y - J_x\right) \end{cases}$$

  - *Exact same* result we got from conceptual arguments (Terrell rotation + counterrotation)!
- Must consider translations for rest of Galilei group

Poincaré algebra: instant form

- 10 generators: 6 Lorentz transformations $M^{\mu\nu} = -M^{\nu\mu}$ + 4 translations P^{μ}
- Algebra given by commutation rules:

$$\begin{aligned}
\left[M^{\mu\nu}, M^{\rho\sigma}\right] &= i\left(g^{\mu\sigma} M^{\nu\rho} + g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho}\right)\\
\left[M^{\mu\nu}, P^{\rho}\right] &= i\left(g^{\nu\rho} P^{\mu} - g^{\mu\rho} P^{\nu}\right)\\
\left[P^{\mu}, P^{\nu}\right] &= 0
\end{aligned}$$

- More explicit rules (exercise for you to show!):

$$\begin{aligned}
[J_i, P^j] &= i\epsilon_{ijk} P^k\\
[K_i, P^j] &= -i\delta_{ij} P^0\\
[J_i, P^0] &= 0\\
[K_i, P^0] &= -i P^i
\end{aligned}$$

Poincaré algebra: light front

- Full Poincaré group in light front coordinates (exercise for you to show!):

$$[R_i, P^j] = -i\epsilon_{3ij} P^-$$
$$[R_i, P^+] = i\epsilon_{3ij} P^j$$
$$[R_i, P^-] = 0$$
$$[B_-, P^j] = 0$$
$$[B_-, P^+] = iP^+$$
$$[B_-, P^-] = -iP^-$$
$$[B_-, B_i] = iB_i$$
$$[R_i, B_-] = 0$$
$$[R_i, B_j] = -i\delta_{ij} R_-$$
$$[R_i, R_j] = 0$$

$$[R_-, P^i] = i\epsilon_{3ij} P^j$$
$$[R_-, B_i] = i\epsilon_{3ij} B_j$$
$$[R_-, P^-] = 0$$
$$[B_i, P^j] = -i\delta_{ij} P^+$$
$$[B_i, B_j] = 0$$
$$[B_i, P^-] = -iP^i$$
$$[P^i, P^-] = 0$$
$$[R_-, P^+] = 0$$
$$[B_i, P^+] = 0$$
$$[P^i, P^+] = 0$$
$$[P^-, P^+] = 0$$

The Galilei subgroup

Galilei subgroup: what's in and out

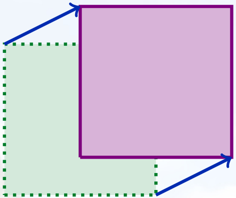
- What's in the Galilei subgroup?
  - $\boldsymbol{B}_\perp$ — transverse light front boosts, which commute
  - R^- — Rotation around z axis (aka, rotation around x^- axis with x^+ = fixed)
  - $\boldsymbol{P}_\perp$ — translations in transverse plane
  - P^- — time evolution (Hamiltonian)
  - P^+ — **central charge**; commutes with everything else, kind of like a mass

...that's **7 out of 10** generators!

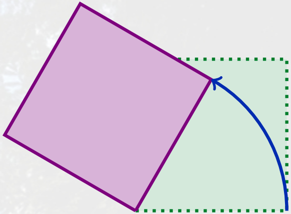
- What's *not* in the subgroup?
  - B_- — longitudinal boosts (they change P^+, so fail to commute with it)
  - $\boldsymbol{R}_\perp$ — rotations around x and y axes (they reorient the light front, so change P^+)

...basically, anything that fails to commute with P^+

Galilei group in 2D non-relativistic physics



2 translations — $\boldsymbol{P}$



1 rotation — R

2 boosts — $\boldsymbol{B}$

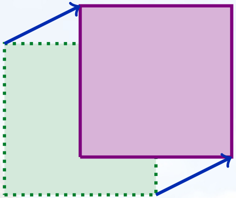
1 Hamiltonian — H

1 mass — M

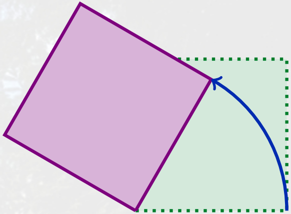
- 7 total generators
- All non-zero commutators:

$$[R, P_i] = i\epsilon_{ij3}P_j \qquad [R, B_i] = i\epsilon_{ij3}B_j \qquad [B_i, P_j] = -i\delta_{ij}M \qquad [H, B_i] = iP_i$$

Galilei group in 2D light front physics



2 translations — $\boldsymbol{P}_\perp$



1 rotation — R_-

2 boosts — $\boldsymbol{B}_\perp$

1 Hamiltonian — P^-

1 longitudinal momentum — P^+

- 7 total generators
- All non-zero commutators:

$$[R_-, P_i] = i\epsilon_{ij3}P_j \qquad [R_-, B_i] = i\epsilon_{ij3}B_j \qquad [B_i, P_j] = -i\delta_{ij}P^+ \qquad [P^-, B_i] = iP_i$$

