New Variational Method for QFT

O. W. Greenberg greenberg@umdhep.umd.edu

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1. Introduction

Variational principle in QM

$\psi \in L^2$

$$E_{ground} \leq \frac{(\psi, H\psi)}{(\psi, \psi)}$$

Variational principle in QFT

$\Psi \in$ space of functionals

$$E_{ground} \leq \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

New variational principle in QFT

$\phi = \phi(\text{in fields})$

quantized field

$H = H(\text{Haag amplitudes, in fields})$

The exact answer diagonalizes H.

Minimize the non-diagonal terms.

2. Haag expansion - general idea
(First assume no massless particle, not a gauge theory.)

Evolve a state at finite time forward to $t \to \infty$ or backward to $t \to -\infty$ with the full H.

<u>No</u> <u>adiabatic</u> <u>switching.</u>

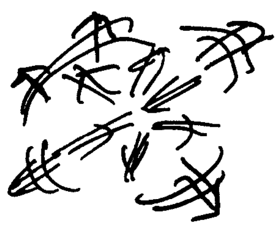
Particles either separate widely, their interactions decrease as e^{-mr}, so they become non-interacting,

or

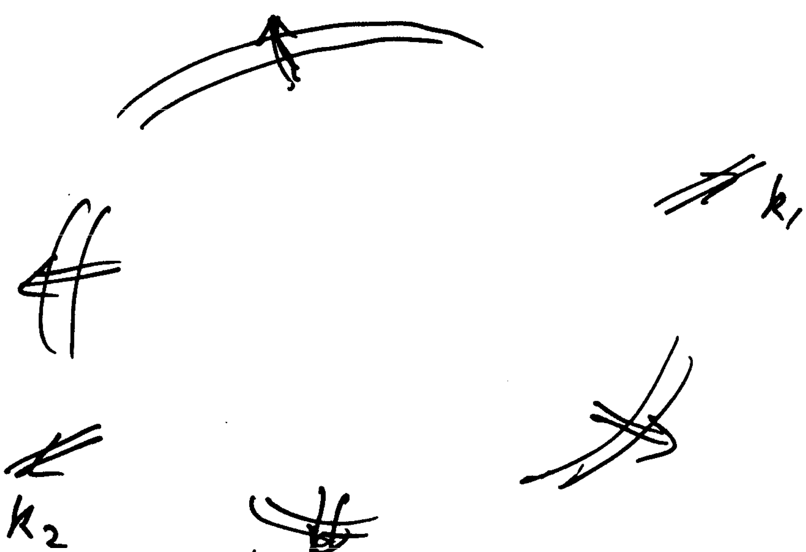
They don't separate and form a bound state.

Introduce in (or out) fields for each particle and stable bound state.

$t \to -\infty$



finite t



$t \to \infty$

Assume every state is a superposition of in (or out) states. Asymptotic completeness

Must include fields for bound states among the asymptotic fields.

Assume every operator can be expanded in (normal-ordered) products of in (or out) fields. Haag expansion

Make this expansion for the Heisenberg fields.

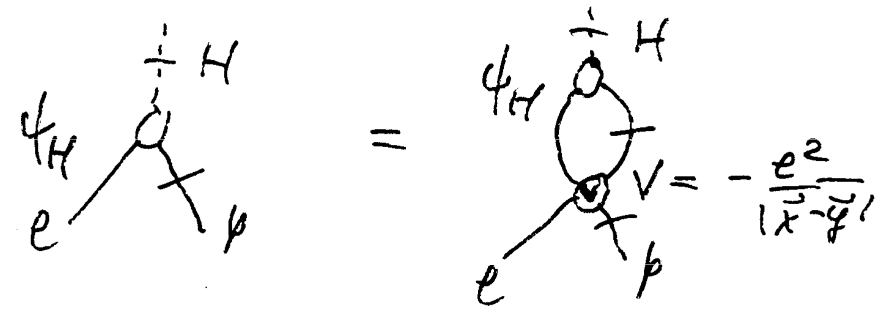
Insert the expansions in the operator equations of motion, renormal order, equate the coefficients of corresponding normal-ordered terms to get equations for the coefficient functions. Haag amplitudes

