Improved Actions for Lattice QCD

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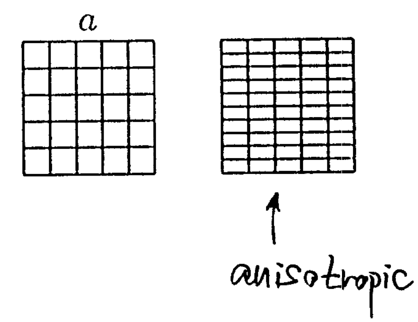
Motivation

- **"Solve QCD"**: Calculate proton mass (all hadron masses, decay rates, etc) from **first principles**.
- **Test SM, Determine SM Par's, Search for New Physics**
  - KM matrix, CP violation, SUSY
  - All involve **non-perturbative** matrix elements between hadronic states.
- **Problem**: Cost of (full) QCD simulation grows like $1/a^{10}$
  - Instead of decreasing the lattice spacing, it is **much more efficient to improve the action**: $\mathrm{O}(a, a^2) \to \mathrm{O}(a^2, \alpha^2 a^4)$
  - Using $a = 0.1 - 0.25$ fm instead of $0.05 - 0.1$ fm saves factor of $10^3 - 10^4$!

Outline (History of Improved Actions)

- Recap of Lattice QCD
- Improvement of Lattice Actions (Symanzik, et al (on-shell) early 80's)
- Improvement of Pure QCD (Lüscher/Weisz, 1985)
  — and why its first incarnation did not work
- Perturbatively and Tadpole Improved Glue (1994)
- (Improved) Quarks on the Lattice
  – Solving the "doubler" and "ghost" problems: Wilson action (1975)
  – Improved quarks via field transf's: SW (1985) and D234 (1995) actions
- Non-Perturbative Improvement of Quark Actions at $O(a)$ (1996/97)
- Simulation Results
- Conclusions and Outlook:
  — Solving quenched QCD; towards full QCD; anisotropic lattices

Lattice QCD



- Space-time → (hypercubic) lattice
- Fermion fields $\psi(x), \bar{\psi}(x)$ live on **sites**.
- Only known way to preserve exact gauge invariance:
  $A_\mu(x) \rightarrow$ **link variable**

$$U_\mu(x) \;=\; \mathrm{P}\,\exp\left[-iag\int_x^{x+\mu} dx'_\mu\, A_\mu(x')\right] \;\in\; \mathrm{SU}(N)$$

Parallel transporter from $x+\mu$ to x (→ covariant derivative).

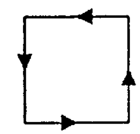
- Gauge transformations:

$$\psi(x) \;\rightarrow\; \Lambda(x)\,\psi(x)\,, \quad \bar{\psi}(x) \;\rightarrow\; \bar{\psi}(x)\,\Lambda^{-1}(x)$$
$$U_\mu(x) \;\rightarrow\; \Lambda(x)\,U_\mu(x)\,\Lambda^{-1}(x+\mu)$$

Wilson Gauge Action

- Simplest gauge inv't operator uses the **plaquette variable**:

$$U(\Box) \equiv P_{\mu\nu}(x) \equiv \prod_{\text{links}}$$



- In a small a expansion:

$$P_{\mu\nu} = 1 - iga^2 F_{\mu\nu}(x) - \tfrac{1}{2} g^2 a^4 F_{\mu\nu}^2 + \ldots$$

- So:

$$\underbrace{\frac{2N}{g^2}}_{\underline{\underline{\beta}}} \sum_{\Box} \left(1 - \frac{1}{N} \mathrm{Re}\,\mathrm{Tr}\, U(\Box)\right) = \int d^4x\, \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}^2 + \mathcal{O}(a^2)$$

$\uparrow$

Improved Actions (classical)

- Symanzik: Add higher dimensional operators to the naive lattice action to cancel its a^2 (bosons) discretization errors.
- Classical improvement
  - For bosons: Just calculus
  - For fermions: Technical complication due to doublers → later ...
- Example: Classically improving free scalar field ($d = 1$):
  - Naive lattice laplacian:

$$\begin{aligned} \Delta\phi(x) &= \frac{1}{a^2}\left[\phi(x+a) + \phi(x-a) - 2\phi(x)\right] \\ &= \left[e^{a\partial} + e^{-a\partial} - 2\right]\phi(x) = \partial^2\phi(x) + \mathcal{O}(a^2) \end{aligned}$$

  - Improve: $\partial^2 = \Delta - \frac{a^2}{12}\Delta^2 + \frac{a^4}{90}\Delta^3 + \ldots \equiv P(\Delta)$
  - Actions are **on- and off**-shell improved, *i.e.* lattice field $\phi(x)$ is improved to same order as observables (spectrum → plot).

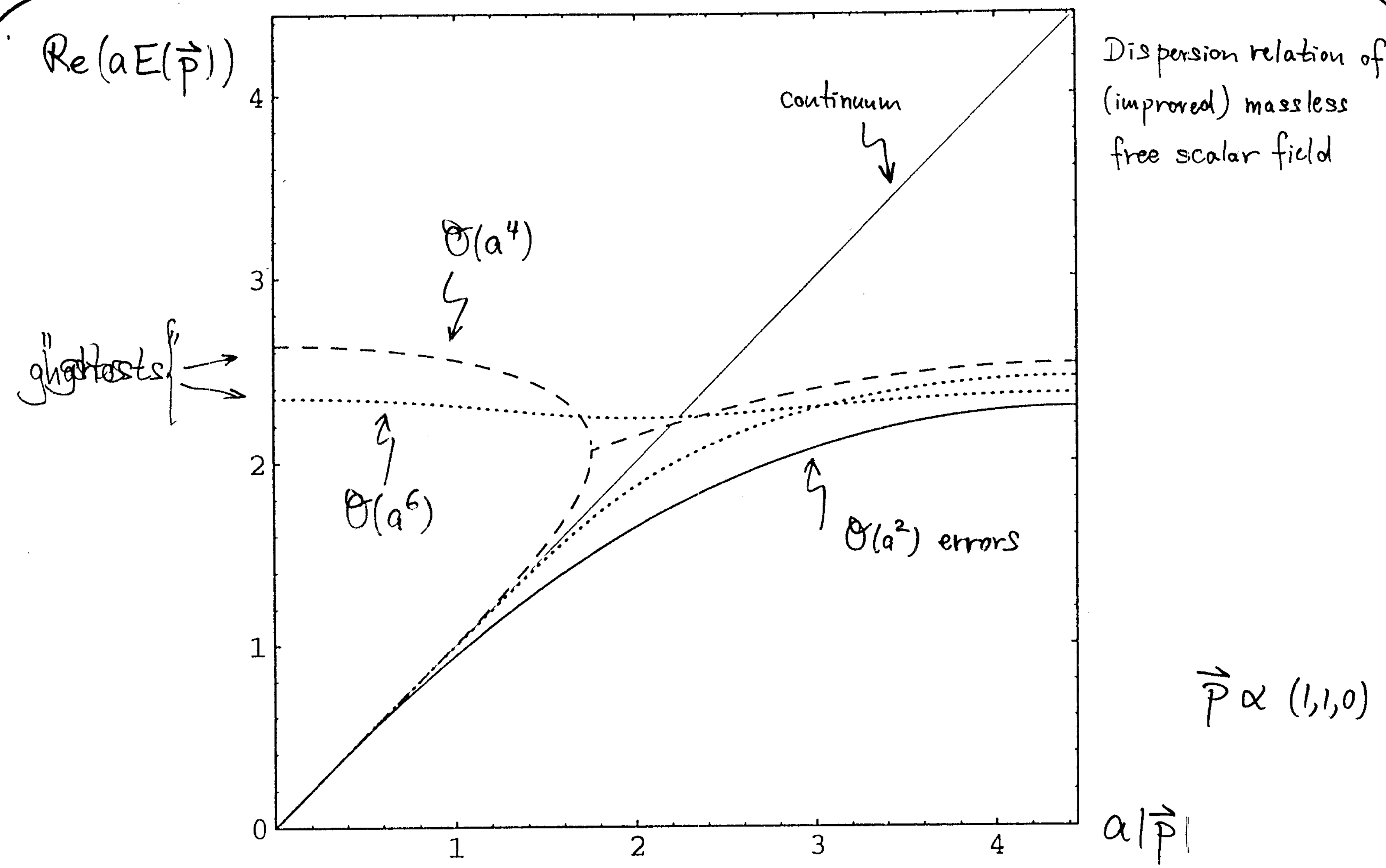


Figure 1: (Real part of the) energy $aE = aE(\mathbf{p})$ versus $a|\mathbf{p}|$, with $\mathbf{p} \propto (1,1,0)$, for free scalar field improved to have $\mathcal{O}(a^2)$ (solid), $\mathcal{O}(a^4)$ (dashed), and $\mathcal{O}(a^6)$ (dotted) errors.