Transverse position operator

- All non-zero commutators:

$$[R_-, P_i] = i\epsilon_{ij3}P_j \qquad [R_-, B_i] = i\epsilon_{ij3}B_j \qquad [B_i, P_j] = -i\delta_{ij}P^+ \qquad [P^-, B_i] = iP_i$$

- Position operator should obey:

$$[X_\perp^i, X_\perp^j] = 0 \qquad [X_\perp^i, P_\perp^j] = i\delta^{ij} \qquad i[P^-, X_\perp^i] = V_\perp^i$$

- Actually, the following works!:

$$X_\perp^i = -\frac{B_i}{P^+}$$

  - Transverse boosts commute
  - P^+ commutes with entire subgroup
  - $P_\perp^i = P^+ V_\perp^i$
- Exercise for you: prove the purple commutation relations

Transverse position operator and boosts

- Transverse position operator:

$$X_\perp^i = -\frac{B_i}{P^+}$$

- Recall:

$$[B_i, B_j] = 0 \qquad [B_i, P^+] = 0 \qquad [B_-, B_i] = iB_i \qquad [B_-, P^+] = iP^+$$

- Simply enough:

$$[B_i, X^j] = \overbrace{\left[B_i, -(P^+)^{-1}B_j\right]}^{[A,BC]=B[A,C]+[A,B]C} = -(P^+)^{-1}\underbrace{[B_i, B_j]}_{=0} - \underbrace{\left[B_i, (P^+)^{-1}\right]}_{=0}B_j = 0$$

- With a bit more algebra:

$$[B_-, X^i] = \overbrace{\left[B_-, -(P^+)^{-1}B_i\right]}^{[A,BC]=B[A,C]+[A,B]C} = -(P^+)^{-1}\underbrace{[B_-, B_i]}_{=iB_i} - \overbrace{\left[B_-, (P^+)^{-1}\right]}^{[A,B^{-1}]=-B^{-1}[A,B]B^{-1}}B_i$$

$$= -i(P^+)^{-1}B_i + (P^+)^{-1}\underbrace{[B_-, P^+]}_{=iP^+}(P^+)^{-1}B_i = 0$$

Invariance of transverse position operator

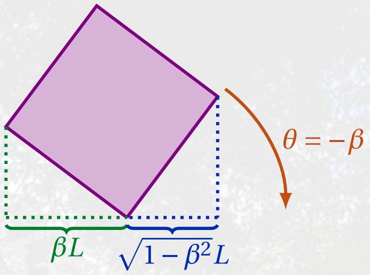
- $X_\perp^i$ commutes with transverse *and* longitudinal boosts

$$[B_i, X_\perp^j] = [B_-, X_\perp^j] = 0$$

- Can reach any momentum from rest by a sequence of boosts:

$$p = \mathscr{B}_\perp(\boldsymbol{v}_\perp)\underbrace{\mathscr{K}_z(\eta)}_{=\mathscr{B}_-(-\eta)}p_{\text{rest}}$$

$=\mathscr{B}_-(-\eta)$ because $B_- = M^{+-} = -M^{03} = K_z$ ... artifact of $x^- \sim x^0 - x^3$



- Light front provides a transverse spatial picture that ...
  - ... is quantum-mechanical (canonical commutation relations)
  - ... applies to relativistic systems
  - ... is invariant under motion
- Ultimately a result of Terrell counter-rotations (conceptually), or of the Galilei subgroup (formally)

Summary so far

- The Poincaré group has a Galilei subgroup
  - Its algebra is identical to 2D non-relativistic physics
  - Formally explains all our classical-looking formulas
- Light front time is **invariant under this subgroup**
- Transverse separations are **invariant under this subgroup**
- This will allow a spatio-temporal description of *internal properties of hadrons*


One-particle Wave Functions

One-particle Schödinger equation

- Recall the on-shell relation:

$$p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$$

- First quantization—convert to operators:

$$[X^-, P^+] = -i \qquad [X_\perp^i, P_\perp^j] = i\delta^{ij} \qquad [X^-, P_\perp^j] = [X_\perp^j, P^+] = 0$$

  - Minus sign in null coordinate—because no minus when raising/lowering index
  - This is *not* light front quantization—that'll come in a later lecture

- Momenta become derivatives in coordinate representation:

$$\boldsymbol{p}_\perp \to -i\boldsymbol{\nabla}_\perp \qquad p^- \to i\partial_+ \qquad p^+ \to i\partial_-$$

- Get a light front Schödinger equation:

$$-2\partial_+\partial_-\psi(x) = m^2\psi(x) - \boldsymbol{\nabla}_\perp^2\psi(x)$$

Longitudinal position operator

- Can define a longitudinal position operator:

$$X^- \equiv -\frac{1}{2}\left(\frac{1}{P^+}B_- + B_-\frac{1}{P^+}\right)$$

  - Defined in analogy to $X_\perp^j = -\frac{B_j}{P^+}$
  - Need average between orders make Hermitian—since $[B_-, P^+] = iP^+ \neq 0$
- Helpfully,

$$\left[P^-, (P^+)^{-1}B_i\right] = \left[P^-, B_i(P^+)^{-1}\right] = i\frac{P^-}{P^+} \qquad \Longrightarrow \qquad i\left[P^-, X^-\right] = \frac{P^-}{P^+} = V^-$$

  - Using $[B_-, P^-] = -iP^-$ from slide 47
  - Recall NR-like relation $(\boldsymbol{p}_\perp, p^-) = p^+(\boldsymbol{v}_\perp, v^-)$
- Also helpfully,

$$\left[X^-, X_\perp^i\right] = 0$$

  - Let's work it out (next slide) ...