3. Illustration - H atom - nonrelativistic

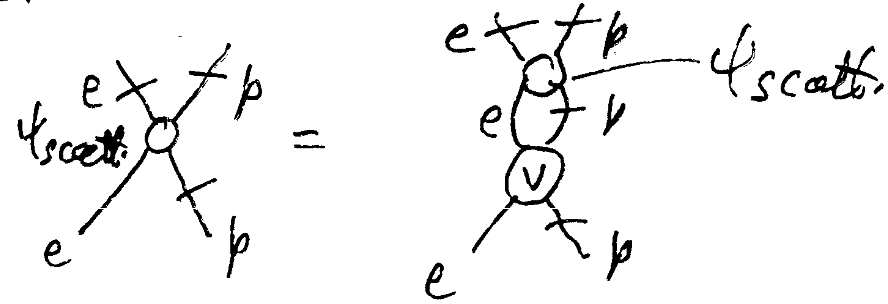
$$H = -\int \phi_e^{+} \frac{\nabla^2}{2m} \phi_e \, d^3x - \int \phi_p^{+} \frac{\nabla^2}{2M} \phi_p$$

$$- e^2 \int \phi_p^{+}(x) \, \phi_e^{+}(y) \frac{1}{|\vec{x}-\vec{y}|} \phi_e(y) \, \phi_p(x) \, d^3x \, d^3y$$

$$\phi_e \sim \phi_e^{in} + \int \psi_{H\text{atom}} \, \phi_p^{+in} \phi_H^{in} +$$



$$+ \int \psi_{scatt.} \, \phi_p^{in+} \phi_p^{in} \phi_e^{in} + \ldots$$



$\psi_{H\text{-atom}}$ obeys the usual Schrödinger equation.

H atom - relativistic

$$(i\not\partial - e\not A - m)\,\phi_e = 0$$

$$(i\not\partial + e\not A - M)\,\phi_p = 0$$

$$\Box A_\mu = e(\bar\phi_e \gamma_\mu \phi_e - \bar\phi_p \gamma_\mu \phi_p)$$

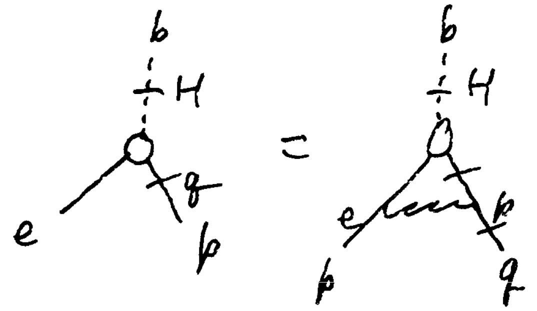
$$\tilde A_\mu(p) = \tilde A_\mu^{in}(p) - \frac{e}{(p+i\epsilon^0)^2}\int(\bar\phi_e^{in}\gamma^\mu\phi_e^{in} - \bar\phi_p^{in}\gamma^\mu\phi_p^{in})$$

$$\tilde\phi_e(p) = \tilde\phi_e^{in}(p)\,\delta(p^2-m^2)$$

$$+\int d^4q\, d^4b\, \delta(p+q-b)\,\bar\phi_p^{in}(q)\,\delta(q^2-M^2)\, f_H(q,b)\, H^{in}(b)\,\delta(b^2-M_H^2)$$

$$+ \;\cdots$$

$$(\not b - \not q - m)\, f_H(q,b) = e^2\int d^4q'\,\delta(q'^2-M^2)\,\gamma_\mu \frac{f_H(q',b)}{(q-q')^2}\gamma^\mu \frac{(\not q+M)}{2M}$$



Invariant functions depend on one variable.