Improved Actions (quantum)

improve to finite dim (order in 'a')

- Organize improved action (and fields) by dim of operators. Makes sense at least for asymptotically free theories.
- Quantum effects will generate all operators of allowed symmetries; they mimick effects of UV modes beyond cut-off.
- For **on-shell** improvement can use field transformations to set coefficients of certain operators ("**redundant**") to zero, or achieve some other aim, like eliminate doublers. Points to note:
  - Any interpolating fields will give correct spectral quantities (masses, decay rates, etc).
  - With improved composite fields get matrix elements between physical states.

So, can get all we need!

Weisz

Tim Klassen

Improved Actions (quantum) cont'd

- **Classically** it is easy to achieve on- and off-shell improvement (at worst: keep track of field transformations).
- On **quantum** level demanding only **on-shell** improvement is tremendous **simplification**.
- Fix improvement coefficients by matching to suitable set of on-shell conditions. In practice:
  - Lattice PT (is complicated, especially for improved actions).
  - If possible, impose **non-perturbative** improvement conditions: **Rotational** and **chiral** symmetry.

    } examples later. E.g. pot'l in connection with non-pert tuning of rot'l symmetry!
- In many cases: At $O(a^2)$ classical + tadpole improvement works well ($\rightarrow$ later).

Improvement of Pure QCD (Weisz/Lüscher)

- Wilson: $\beta \sum_{\Box} \left(1 - \frac{1}{N} \mathrm{Re}\,\mathrm{Tr}\, U(\Box)\right) = \int \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}^2 + \mathcal{O}(a^2)$

- Classical improvement of gluon action

$$\mathrm{Re}\,\mathrm{Tr}\, U(\Box) \propto \sum_{\mu\nu} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} + \#\, a^2 \underbrace{\sum_{\mu\nu} \mathrm{Tr}\, D_\mu F_{\mu\nu} D_\mu F_{\mu\nu}}_{\text{violates rotational sym (good: easy to tune)}} + \mathcal{O}(a^4)$$

- At quantum level two more operators can appear at $\mathcal{O}(a^2)$:

$$\sum_{\mu\nu\sigma} \mathrm{Tr}\, D_\mu F_{\nu\sigma} D_\mu F_{\nu\sigma}$$

$$\sum_{\mu\nu\sigma} \mathrm{Tr}\, D_\mu F_{\mu\sigma} D_\nu F_{\nu\sigma} \quad \leftarrow \text{redundant:} \quad A_\sigma \to A_\sigma + \#\, a^2 \sum_\nu D_\nu F_{\nu\sigma}$$

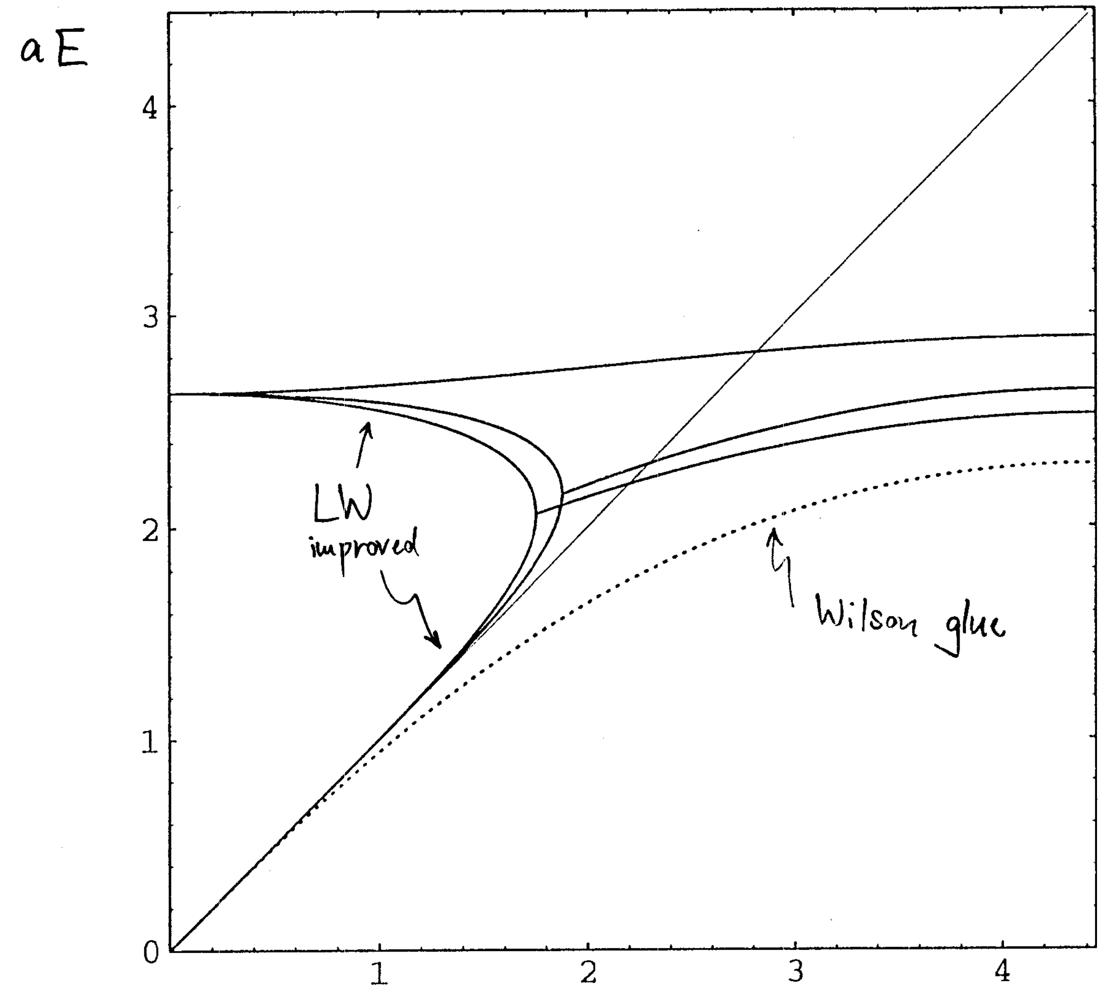
Improvement of Pure QCD (cont'd)

- For $\mathcal{O}(a^2)$ on-shell quantum improvement it suffices to add two terms to $\Box$, for example (there are many other possibilities):

$$S_g[U] = \beta_{\text{pl}} \sum \Box + \beta_{\text{rt}} \sum \boxed{} + \beta_{\text{pg}} \sum \lozenge$$

$$\stackrel{!}{=} \frac{1}{2} \int \operatorname{Tr} F_{\mu\nu}^2 + \mathcal{O}(a^4)$$

- At tree level only have to add $\boxed{}$ to $\Box$, with $\frac{\beta_{\text{rt}}}{\beta_{\text{pl}}} = -\frac{1}{20}$. Then the action is also off-shell improved.
- Dispersion Relation $\rightarrow$ plot.