Longitudinal position operator (continued)

- Helpful to recall:

$$X^- = -\frac{1}{2}\left(\frac{1}{P^+}B_- + B_-\frac{1}{P^+}\right) \qquad X_\perp^i = -\frac{B_i}{P^+} \qquad \underbrace{[B_-, B_i] = iB_i \qquad [B_-, P^+] = iP^+}_{\text{slide 47}}$$

- Let's work out ...

$$[X^-, X_\perp^i] = \left[-\tfrac{1}{2}\left(\tfrac{1}{P^+}B_- + B_-\tfrac{1}{P^+}\right), -\tfrac{1}{P^+}B_i\right] = \tfrac{1}{2}\left(\tfrac{1}{P^+}\left[B_-, \tfrac{1}{P^+}B_i\right] + \left[B_-, \tfrac{1}{P^+}B_i\right]\tfrac{1}{P^+}\right)$$

  - Working out the common commutator ...

$$\overbrace{\left[B_-, \tfrac{1}{P^+}B_i\right]}^{[A,BC]=B[A,C]+[A,B]C} = \tfrac{1}{P^+}\underbrace{[B_-, B_i]}_{=iB_i} + \overbrace{\left[B_-, \tfrac{1}{P^+}\right]}^{[A,B^{-1}]=-B^{-1}[A,B]B^{-1}}B_i = i\tfrac{1}{P^+}B_i - \tfrac{1}{P^+}\underbrace{[B_-, P^+]}_{=iP^+}\tfrac{1}{P^+}B_i = i\tfrac{1}{P^+}B_i - i\tfrac{1}{P^+}B_i = 0$$

  - So the whole thing equals zero.

- Much easier to show that

$$[X^-, P^+] = -i$$

  - I leave proof as an exercise

Coordinate vs. momentum representations

- For transverse components, we can say:

Operators		Coordinate representation		Momentum representation	
$\boldsymbol{X}_\perp$,	$\boldsymbol{P}_\perp$	$\boldsymbol{x}_\perp$,	$-i\nabla_\perp^{(x)}$	$i\nabla_\perp^{(p)}$,	$\boldsymbol{p}_\perp$

  - Similar to all three components in non-relativistic QM!

- This doesn't work for longitudinal components—instead:

Operators		Coordinate representation		Momentum representation	
X^-,	P^+	x^-,	$i\partial_-$	$-i\dfrac{\partial}{\partial p^+} + f(p^+)$,	p^+

  - Can show this and determine $f(p^+)$ using the momentum-space integration element

Momentum-space normalization

- Lorentz-invariant one-particle completeness relation:

$$1 = \overbrace{\int \frac{\mathrm{d}^4 p}{(2\pi)^4}}^{\text{always invariant}} \underbrace{(2\pi)\delta^{(1)}(p^2 - m^2)}_{\text{invariant on-shell condition}} |p^+, \boldsymbol{p}_\perp\rangle\langle p^+, \boldsymbol{p}_\perp|$$

  - Examining the delta further:

$$\delta^{(1)}(p^2 - m^2) = \delta^{(1)}(2p^+p^- - \boldsymbol{p}_\perp^2 - m^2) = \frac{1}{2p^+}\delta^{(1)}\left(p^- - \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}\right)$$

  - Allows the p^- integral to be eliminated:

$$1 = \underbrace{\int \frac{\mathrm{d}p^+ \, \mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}}_{\text{invariant momentum-space integration element}} |p^+, \boldsymbol{p}_\perp\rangle\langle p^+, \boldsymbol{p}_\perp|$$

- Inner products in momentum space:

$$\langle \phi | \psi \rangle = \int \frac{\mathrm{d}p^+ \, \mathrm{d}^2 p_\perp}{2p^+(2\pi)^3} \overbrace{\langle \phi | p^+, \boldsymbol{p}_\perp \rangle}^{\phi^*(p^+, \boldsymbol{p}_\perp)} \overbrace{\langle p^+, \boldsymbol{p}_\perp | \psi \rangle}^{\psi(p^+, \boldsymbol{p}_\perp)}$$

- The longitudinal position is Hermitian, so ...

$$\langle\phi|X^-|\psi\rangle = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\langle\phi|p^+,\boldsymbol{p}_\perp\rangle\langle p^+,\boldsymbol{p}_\perp|X^-|\psi\rangle = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\langle\phi|(X^-)^\dagger|p^+,\boldsymbol{p}_\perp\rangle\langle p^+,\boldsymbol{p}_\perp|\psi\rangle$$