Single-time, but covariant.

No negative norm solutions; no spurious solutions.

4. Previous work using the Haag expansion

ϕ^4 theory, OWG, PR

Deuteron in pseudoscalar meson theory, OWG and R. Goodwin, PR

Higher sectors of the Lee model, A. Pagnamenta, JMP

H-atom Thesis } A. Raychaudhuri
Bound states of $(\phi\phi)$ in $\phi^2\chi$ model, } PR

Form factor of K^0 and $\bar{K}^0$, OWG, S. Nussinov, J. Sucher, PLB

Goldstone and Higgs problems, OWG, PTP Suppl. and PRD

Nambu--Jona-Lasinio model, OWG and P.K. Mohapatra PRD
" " " " with isospin, OWG and L. Orr PRD

Bound state of $(\phi_1\phi_2)$ in $(\phi_1^2+\phi_2^2)\chi$ model, OWG, R. Ray and F. Schlumpf PLB

Related work, F. Gross, PR
K. Johnson, PRD
M. Bander, PRL

BCS model of superconductivity (finite T), OWG Can. J. Phys

Bound states in an exactly Galilean-invariant theory S. R. Corley and OWG JMP

5. New Variational Principle

The asymptotic fields transform the same way as the interacting fields (both in the Heisenberg picture), for a scalar field,

$$U(a,\Lambda)\, \phi(x)\, U(a,\Lambda)^{+} = \phi(\Lambda x + a)$$

$$U(a,\Lambda)\, \phi^{\substack{out\\ in}}(x)\, U(a,\Lambda)^{+} = \phi^{\substack{out\\ in}}(\Lambda x + a)$$

The asymptotic fields obey free field CR's

$$[\phi^{\substack{out\\ in}}(x), \phi^{\substack{out\\ in}}(y)] = i\,\Delta(x-y; m^2)$$

This implies (assuming the vacuum has zero energy)

$$P^0 = H = \sum_i H_{free}(\phi_i^{\substack{out\\ in}})$$

The Haag expansion in momentum space (all fields scalar, for simplicity) is

$$\tilde{\phi}(k) = (2\pi)^{3/2} v\, \delta(k) + \tilde{\phi}^{in}(k)\, \delta(k^2 - m^2) +$$

$$+ \sum_2^{\infty} \frac{1}{n!} \int f^{(n)}(k_1, \dots, k_n) : \prod \tilde{\phi}_j^{in}(k_j)\, \delta(k_j^2 - m_j^2) : \delta(k - \Sigma k_j) \prod d^4 k_j$$

Put this into the formula for H. The result will have all kinds of normal-ordered terms,

$$
\begin{aligned}
H = {} & F^{(0)} + F_-^{(1)} a^{in}(\vec{0}) + F_+^{(1)} a^{in}(\vec{0})^+ \\
& + \int \frac{d^3k}{2\omega_k} \Big[F_{--}^{(2)}(\vec{k}) a^{in}(\vec{k}) a^{in}(-\vec{k}) e^{-2i\omega_k x^0} + \\
& + F_{+-}^{(2)}(\vec{k}) a^{in+}(\vec{k}) a^{in}(\vec{k}) + F_{++}^{(2)}(\vec{k}) a^{in+}(\vec{k}) a^{in+}(-\vec{k}) e^{2i\omega_k x^0} \Big] \\
& + \sum_{n>2} \int \frac{d^{3n}k}{\prod \omega_i} F^{(n)}(\vec{k}_i) : \prod a^{in}(\vec{k}_i)^{(\,)} : \delta(\Sigma \pm \vec{k}_i)\, e^{\Sigma \pm \omega_i \cdot x^0},
\end{aligned}
$$

where

$$\phi^{in}(k)\delta(k^2-m^2) = \theta(k^0)\frac{a^{in}(\vec{k})}{2\omega_k}\delta(k^0-\omega_k) + \theta(-k^0)\frac{a^{in+}(\vec{k})}{2\omega_k}\delta(k^0+\omega_k),$$

$a^{in}(\vec{k})^{(\,)}$ stands for either a^{in} or a^{in+}.