Improved vs Wilson glue dispersion relation (isotropic)

Figure 2: Energy $aE(\mathbf{k})$ of the two gluon polarizations as a function of $a|\mathbf{k}|$, with $\mathbf{k} \propto (1,1,0)$, for the Wilson (dotted) and Lüscher-Weisz improved (solid) actions on an isotropic lattice.

Improvement of Pure QCD (cont'd)

- Lüscher/Weisz (1985) for SU(3) (from on−shell scattering amplitudes in twisted, compactified universe)

$$\frac{\beta_{\rm rt}}{\beta_{\rm pl}} = -\frac{1}{20}\Big(1 + 2.0146\,\alpha_0\Big) + \mathcal{O}(\alpha^2) \qquad \alpha_0 \equiv \frac{g^2}{4\pi}$$

$$\frac{\beta_{\rm pg}}{\beta_{\rm pl}} = \underbrace{0}_{\rm classical} -0.03325\,\alpha_0 + \mathcal{O}(\alpha^2)$$

- This action was tried in MC simulations — and did **not** seem to give significant improvements (relative to cost) !?
- The basic reason became clear through the work of Lepage/Mackenzie (1992) as part of the solution to the puzzle of why **lattice PT did not seem to work**.

The Bare Coupling is Bad

- For pure SU(3)

$$\alpha_0 = \alpha_{\overline{\rm MS}}(\underbrace{28.81}/a) + \mathcal{O}(\alpha^3)$$

- α_0 is **anomalously small**; PT in α_0 underestimates quantum effects and is badly convergent ($\rightarrow$ plot).

- Reason: $U_\mu = e^{iagA_\mu}$ has all powers of A in its expansion. **Tadpole** diagrams in lattice PT are often dominant ($75 - 90\%$), destroying contact with continuum-like PT where such diagrams are absent.

- Always use "physical" coupling like $\alpha_{\overline{\rm MS}}(q), \alpha_V(q)$ at appropriate scale $q = q^*$ ($\rightarrow$ Lepage/Mackenzie).

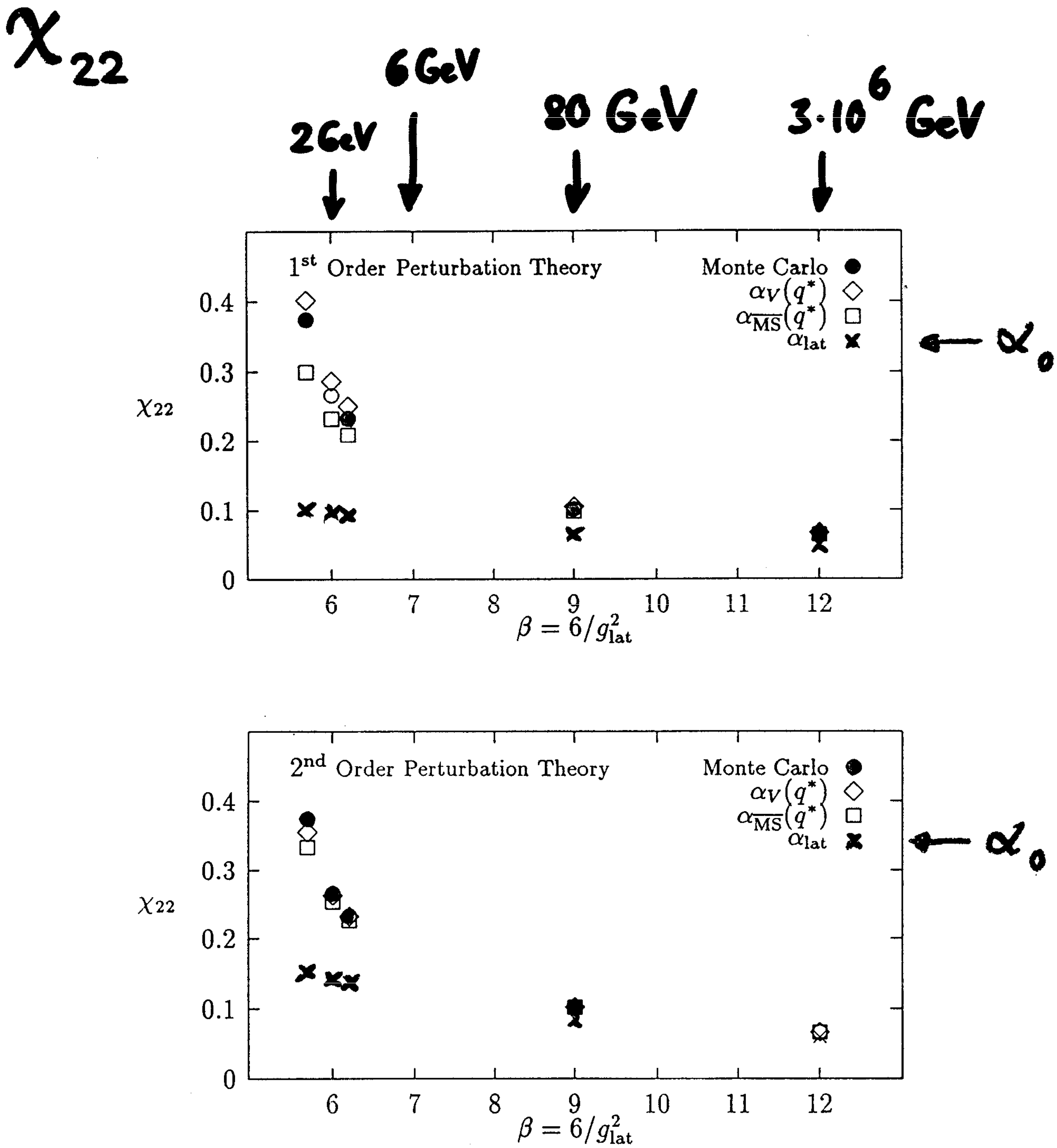


Figure 1: Results for Creutz ratio χ_{22} at different couplings β from Monte Carlo simulations (circles), and from perturbation theory (using $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{\rm MS}}(q^*)$ (boxes), and $\alpha_{\rm lat}$ (crosses)). The first plot shows perturbation theory through one-loop order, and the second through two-loop order. Statistical errors in the Monte Carlo results are negligible.

Tadpole Improvement of Improved Glue

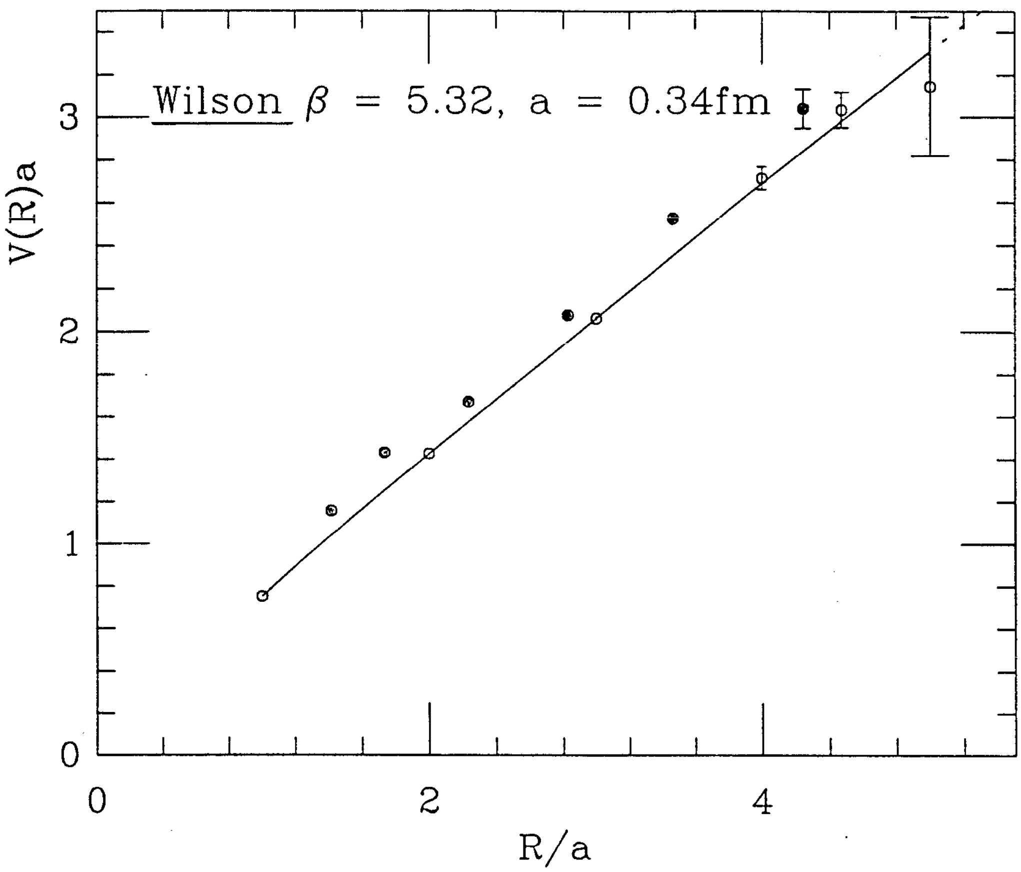
- Coefficients in action become at one-loop