- Then, using:

$$\langle p^+,\boldsymbol{p}_\perp|X^-|\psi\rangle = -i\frac{\partial\psi(p^+,\boldsymbol{p}_\perp)}{\partial p^+} + f(p^+)\psi(p^+,\boldsymbol{p}_\perp)$$

$$\langle\phi|(X^-)^\dagger|p^+,\boldsymbol{p}_\perp\rangle = i\frac{\partial\phi^*(p^+,\boldsymbol{p}_\perp)}{\partial p^+} + f^*(p^+)\phi^*(p^+,\boldsymbol{p}_\perp)$$

- We find:

$$\int\frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\phi^*\left(-i\frac{\partial\psi}{\partial p^+} + f(p^+)\psi\right) = \int\frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\left(i\frac{\partial\phi^*}{\partial p^+} + f^*(p^+)\phi^*\right)\psi$$

Longitudinal position operator (continued)

- We found:

$$\int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\phi^*\left(-i\frac{\partial\psi}{\partial p^+} + f(p^+)\psi\right) = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\left(i\frac{\partial\phi^*}{\partial p^+} + f^*(p^+)\phi^*\right)\psi$$

- Rearranged:

$$\int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\frac{\partial}{\partial p^+}\left[\phi^*\psi\right] = 2\int \frac{\mathrm{d}p^+\,\mathrm{d}^2p_\perp}{2p^+(2\pi)^3}\mathrm{Im}[f(p^+)]\phi^*\psi$$

- Using integration by parts gives us:

$$f(p^+) = \frac{i}{2p^+} \qquad \Longrightarrow \qquad X^- = -i\left(\frac{\partial}{\partial p^+} - \frac{1}{2p^+}\right)$$

  - Somewhat akin to the Newton-Wigner operator in instant form

- The position and momentum operators in the various representations are:

Operators		**Coordinate representation**		**Momentum representation**	
$\boldsymbol{X}_\perp$,	$\boldsymbol{P}_\perp$	$\boldsymbol{x}_\perp$,	$-i\boldsymbol{\nabla}_\perp^{(x)}$	$i\boldsymbol{\nabla}_\perp^{(p)}$,	$\boldsymbol{p}_\perp$
X^-,	P^+	x^-,	$i\dfrac{\partial}{\partial x^-}$	$-i\left(\dfrac{\partial}{\partial p^+} - \dfrac{1}{2p^+}\right)$,	p^+

- Often we use a **mixed representation** — transverse position + longitudinal momentum

$$i\partial_+\psi(\boldsymbol{x}_\perp, p^+) = \frac{m^2 - \boldsymbol{\nabla}_\perp^2}{2p^+}\psi(\boldsymbol{x}_\perp, p^+)$$

  - We get a nice-looking linear Schrödinger equation
  - We'll see interesting developments when we introduce a potential (next section!)

Fourier transforms

- Fourier transform utilizes the Lorentz-invariant integration element:

$$\psi(\boldsymbol{x}_\perp, x^-; x^+) = \int \frac{\mathrm{d}p^+\, \mathrm{d}^2 p_\perp}{2p^+ (2\pi)^3} \psi(\boldsymbol{p}_\perp, p^+)\, \mathrm{e}^{i(\boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp - p^+ x^-)}\, \mathrm{e}^{-ip^- x^+}$$

  - Time dependence generated by $p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+}$
- Inverse transform:

$$\psi(\boldsymbol{p}_\perp, p^+) = 2p^+\, \mathrm{e}^{ix^+ p^-} \int \mathrm{d}^2 x_\perp\, \mathrm{d}x^-\, \psi(\boldsymbol{x}_\perp, x^-; x^+)$$

- Can also do partial transforms — only transform longitudinal/transverse:

$$\psi(\boldsymbol{x}_\perp, x^-) = \int \frac{\mathrm{d}p^+}{4\pi p^+} \psi(\boldsymbol{x}_\perp, p^+)\, \mathrm{e}^{-ip^+ x^-} \quad \Longrightarrow \quad \psi(\boldsymbol{x}_\perp, p^+) = 2p^+ \int \mathrm{d}x^-\, \psi(\boldsymbol{x}_\perp, x^-)\, \mathrm{e}^{ip^+ x^-}$$

$$\psi(\boldsymbol{x}_\perp, p^+) = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \psi(\boldsymbol{p}_\perp, p^+)\, \mathrm{e}^{i\boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp} \quad \Longrightarrow \quad \psi(\boldsymbol{p}_\perp, p^+) = \int \mathrm{d}^2 x_\perp\, \psi(\boldsymbol{x}_\perp, p^+)\, \mathrm{e}^{-i\boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp}$$

  - Formulas for $x^+ = 0$ for simplicity
  - Basically associate $2p^+(2\pi)$ with the $x^- \leftrightarrow p^+$ transform

- Momentum-space normalization follows from completeness relation:

$$\langle\psi|\psi\rangle = \int \frac{\mathrm{d}p^+\,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}\left|\psi(\boldsymbol{p}_\perp, p^+)\right|^2 \equiv 1$$