Since the correct Haag expansion diagonalize H, all the non-quadratic terms should vanish, the $a^{in}a^{in}$ and $a^{in+}a^{in+}$ terms should also vanish, and the coefficient of the $a^{in+}a^{in}$ term should be minimum.

The variational principle is to make these unwanted terms as small as possible.

6. Free field

$$H = \int d^3x \, (\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2)$$

Try $\phi = \phi^{in}(x;\mu^2) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} [A(k)e^{-ik\cdot x} + A^{\dagger}(k)e^{ik\cdot x}]$,

$E_k = \sqrt{\vec{k}^2 + \mu^2}$, $\quad [A(k), A^{\dagger}(l)]_- = 2E_k\,\delta(\vec{k}-\vec{l})$.

Get

$$H = \int \frac{d^3k}{2E_k} \Big[\frac{2\vec{k}^2 + m^2 + \mu^2}{2E_k} (A^{\dagger}(k)A(k) + A(k)A^{\dagger}(k))$$
$$+ \Big(\frac{m^2-\mu^2}{2E_k} A(k)A(-k)e^{-2iE_k x^0} + h.c. \Big) \Big]$$

Minimize $|\ \ |^2$ of coefficients $\Rightarrow \mu^2 = m^2$.
Get same result from each coefficient.

A similar calculations gives $\mu^2 = m^2$ for the Dirac field.

7. Gradient coupling model

$$\mathcal{L} = Z_2 \bar\Psi (i\not\partial + g\not\partial\phi - M)\psi + \tfrac{1}{2}(\partial_\mu\phi\cdot\partial^\mu\phi - m^2\phi^2)$$

$$H = \int d^3x \left[Z_2\bar\Psi(i\gamma^j\partial^j + g\,\gamma^j\partial^j\phi + M)\psi + \tfrac{1}{2}(\dot\phi^2 + (\partial^j\phi)^2 + m^2\phi^2)\right]$$

As trial fields, take

$\phi = \phi_0, \quad \psi = \,:f(\phi_0):\, \psi_0$, all at the same x.

Get

$$H = \int d^3x \Big[Z_2\bar\psi_0 : f(\phi_0)^+ : \cdot$$

$$\cdot\Big(\underbrace{i : f'(\phi_0)\gamma^j\partial^j\phi_0 :}_{1} + \underbrace{g\gamma^j\partial^j\phi_0 : f(\phi_0):}_{2} + i : f(\phi_0): \gamma^j\partial^j + M : f(\phi_0):\Big)\psi_0$$

$$+ \tfrac{1}{2}(\dot\phi_0^2 + (\partial^j\phi_0)^2 + m^2\phi_0^2)\Big]$$

Can get a relation between 1 and 2 if can take $\partial^j\phi_0$ out of $:1:$. This can be done, because $\langle \partial^j\phi_0\,\phi_0\rangle_0 = 0$ by symmetry $k\to -k$ in k-space.

Then 1 and 2 cancel if

$$f' + g f = 0, \qquad f(x) = e^{-igx}.$$

Must evaluate $:f(\phi_0)^+:\,:f(\phi_0):$. Use Baker-Hausdorff-Campbell, get

$$\lim_{x\to y} e^{g^2\langle\phi_0(x)\,\phi_0(y)\rangle_0} = \frac{1}{Z_2}.$$

Thus the variational principle leads to the exact solution,

$$\psi(x) = :e^{ig\phi_0(x)}: \psi_0(x), \quad \phi(x) = \phi_0(x).$$

This is an infinite loop-order result.