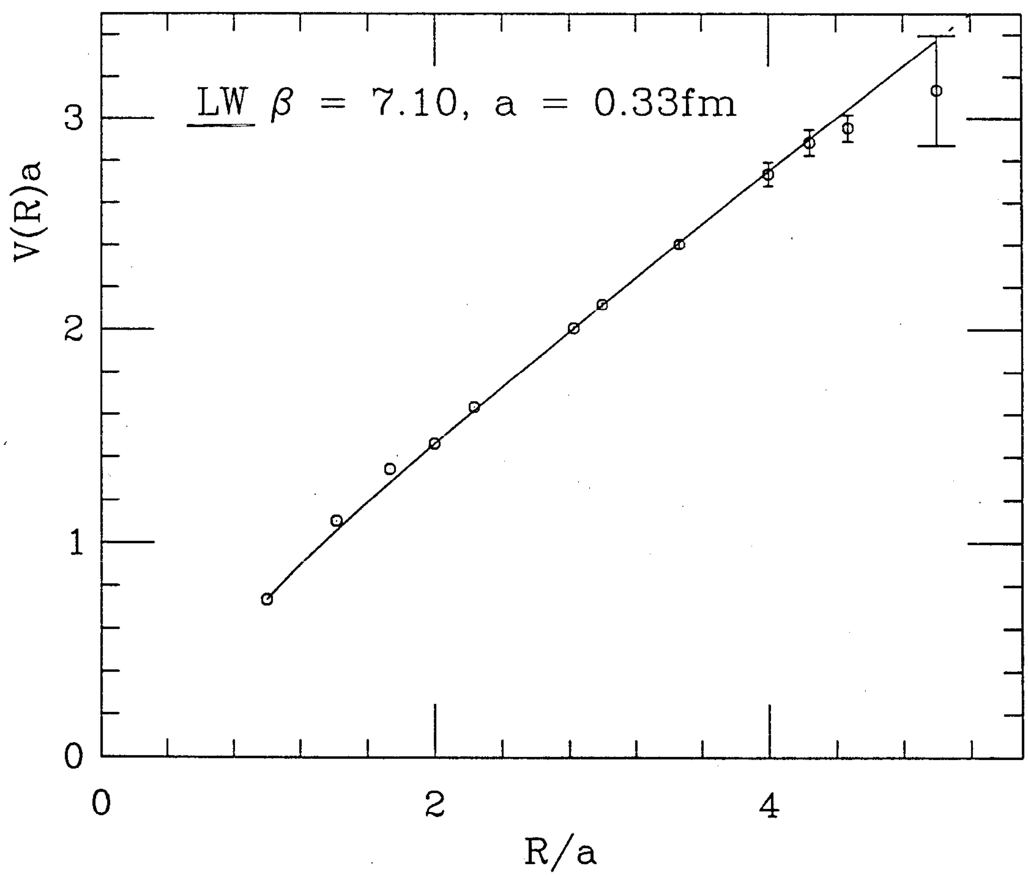
$$\frac{\beta_{\rm rt}}{\beta_{\rm pl}} = -\frac{1}{20}\left(1 + 2.0146\,\alpha_0\right) = -\frac{1}{20\,u_0^2}\left(1 + \underbrace{0.4805}\,\alpha\right)$$

$$\frac{\beta_{\rm pg}}{\beta_{\rm pl}} = -0.03325\,\alpha_0 = -\frac{1}{u_0^2}\,0.03325\,\alpha$$

- Tadpole improvement gives large enhancement of correction terms (factors of up to ≈ 2 on coarse lattices).

Static Potential $V(\vec{R})$ for pure QCD

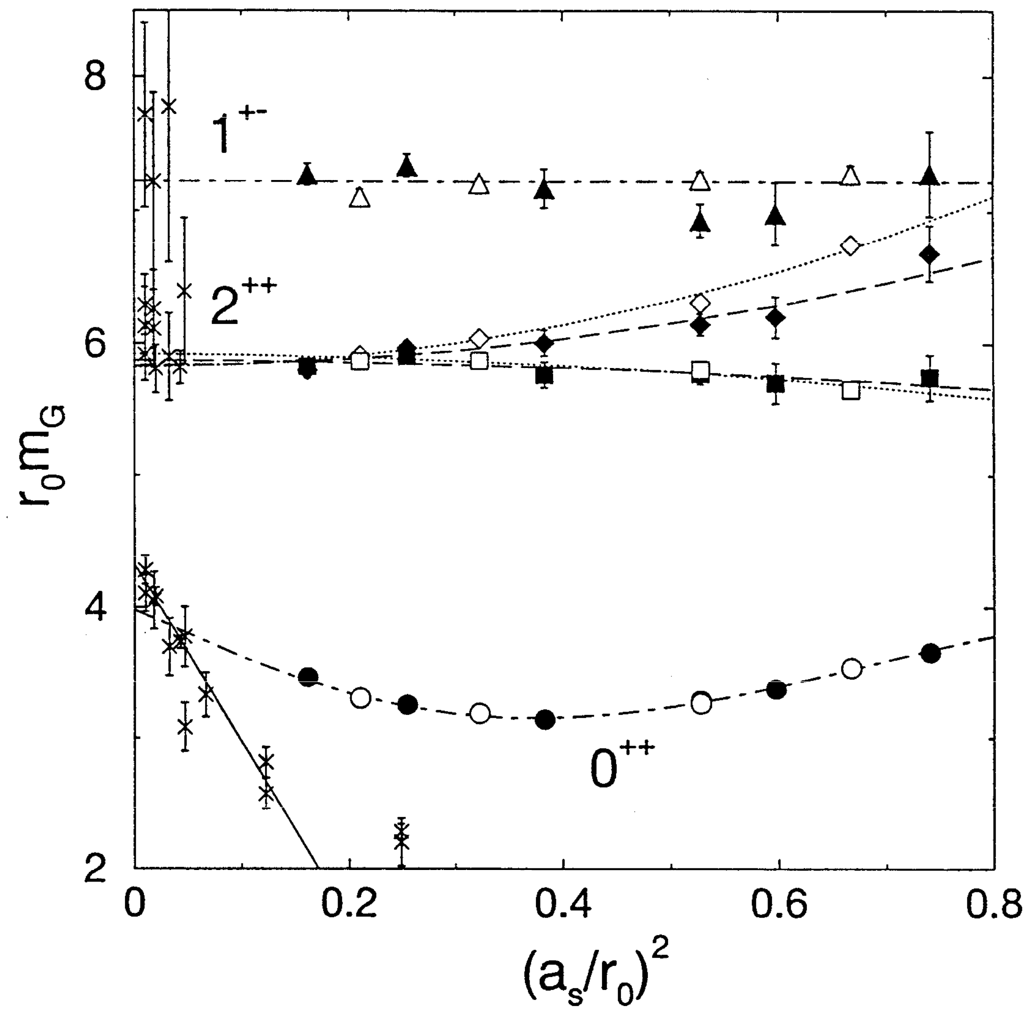




Could use thi as non-pert improvement condition....