- Coordinate-space normalization derived via Fourier transform:

$$\int \mathrm{d}^2 x_\perp\,\mathrm{d}x^-\,\psi^*(\boldsymbol{x}_\perp, x^-; x^+)\, i\overleftrightarrow{\partial}_-\psi(\boldsymbol{x}_\perp, x^-; x^+)$$

$$= \int \frac{\mathrm{d}p^+\,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}\int \frac{\mathrm{d}k^+\,\mathrm{d}^2 k_\perp}{2k^+(2\pi)^3}(p^+ + k^+)\psi^*(\boldsymbol{k}_\perp, k^+)\psi(\boldsymbol{p}_\perp, p^+)\underbrace{\int \mathrm{d}^2 x_\perp\,\mathrm{d}x^-\,e^{i\left((\boldsymbol{p}_\perp-\boldsymbol{k}_\perp)\cdot\boldsymbol{x}_\perp-(p^+-k^+)x^-\right)}}_{=(2\pi)^3\delta^{(1)}(p^+-k^+)\delta^{(2)}(\boldsymbol{p}_\perp-\boldsymbol{k}_\perp)}$$

$$= \int \frac{\mathrm{d}p^+\,\mathrm{d}^2 p_\perp}{2p^+(2\pi)^3}|\psi(\boldsymbol{p}_\perp, p^+)|^2 = 1$$

- The mixed-representation normalization rule (try proving as an exercise):

$$\int \frac{\mathrm{d}p^+\,\mathrm{d}^2 x_\perp}{4\pi p^+}\left|\psi(\boldsymbol{x}_\perp, p^+)\right|^2 = 1$$

Summary so far

- Quantum mechanics looks pretty normal in transverse components
- A longitudinal position operator exists, but exhibits weird behavior

Operators		**Coordinate representation**		**Momentum representation**	
$\boldsymbol{X}_\perp$,	$\boldsymbol{P}_\perp$	$\boldsymbol{x}_\perp$,	$-i\boldsymbol{\nabla}_\perp^{(x)}$	$i\boldsymbol{\nabla}_\perp^{(p)}$,	$\boldsymbol{p}_\perp$
X^- ,	P^+	x^- ,	$i\dfrac{\partial}{\partial x^-}$	$-i\left(\dfrac{\partial}{\partial p^+} - \dfrac{1}{2p^+}\right)$,	p^+

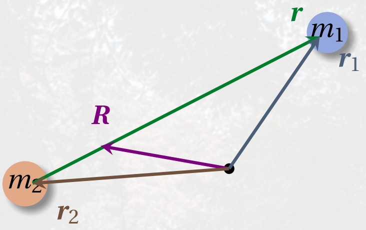
- It's common to use a **mixed representation** when doing light front QM — $p+$ and $\boldsymbol{x}_\perp$
  - Fully momentum-space representation is also common
  - Basically nobody works in the x^- coordinate representation

$$\int \frac{\mathrm{d}p^+ \mathrm{d}^2 x_\perp}{4\pi p^+} \left| \psi(\boldsymbol{x}_\perp, p^+) \right|^2 = 1$$


Separation of Variables

- Let's begin with a review—**separation of variables** for non-relativistic two-body systems:

$$\underbrace{\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}}_{\text{barycenter (center of mass)}} \qquad \underbrace{\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2}_{\text{relative separation}}$$



- Canonically conjugate momenta:

$$\underbrace{\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2}_{\text{total momentum}} \qquad \underbrace{\boldsymbol{k} = \frac{m_2 \boldsymbol{p}_1 - m_1 \boldsymbol{p}_2}{m_1 + m_2}}_{\text{relative momentum}}$$

  - Can be proved by requiring:

$$[r^i, k^j] = i\delta^{ij} \qquad [r^i, P^j] = 0$$

$$[R^i, P^j] = i\delta^{ij} \qquad [R^i, k^j] = 0$$

Separation of variables (non-relativistic, continued)

- Transformation formulas:

$$\underbrace{\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}}_{\text{barycenter (center of mass)}} \qquad \underbrace{\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2}_{\text{relative separation}} \qquad \underbrace{\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2}_{\text{total momentum}} \qquad \underbrace{\boldsymbol{k} = \frac{m_2 \boldsymbol{p}_1 - m_1 \boldsymbol{p}_2}{m_1 + m_2}}_{\text{relative momentum}}$$

- Inversion formulas:

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\boldsymbol{r} \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\boldsymbol{r} \qquad \boldsymbol{p}_1 = \frac{m_1}{m_1 + m_2}\boldsymbol{P} + \boldsymbol{k} \qquad \boldsymbol{p}_2 = \frac{m_2}{m_1 + m_2}\boldsymbol{P} - \boldsymbol{k}$$