8. In QED, the soft photons, following Kulish and Faddeev, should have the form

$$\psi(x) = \int \frac{d^3p}{\sqrt{2p^0}} e^{-ip\cdot x} \exp\left\{ \frac{e}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k^0}} \left[f^{\mu}(k,p)\, a_{\mu}^{+}(k) - h.c. \right] \right\} \cdot \psi_{in}(p)$$

Have not yet succeeded in deriving this from the variational principle.

Use this to replace the usual in field.

9. Comparison with ground-state wavefunctional variational principle

G. Tiktopoulos hep/th/9705230 v2

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \frac{1}{2}m^2\varphi_u^2 - \frac{\lambda}{4}\varphi^4$$

in 1+1.

Try (simplest approximation)

$$\Psi(\varphi, v) = \exp\left\{-\frac{1}{4}\int dx\,dy\,(\varphi(x)-v)\,K(x-y)\,(\varphi(y)-v)\right\}$$

$$E_\Psi = \frac{1}{V}\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

$$= \frac{1}{8}D^{-1}(0) + \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + m_u^2\right)D(x)\Big|_{x=0} + \frac{1}{2}m_u^2 v^2$$

$$+ \frac{\lambda}{24}v^4 + \frac{\lambda}{4}v^2 D(0) + \frac{\lambda}{8}D(0)^2$$

$$D(x-y) = \frac{\langle\Psi|\varphi(x)\varphi(y)|\Psi\rangle}{\langle\Psi|\Psi\rangle} - v^2 = K^{-1}(x-y)$$

$$\frac{\delta E_\Psi}{\delta D(x)} = 0 \implies \tilde{D}(k) = \frac{1}{2\sqrt{k^2+m^2}}$$

Gap eq.: $\quad m^2 = m_u^2 + \frac{\lambda}{2}v^2 + \frac{\lambda}{2}\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\,\tilde{D}(k)$

Calculation of the same model using the Haag expansion variational principle:

Try $\phi = \varphi_0 + v$, where φ_0 is a generalized free field. (Here, I would start with a free field of unknown mass m^2, since I expect states to have discrete mass for a single particle, but I did the more general calculation to see how things would go if there were no discrete-mass one-particle state, as we expect in QCD for quarks and gluons.)

$$\mathcal{H} = \frac{1}{2}\langle \dot{\varphi}_0^2(0) + \varphi_0'^2(0) + m_u^2\, \varphi_0^2(0)\rangle + \frac{1}{2} m_u^2 v^2$$

$$+ \frac{\lambda}{8}\langle \varphi_0^2(0)\rangle^2 + \frac{\lambda}{4} v^2 \langle \varphi_0^2(0)\rangle + \frac{\lambda}{24} v^2$$

$$+ \left(m_u^2 v + \frac{\lambda}{2} v \langle \varphi_0^2(0)\rangle + \frac{\lambda}{6} v^2 \right) : \varphi_0(x):$$

$$+ \frac{1}{2}\left(:\dot{\varphi}_0^2(x): + :\varphi_0'^2(x): + \left(\underbrace{m_u^2 + \frac{\lambda}{2}\langle \varphi_0^2(0)\rangle + \frac{\lambda}{2} v^2}_{m^2} \right) : \varphi_0^2(x):\right.$$

$$+ \frac{\lambda v}{6} : \varphi_0^3(x): + \frac{\lambda}{24} : \varphi_0^4(x):$$

Let m_u^2 depend on Λ so as to make m^2 finite.