Glueballs with improved, anisotropic gauge actions

Morningstar Peardon



× : Wilson

FIG. 12. Glueball mass estimates in terms of r_0 against the lattice spacing $(a_s/r_0)^2$. Results from the $\xi = 5$ simulations for the lattice irreps A_1^{++}, E^{++}, T_2^{++} and T_1^{+-} are labeled ○, □, ◇, and △, respectively. The corresponding solid symbols indicate the results from the $\xi = 3$ simulations. Data from Wilson action simulations taken from Refs. [19, 22, 20, 23] are shown using crosses. The dashed, dotted, and dash-dotted curves indicate extrapolations to the continuum limit obtained by fitting to the $\xi = 3$ data, the $\xi = 5$ data, and all data, respectively. The solid line indicates the extrapolation of the Wilson action data to the continuum limit.

Fermions (non-chiral) on the Lattice

- Aim: Discretize Dirac eqn $(\not{D} + m)\psi(x) = 0$.
- It's much harder to discretize Fermions than Bosons: Consider (anti-)hermitean first and second order lattice derivatives ($A_\mu \equiv 0$)

$$\nabla_\mu \psi(x) \equiv \frac{1}{2a_\mu}\left[\psi(x+\mu) - \psi(x-\mu)\right]$$

$$\Delta_\mu \phi(x) \equiv \frac{1}{a_\mu^2}\left[\phi(x+\mu) + \phi(x-\mu) - 2\phi(x)\right]$$

- In p-space

$$\nabla_\mu \leftrightarrow i\bar{p}_\mu , \qquad \bar{p}_\mu \equiv \frac{1}{a_\mu}\sin(a_\mu p_\mu)$$

$$\Delta_\mu \leftrightarrow -\hat{p}_\mu^2 , \qquad \hat{p}_\mu \equiv \frac{2}{a_\mu}\sin(a_\mu p_\mu/2)$$

- The effective lattice spacing of $\bar{p}$ is twice that of $\hat{p}$ — ∇_μ **decouples even and odd sites** ($\rightarrow$ doublers). The a^2 error of $\bar{p}$ is $4\times$ that of $\hat{p}$.

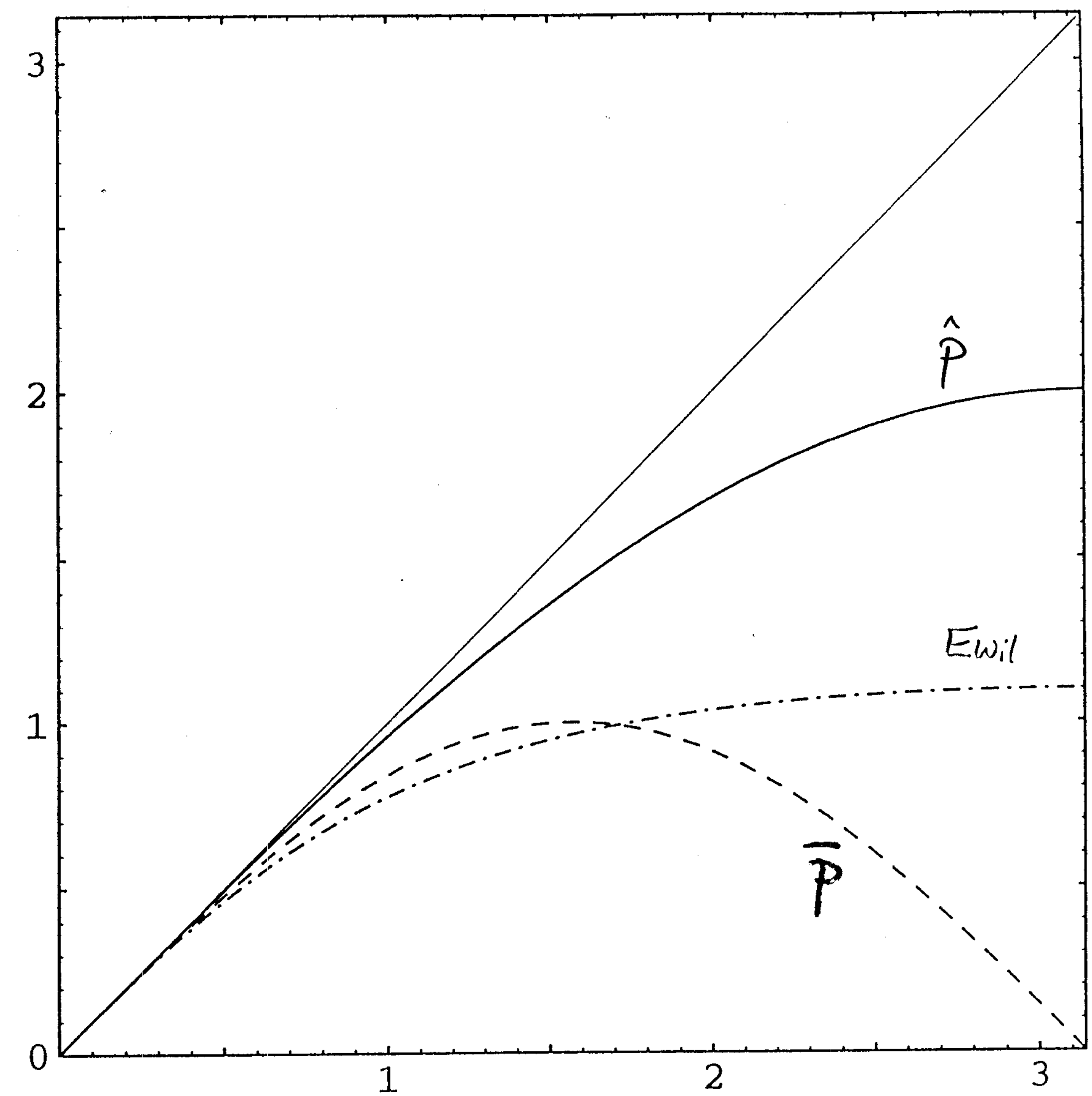


Figure 1: $\hat{p}$ (solid), $\bar{p}$ (dashed) and E_{Wil} (dot-dashed).

Lattice Covariant Derivative Operators

- Conceptual building blocks of our lattice quark actions are the (anti-) hermitean first and second order derivatives:

$$\nabla_\mu \psi(x) \equiv \frac{1}{2a_\mu}\left[U_\mu(x)\,\psi(x+\mu) - U_{-\mu}(x)\,\psi(x-\mu)\right]$$

$$\Delta_\mu \psi(x) \equiv \frac{1}{a_\mu^2}\left[U_\mu(x)\,\psi(x+\mu) + U_{-\mu}(x)\,\psi(x-\mu) - 2\psi(x)\right]$$

- ∇_μ has a "doubler problem".
- Δ_μ does not (it's like a mass term).
  But it does **break chiral symmetry**.

Eliminating Doublers

staggered ...

- Any chirally symmetric action of the form

$$M = \sum_\mu \gamma_\mu \nabla_\mu \left(1 + b_\mu\, a_\mu^2\, \Delta_\mu + d_\mu\, a_\mu^4\, \Delta_\mu^2 + \ldots\right)$$

  will have doublers (Nielsen-Ninomiya).