- Main perk — clean separation of the Hamiltonian:

$$H = \frac{\boldsymbol{p}_1^2}{2m_1} + \frac{\boldsymbol{p}_2^2}{2m_2} + V(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \overbrace{\frac{\boldsymbol{P}^2}{2M}}^{\text{free Hamiltonian for barycentric motion}} + \underbrace{\frac{\boldsymbol{k}^2}{2\mu} + V(\boldsymbol{r})}_{\text{internal structure independent of } \boldsymbol{P}}$$

$$M = m_1 + m_2$$

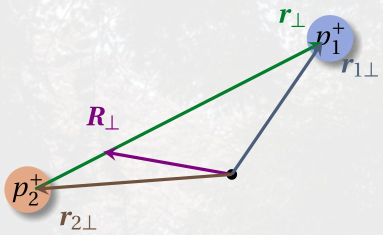
$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

- **Separation of variables** proceeds similarly for transverse light front coordinates:

$$\underbrace{\boldsymbol{X}_\perp = \frac{p_1^+ \boldsymbol{x}_{1\perp} + p_2^+ \boldsymbol{x}_{2\perp}}{p_1^+ + p_2^+}}_{\text{barycenter (center of } p^+)} \qquad \underbrace{\boldsymbol{r}_\perp = \boldsymbol{x}_{1\perp} - \boldsymbol{x}_{2\perp}}_{\text{relative separation}}$$

- But we've got p^+ in place of the mass
- Canonically conjugate momenta:

$$\underbrace{\boldsymbol{P}_\perp = \boldsymbol{p}_{1\perp} + \boldsymbol{p}_{2\perp}}_{\text{total momentum}} \qquad \underbrace{\boldsymbol{k} = \frac{p_2^+ \boldsymbol{p}_{1\perp} - p_1^+ \boldsymbol{p}_{2\perp}}{p_1^+ + p_2^+}}_{\text{relative momentum}}$$



  - Can be proved by requiring:

$$[r_\perp^i, k_\perp^j] = i\delta^{ij} \qquad [r_\perp^i, P_\perp^j] = 0$$
$$[R_\perp^i, P_\perp^j] = i\delta^{ij} \qquad [R_\perp^i, k_\perp^j] = 0$$

Separation of variables (light front, continued)

- Transformation formulas:

$$\underbrace{\boldsymbol{X}_\perp = \frac{p_1^+ \boldsymbol{x}_{1\perp} + p_2^+ \boldsymbol{x}_{2\perp}}{p_1^+ + p_2^+}}_{\text{barycenter (center of mass)}} \qquad \underbrace{\boldsymbol{r}_\perp = \boldsymbol{x}_{1\perp} - \boldsymbol{x}_{2\perp}}_{\text{relative separation}} \qquad \underbrace{\boldsymbol{P}_\perp = \boldsymbol{p}_{1\perp} + \boldsymbol{p}_{2\perp}}_{\text{total momentum}} \qquad \underbrace{\boldsymbol{k}_\perp = \frac{p_2^+ \boldsymbol{p}_{1\perp} - p_1^+ \boldsymbol{p}_{2\perp}}{p_1^+ + p_2^+}}_{\text{relative momentum}}$$

- Inversion formulas:

$$\boldsymbol{x}_{1\perp} = \boldsymbol{X}_\perp + \frac{p_2^+}{p_1^+ + p_2^+}\boldsymbol{r}_\perp \qquad \boldsymbol{p}_{1\perp} = \frac{p_1^+}{p_1^+ + p_2^+}\boldsymbol{P}_\perp + \boldsymbol{k}_\perp$$

$$\boldsymbol{x}_{2\perp} = \boldsymbol{X}_\perp - \frac{p_1^+}{p_1^+ + p_2^+}\boldsymbol{r}_\perp \qquad \boldsymbol{p}_{2\perp} = \frac{p_2^+}{p_1^+ + p_2^+}\boldsymbol{P}_\perp - \boldsymbol{k}_\perp$$

- What does this do to the Hamiltonian?

Separation of variables and light front Hamiltonian

- Start working it out ...

$$P^- = \frac{m_1^2 + \boldsymbol{p}_{1\perp}^2}{2p_1^+} + \frac{m_2^2 + \boldsymbol{p}_{2\perp}^2}{2p_2^+} + \underbrace{P^-_{\text{int}}(\ldots)}_{\text{interaction Hamiltonian}} = \frac{p_2^+ m_1^2 + p_1^+ m_2^2}{2p_1^+ p_2^+} + \frac{\boldsymbol{P}_\perp^2}{2(p_1^+ + p_2^+)} + \frac{\boldsymbol{k}_\perp^2}{2\frac{p_1^+ p_2^+}{p_1^+ + p_2^+}} + P^-_{\text{int}}(\ldots)$$

  - In contrast to a reduced mass, we define **momentum fractions**:

$$\underbrace{P^+ = p_1^+ + p_2^+}_{\text{total } P^+} \qquad \underbrace{\xi_1 = \frac{p_1^+}{P^+} \qquad \xi_2 = \frac{p_2^+}{P^+}}_{\text{momentum fractions}}$$

- Gives, with minor rearrangement:

$$\underbrace{M^2 = 2P^+P^- - \boldsymbol{P}_\perp^2}_{\text{free Hamiltonian for barycentric motion}} = \overbrace{\frac{m_1^2}{\xi_1} + \frac{m_2^2}{\xi_2} + \frac{\boldsymbol{k}_\perp^2}{\xi_1 \xi_2} + \underbrace{2P^+P^-_{\text{int}}(\ldots)}_{\equiv V(\ldots)}}^{\text{Internal structure — must be boost-invariant, independent of } (P^+; \boldsymbol{P}_\perp).}$$