Define

$$M^2 = m_u^2 + \frac{\lambda}{2} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \frac{1}{2\sqrt{k^2+M^2}} = M^2$$

Gap eq. becomes

$$m^2 = M^2 + \frac{1}{2} \lambda v^2 + \frac{1}{8\pi} \lambda \log \frac{M^2}{m_u^2}$$

$$E_{\Phi, min} = \frac{\Lambda^2}{4\pi} + \left(\frac{M^2}{8\pi} - \frac{\lambda}{2^7 \pi^2} \log \frac{4\Lambda^2}{M^2} \right) \log \frac{4\Lambda^2}{M^2}$$

$$+ \left(\frac{M^2}{8\pi} + \frac{\lambda}{2^7\pi^2} \log \frac{M^2}{m^2} \right) \log \frac{M^2}{m^2}$$

$$+ \frac{1}{2} \left(M^2 + \frac{\lambda}{8\pi} \log \frac{M^2}{m^2} \right) v^2 + \frac{\lambda}{24} v^4$$

For $\frac{\lambda}{M^2}$ big enough, get radiatively induced symmetry breaking.

Presence of logs is non-trivial.

Use the Källén-Lehmann representation:

$$\langle \varphi_0(x)\,\varphi_0(y)\rangle = \frac{1}{2\pi}\int_0^\infty d\kappa^2 \rho(\kappa^2) \int \frac{dk}{2\sqrt{k^2+\kappa^2}}\, e^{ik(x-y)}$$

Find

$$E_\Psi = \frac{1}{2}\,\frac{1}{2\pi}\int d\kappa^2 \rho(\kappa^2)\int \frac{dk\,(2k^2+\kappa^2)}{2\sqrt{k^2+\kappa^2}}$$

$$+\ \frac{1}{2}\left(m_u^2+\frac{\lambda}{2}v^2\right)\frac{1}{2\pi}\int d\kappa^2\rho(\kappa^2)\int\frac{dk}{2\sqrt{k^2+\kappa^2}}$$

$$+\ \frac{\lambda}{8}\left(\frac{1}{2\pi}\int d\kappa^2\rho(\kappa^2)\int\frac{dk}{2\sqrt{k^2+\kappa^2}}\right)^2 + \frac{1}{2}m_u^2 v^2 + \frac{\lambda}{24}v^4$$

Using $m^2 = m_u^2 + \frac{\lambda}{2}\langle\varphi_0^2(0)\rangle + \frac{\lambda}{2}v^2$, the minimum condition becomes

$$\frac{\delta\langle\mathcal{H}\rangle}{\delta\rho(\kappa^2)} = \frac{1}{8\pi}\frac{\delta}{\delta\rho(\kappa^2)}\int d\kappa^2\rho(\kappa^2)\int dk\,\frac{k^2+\frac{1}{2}(\kappa^2+m^2)}{\sqrt{k^2+\kappa^2}}$$

$\rho \geq 0$, so the minimum occurs when

$$\frac{\partial}{\partial\kappa^2}\,\frac{k^2+\frac{1}{2}(\kappa^2+m^2)}{\sqrt{k^2+\kappa^2}} = \frac{1}{4}\frac{(\kappa^2-m^2)}{\sqrt{k^2+\kappa^2}} = 0.$$

Choose $\rho(\kappa^2) = \delta(\kappa^2-m^2)$.

So $\varphi_0(x)$ turns out to be a free field as expected.

The vacuum energy at the minimum is exactly the same as Tiktopoulos found using the vacuum wavefunctional.

How improve the calculation?

In the vacuum wavefunctional method, can't multiply the Gaussian by $\varphi(x_1)\varphi(x_2)\ldots$, etc. because different powers of V would enter. Can try a superposition of Gaussians, but not clear this will converge.

In the Haag expansion method, can add higher terms in the Haag expansion. Should converge to the exact solution.
Can put in the particle structure, including bound states.

10. Conclusion:

Variational principle based on the Haag expansion is promising. Much to be done:

1. φ^4 in 1+1

2. Soft photons in QED

3. Solvable nonrelativistic models in 1+3

4. $\phi^2\chi$ model in 1+3 (with Y. Umino)

5. Try to do the calculation of the QCD β-function of W. E. Brown and I. I. Kogan using the Haag expansional variational principle.