- Adding a $\sum_\mu \Delta_\mu$ term to the action eliminates them (Wilson), **but:**
  - Adding $\sum_\mu \Delta_\mu$ naively wrecks improvement at $O(a)$.
  - These $O(a)$ errors **break chiral** symmetry.

- Solution:
  - **Classically**: Introduce $\sum_\mu \Delta_\mu$ by a **field transformation.** ①
  - To eliminate **all** $O(a)$ errors tune the quark action **non-perturbatively** via the demand that the **PCAC** relation hold at zero (small) quark masses. ②

} Rest of talk ...

Deriving Doubler-Free Improved Quark Actions

(Alford/TK/Lepage: hep-lat/9611010)

- Start with the continuum action

$$\bar{\psi}_c \, M_c \, \psi_c \;\equiv\; \sum_x \bar{\psi}_c(x) \, (\not{D} \underline{\underline{+}} \, m_c) \, \psi_c(x)$$

- Our "canonical" field transformation eliminating doublers is

$$\begin{aligned}
\psi_c &= \Omega \, \psi \\
\bar{\psi}_c &= \bar{\psi} \, \bar{\Omega} \\
\bar{\psi}_c \, M_c \, \psi_c &= \bar{\psi} \, M_\Omega \, \psi \,, \qquad M_\Omega \equiv \bar{\Omega} \, M_c \, \Omega \\
\bar{\Omega} &= \Omega \,, \quad \bar{\Omega}\,\Omega = 1 - \tfrac{1}{2} \underbrace{r a_0}_{\text{convention}} (\not{D} \underline{\underline{-}} \, m_c)
\end{aligned}$$

- r is a free parameter ("redundant"); will be tuned later.

- This gives the transformed action

$$M_\Omega = \not{D} + m_c - \frac{1}{2}\, r a_0 \,(\not{D}^2 - m_c^2)$$

- Note: $\not{D}^2 = \sum_\mu D_\mu^2 + \frac{1}{2}\sigma\cdot F$ ← $\sum_{\mu\nu} \sigma_{\mu\nu} F_{\mu\nu}$

- To obtain a doubler-free quark action improved to any order, just express D_μ, D_μ^2 and $F_{\mu\nu}$ in terms of lattice quantities to the desired order: (leading classical correction (of rotational symm))

$$D_\mu = \nabla_\mu \left(1 - \frac{a_\mu^2}{6}\,\Delta_\mu + \frac{a_\mu^4}{30}\,\Delta_\mu^2 + \ldots\right)$$ as for glue

$$D_\mu^2 = \Delta_\mu - \frac{a_\mu^2}{12}\,\Delta_\mu^2 + \frac{a_\mu^4}{90}\,\Delta_\mu^3 + \ldots$$ a^3 error only

- Final step: To classically also improve **off-shell** quantities, **undo** the field transformation (now on the lattice).

- Jacobian of field transformation only effects $\mathcal{O}(g^2)$.

Sheikholeslami-Wohlert Action

- Expanding to 0-th order, $D_\mu = \nabla_\mu + \mathcal{O}(a_\mu^2)$, $D_\mu^2 = \Delta_\mu + \mathcal{O}(a_\mu^2)$, gives the lattice action:

$$M_{\mathrm{SW}} \;=\; m_c(1 + \tfrac{1}{2} r a_0 m_c) + \not{\nabla} - \frac{1}{2} r a_0 \left(\sum_\mu \Delta_\mu \;+\; \tfrac{1}{2}\sigma\cdot F \right)$$

- Has $\mathcal{O}(a^2)$ classical errors, e.g. $E(0) = m_c + \mathcal{O}(a^2)$.
- No doublers for any $r > 0$, no ghosts for $r=1$.
- For $F_{\mu\nu}$ we can use the **clover** representation; has $\mathcal{O}(a^2)$ errors.
- To obtain Wilson action, one must by hand set $\sigma\cdot F$ to zero, inducing $\mathcal{O}(a)$ errors.

Set that up earlier

Should write Wilson's

leave out 2

D234 Actions

- Expanding to next order gives:

$$M_{\mathrm{D234}} = m_c(1 + \tfrac{1}{2} r a_0 m_c) + \sum_\mu \gamma_\mu \nabla_\mu (1 - b_\mu a_\mu^2 \Delta_\mu)$$
$$- \frac{1}{2} r a_0 \left(\sum_\mu \Delta_\mu + \tfrac{1}{2} \sigma \cdot F \right) + \sum_\mu c_\mu a_\mu^3 \Delta_\mu^2$$

where

$$b_\mu = \frac{1}{6}, \qquad c_\mu = \frac{r a_0}{24 a_\mu}$$

- Has only $\mathrm{O}(a^4)$ classical errors, if one uses an improved $F_{\mu\nu}$.
- For generic r, there will be three ghost branches.
- At the expense of $\mathrm{O}(a^3)$ errors can arrange for only one ghost.

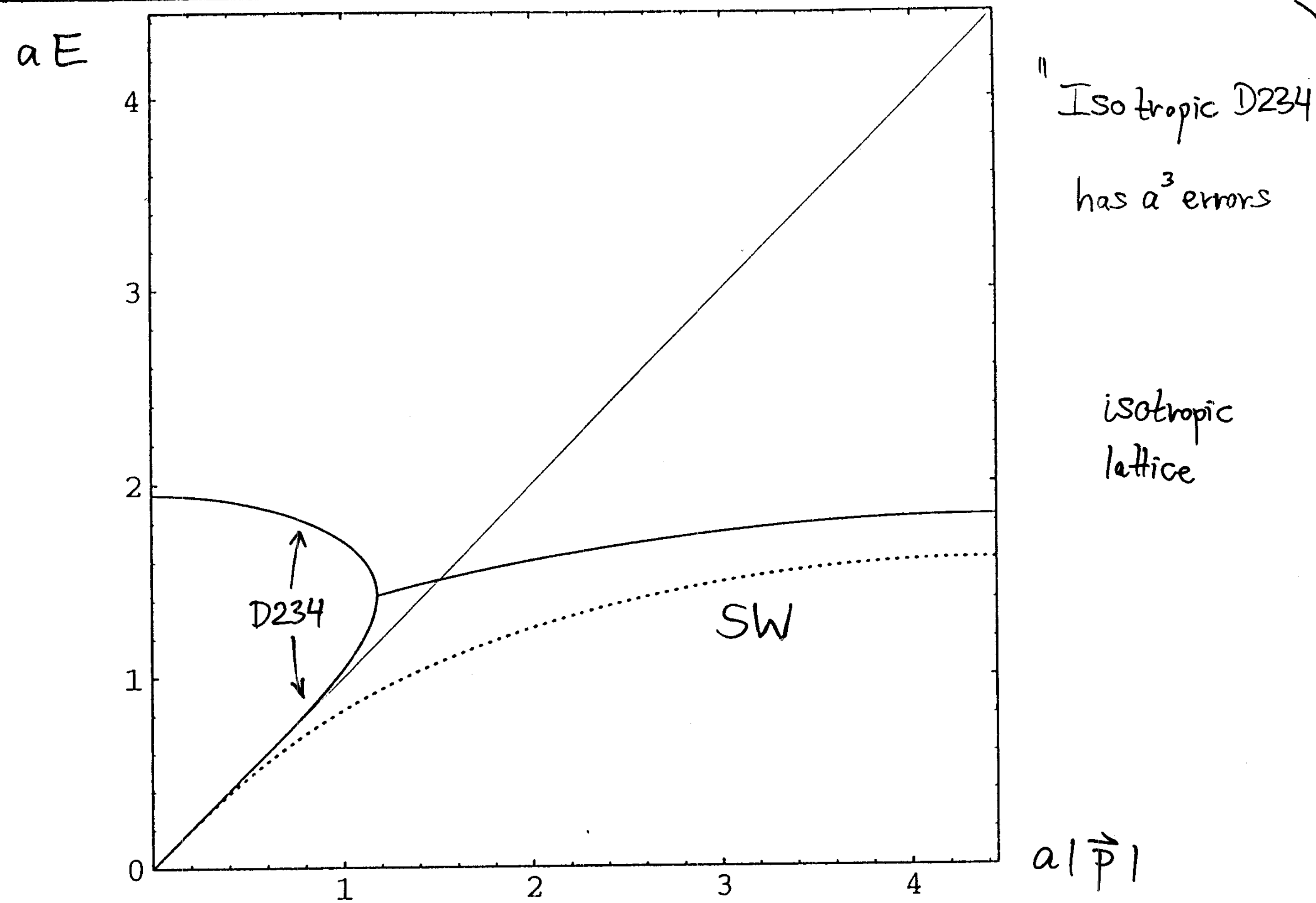


Figure 3: The energy $a_s E(\mathbf{p})$ as a function of $a_s|\mathbf{p}|$, with $\mathbf{p} \propto (1, 1, 0)$, for the massless isotropic D234 (solid) and SW (dotted) actions ($r = 1$). Continuum fermions (thin solid) are shown for comparison.