  - The left-hand side is a Lorentz-invariant
  - Can get to $(P^+; \boldsymbol{P}_\perp) = (M; \boldsymbol{0}_\perp)$ via Galilean boosts
  - Right-hand side *must be* independent of $(P^+; \boldsymbol{P}_\perp)$

Free Hamiltonian under transverse boosts

- Light front Hamiltonian equation (two bodies):

$$M^2 = \underbrace{\frac{m_1^2}{\xi_1} + \frac{m_2^2}{\xi_2} + \frac{\boldsymbol{k}_\perp^2}{\xi_1\xi_2}}_{\text{free Hamiltonian}} + \underbrace{V(\dots)}_{\text{interaction}} = M^2_{\text{free}} + M^2_{\text{int}}$$

  - Brodsky and others call M^2 the Hamiltonian
  - Left-hand side commutes with entire Poincaré group!
  - Whatever free part commutes with must commute with interaction part.

- Helpful to recall:

$$X_\perp^i = -\frac{B_\perp^i}{P^+} \qquad \Longrightarrow \qquad B_\perp^i = -P^+ X_\perp^i$$

  - A boost transforms all objects — so it's *barycentric* $X_\perp^i$ that's used

$$[B_\perp^i, k_\perp^j] = \overbrace{[-P^+ X_\perp^i, k_\perp^j]}^{[AB,C]=A[B,C]+[A,C]B} = -\underbrace{[P^+, k_\perp^j]}_{=0 \text{ because they're all momenta}} X_\perp^i - P^+ \overbrace{[X_\perp^i, k_\perp^j]}^{=0 \text{ from variable separation}} = 0$$

- Also have $[B_\perp^i, \xi_{1,2}] = 0$ basically by definition

(transverse boosts leave plus components invariant)

Free Hamiltonian under transverse boosts

- Light front Hamiltonian equation (two bodies):

$$M^2 = \underbrace{\frac{m_1^2}{\xi_1} + \frac{m_2^2}{\xi_2} + \frac{\boldsymbol{k}_\perp^2}{\xi_1\xi_2}}_{\text{free Hamiltonian}} + \underbrace{V(\ldots)}_{\text{interaction}} = M_{\text{free}}^2 + M_{\text{int}}^2$$

  - Brodsky and others call M^2 the Hamiltonian
  - Left-hand side commutes with entire Poincaré group!
  - Whatever free part commutes with must commute with interaction part.
- Helpful to recall:

$$\mathcal{K}_z(\eta)p_1^+ = e^\eta p_1^+ \qquad \mathcal{K}_z(\eta)p_2^+ = e^\eta p_2^+ \qquad \Longrightarrow \qquad \mathcal{K}_z(\eta)\xi_1 = \xi_1 \qquad \mathcal{K}_z(\eta)\xi_2 = \xi_2$$

- Also have $[B_-, \boldsymbol{k}_\perp] = 0$, basically by definition

  (longitudinal boosts leave transverse components invariant)
- **Free Hamiltonian commutes with boosts!**
  - ***Interaction Hamiltonian must also commute with boosts.***

What commutes with boosts?

- We've already seen the following are "good" variables:

$$\xi_1, \quad \xi_2, \quad \boldsymbol{k}_\perp$$

- Transverse separation $\boldsymbol{r}_\perp$ also works!

$$[B_\perp^i, r_\perp^j] = \overbrace{[-P^+ X_\perp^i, r_\perp^j]}^{[AB,C]=A[B,C]+[A,C]B} = -[P^+, r_\perp^j] X_\perp^i - P^+ \overbrace{[X_\perp^i, r_\perp^j]}^{=0 \text{ from variable separation}} = -\underbrace{[p_1^+, r_\perp^j]}_{=0} X_\perp^i - \underbrace{[p_2^+, r_\perp^j]}_{=0} X_\perp^i = 0$$

- What about longitudinal separation?

$$z_1 - z_2 = \left(\frac{x_1^+ - x_1^-}{\sqrt{2}}\right) - \left(\frac{x_2^+ - x_2^-}{\sqrt{2}}\right) \xrightarrow[\text{fixed } x^+]{} \left(\frac{x_2^- - x_1^-}{\sqrt{2}}\right)$$

  - Commutes with $X_\perp^i$, and ...

$$[P^+, x_2^- - x_1^-] = \underbrace{[p_2^+, x_2^-]}_{= i \text{ by CCR's}} - \underbrace{[p_1^+, x_1^-]}_{=i} = 0$$

  - So this commutes with transverse boosts—*but*:

$$\mathcal{K}_z(\eta)\left(x_2^- - x_1^-\right) = \mathrm{e}^{-\eta}\left(x_2^- - x_1^-\right) \neq \left(x_2^- - x_1^-\right)$$