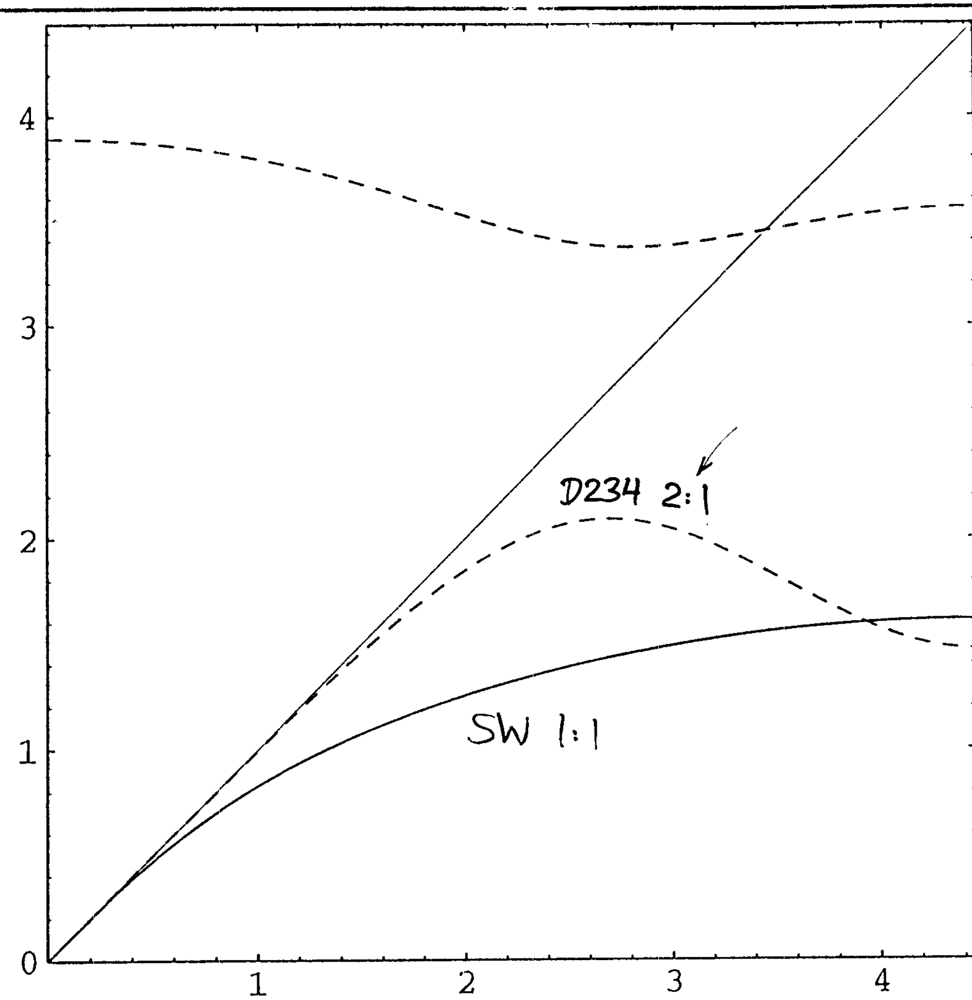
Simulation Results for Dispersion Relations

- For a given hadron define an **"effective velocity of light"**,

$$c(\mathbf{p}): \qquad c(\mathbf{p})^2\mathbf{p}^2 = E(\mathbf{p})^2 - E(0)^2 .$$

- For mesons with $|\mathbf{p}| = 2\pi/aL$, $aL \approx 2.0$ fm at $m_\pi/m_\rho \approx 0.70$ we find on an isotropic lattice with 1-loop and tadpole improved LW glue (Alford/TK/Lepage '95):

		$c^2(\vec{p})$ D234 Action		$c^2(\vec{p})$ SW Action	
β_{LW}	a (fm)	π	ρ	π	ρ
6.8	0.40	0.95(2)	0.93(3)	0.63(2)	0.48(3)
7.1	0.33	0.94(3)	0.96(5)	0.74(3)	0.55(4)
7.4	0.24	0.99(4)	1.00(6)	0.88(2)	0.73(3)

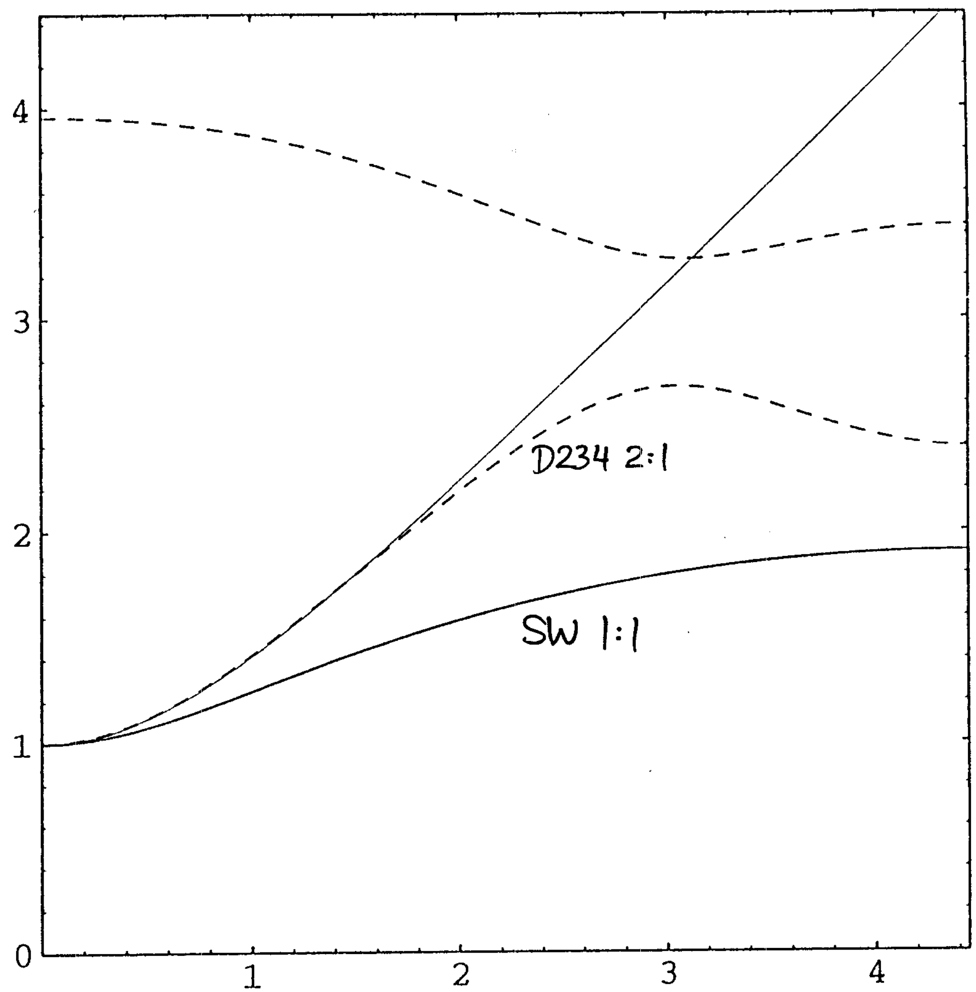


"Anisotropic D234"

has $\mathcal{O}(a_0^3, a^4)$ errors

Figure 4: The energy $a_s E(\mathbf{p})$ as a function of $a_s|\mathbf{p}|$, with $\mathbf{p} \propto (1,1,0)$, for the massless, free SW action ($r = 1$) on a 1:1 lattice (solid), and the D234($\frac{2}{3}$) action on a 2:1 lattice (dashed).

here or later ?