More on longitudinal separation

- So $(x_2^- - x_1^-)$ is not boost-invariant, but the **Miller-Brodsky variable** $\tilde{z}$ is:

$$\tilde{z} \equiv P^+ \left(x_2^- - x_1^-\right)$$

  - Works out because P^+ scales as $\mathrm{e}^{\eta} P^+$ while x^- scales with $\mathrm{e}^{-\eta} x^-$.
  - Can put P^+ on either side since $[P^+, x_2^- - x_1 -] = 0$ (previous slide)
- So $\tilde{z}$ is a good, boost-invariant variable that can appear in the potential.
- Our list of good variables:

$$\begin{array}{ll} \boldsymbol{k}_\perp, & \xi_1 = 1 - \xi_2 \equiv \xi \\ \boldsymbol{r}_\perp, & \tilde{z} \end{array}$$

  - We have $\boldsymbol{k}_\perp$ and $\boldsymbol{r}_\perp$ as canonically conjugate
  - Can helpfully write $\boldsymbol{k}_\perp \to -i\boldsymbol{\nabla}_\perp$ in coordinate representation
  - Is there a similar relationship between ξ and $\tilde{z}$?

- Recall:

$$P^+ = p_1^+ + p_2^+ \qquad \xi = \frac{p_1^+}{p_1^+ + p_2^+}$$

- The derivation will be dry, but important ...

$$x_1^- = -i\left(\frac{\partial}{\partial p_1^+} - \frac{1}{2p_1^+}\right) = -i\left(\frac{\partial P^+}{\partial p_1^+}\frac{\partial}{\partial P^+} + \frac{\partial \xi}{\partial p_1^+}\frac{\partial}{\partial \xi} - \frac{1}{2\xi P^+}\right) = -i\left(\frac{\partial}{\partial P^+} + \frac{1-\xi}{P^+}\frac{\partial}{\partial \xi} - \frac{1}{2\xi P^+}\right)$$

$$x_2^- = -i\left(\frac{\partial}{\partial p_2^+} - \frac{1}{2p_2^+}\right) = -i\left(\frac{\partial P^+}{\partial p_2^+}\frac{\partial}{\partial P^+} + \frac{\partial \xi}{\partial p_2^+}\frac{\partial}{\partial \xi} - \frac{1}{2(1-\xi) P^+}\right) = -i\left(\frac{\partial}{\partial P^+} - \frac{\xi}{P^+}\frac{\partial}{\partial \xi} - \frac{1}{2(1-\xi) P^+}\right)$$

$$x_2^- - x_1^- = \frac{i}{P^+}\left(\frac{\partial}{\partial \xi} - \frac{1-2\xi}{\xi(1-\xi)}\right)$$

- This can be written in a nice, compact form:

$$\tilde{z}\psi(\xi) = P^+\left(x_2^- - x_1^-\right)\psi(\xi) = i\sqrt{\xi(1-\xi)}\frac{\partial}{\partial \xi}\left[\frac{1}{\sqrt{\xi(1-\xi)}}\psi(\xi)\right]$$

Normalization of relative wave function

- The ξ-dependent wave function has a funky normalization
- In terms of longitudinal momentum (suppressing transverse variables):

$$\int \frac{\mathrm{d}p_1^+}{4\pi p_1^+} \int \frac{\mathrm{d}p_2^+}{4\pi p_2^+} \left|\psi(p_1^+, p_2^+)\right|^2 = 1$$

- Jacobian for transformation to (P^+, ξ):

$$\frac{\partial(P^+, \xi)}{\partial(p_1^+, p_2^+)} = \begin{vmatrix} 1 & 1 \\ \frac{1-\xi}{P^+} & -\frac{\xi}{P^+} \end{vmatrix} = \frac{1}{P^+}$$

- Normalization for factorized wave function:

$$\underbrace{\int \frac{\mathrm{d}P^+}{4\pi P^+} \left|\Psi(P^+)\right|^2}_{\equiv 1} \underbrace{\int \frac{\mathrm{d}\xi}{4\pi \xi(1-\xi)} \left|\psi(\xi)\right|^2}_{\equiv 1} = 1$$

Summary so far

- $M^2 = 2P^+P^- - \boldsymbol{P}_\perp^2$ is typically used as the Hamiltonian
  - Capital P signifies total four-momentum of a composite system
- Kinetic and potential energy parts are separately boost-invariant:

$$M^2 = M_{\text{kin}}^2 + M_{\text{pot}}^2 = \sum_{\text{particles}} \frac{m_n^2 - \nabla_{n,\perp}^2}{\xi_n} + V_{\text{LF}}$$

- Typically parametrize wave functions in terms of **momentum fractions**:

$$\xi_n = \frac{p_n^+}{P^+}$$

  - **Caution:** most authors use x instead of ξ
  - I'm using ξ here to avoid confusion with spacetime coordinates
- Wave function normalization (two body systems):

$$\int \frac{\mathrm{d}\xi\,\mathrm{d}^2 r_\perp}{4\pi\xi(1-\xi)} \left|\psi(\boldsymbol{r}_\perp,\xi)\right|^2 = 1$$