"Anisotropic D234"
massive case
$a_s m = 1$

Figure 5: The energy $a_s E(\mathbf{p})$ as a function of $a_s|\mathbf{p}|$, with $\mathbf{p} \propto (1,1,0)$, for the massive, free SW action ($r = 1$) on a 1:1 lattice (solid), and the D234($\frac{2}{3}$) action on a 2:1 lattice (dashed).

(for details → Alford/TK/Lepage: hep-lat/9611010)
Replace this with D234i($\frac{2}{3}$) ?
here, or later?

"Dispersion Relations at charm mass"

(Alford/TK/Lepage: Lattice '96, hep-lat/9608113)

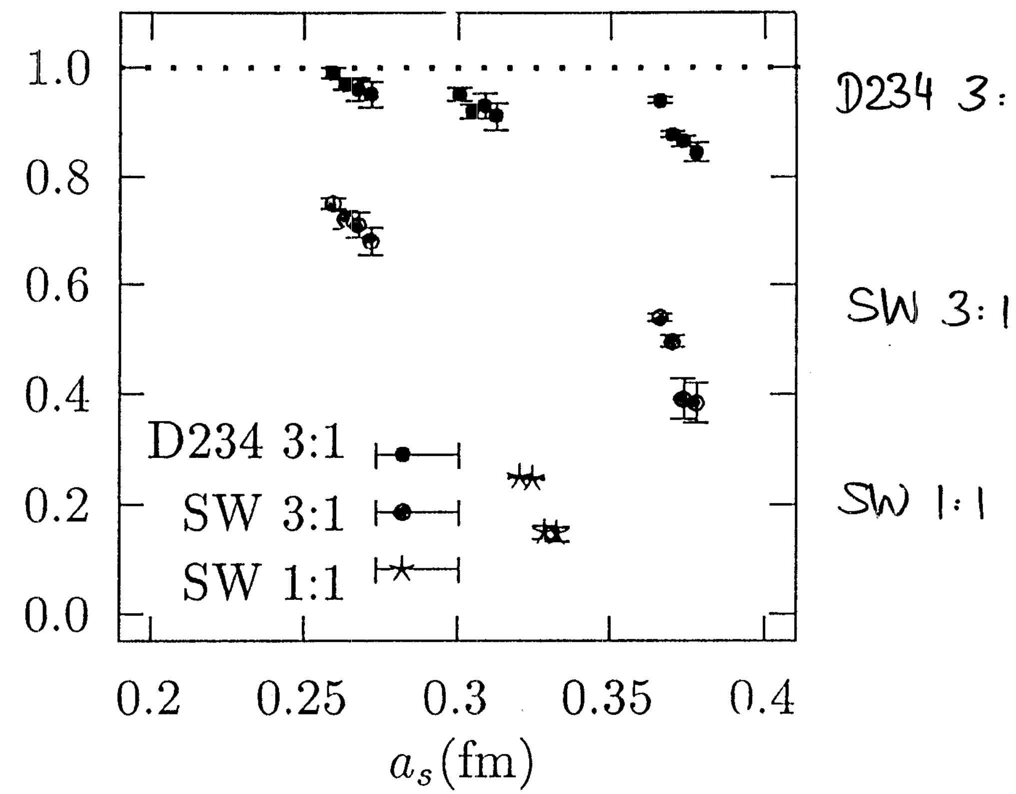


Figure 7: $c(\mathbf{p})$ of various hadrons for the D234($\frac{2}{3}$) and SW actions on various lattices at the charm mass, $|\mathbf{p}| \approx 670\,\mathrm{MeV}$. Each quadruplet represents, from left to right, the η_c ("pion"), J/ψ ("rho"), "nucleon" and "delta".

at end?

Quantum Improved Quark Actions

- Gluons: Errors are $O(a^2, a^4, \ldots)$, **no** $O(a)$.
- Wilson-type Quarks: Have $O(a)$ and $O(a^2)$ errors, which are qualitatively **very different**:
  - $O(a)$ errors break **chiral** but not rotational symmetry.
  - Leading $O(a^2)$ break **rotational** but not chiral symmetry.
- MC data show: The **only** quantities that have strong scaling violations on coarse lattices with both SW and D234 tadpole improved actions are those that depend strongly on $O(a)$ term: $m_{\rho,\phi}/\sqrt{\sigma}$, HFS ! ($\rightarrow$ `hep-lat/9612005`, Bielefeld '96).
- Need **non-perturbative** method to tune $O(a)$ term!

O(a)-Tuning of Wilson-type Quarks

- Write the O(a) terms of an **isotropic** action as

$$-\frac{ar}{2}\left(\sum_{\mu}\Delta_{\mu}+\frac{\omega}{2}\sigma\cdot F\right) \qquad (\underline{c_{SW} = r\,\omega})$$

- The coefficient r, say, can be adjusted at will by a field transformation ("redundancy"). can be ignored for on-shell improvement
- Classically, $\omega = 1$, independent of r.
- To eliminate O(a) quantum errors, ω must be tuned.
- How?

Quark Actions on Anisotropic Lattices

An $O(a)$ on-shell improved Wilson-type quark action can always be written as a discretization of an effective continuum actions of the form:

$$\bar{\psi}(x)\left[m_0 + D_0 + c\not{\mathbf{D}} - \frac{1}{2} r a_0 \left(\sum_\mu D_\mu^2 + \omega_0 \sum_k \sigma_{0k} F_{0k} + \omega \sum_{k<l} \sigma_{kl} F_{kl}\right)\right]\psi(x)$$

Proof: Using field redefinition at $O(a_0, a)$ can eliminate $[\not{\mathbf{D}}, D_0]$ and $D_0^2 - \mathbf{D}^2$.

- Parameters to be tuned at quantum level (all 1 classically):
  - c = bare velocity of light
  - ω_0 = temporal clover coefficient
  - ω = spatial clover coefficient
- Isotropic lattice: $c = 1$, only $\omega \equiv \omega_0$ has to be tuned (or $c_{SW} \equiv r\omega$).
- Tune c by demanding **renormalized** velocity of light $= 1$.
- Tune ω_0, ω using a background electric, resp, magnetic field.

details to be worked out. not completely trivial but doable...

In following restrict to isotropic ....

Tim Klassen

Chiral Symmetry and Non-Perturbative Tuning

- Idea (Lüscher et al, ALPHA collab): The **PCAC** relation

$$\partial_\mu A_\mu^b = 2mP^b$$

for the pseudo-scalar density and the iso-vector axial current

$$\begin{aligned} P^b(x) &\propto \bar{\psi}(x)\gamma_5 \tfrac{1}{2}\tau^b \psi(x) \qquad \tau: \text{ flavor matrices} \\ A_\mu^b(x) &\propto \bar{\psi}(x)\gamma_\mu\gamma_5 \tfrac{1}{2}\tau^b \psi(x) \;+\; a\, \underline{\underline{c_A}}\, \partial_\mu P^b(x)\,. \end{aligned}$$

will hold up to $O(a^2)$ errors for small quark masses **only** if chiral symmetry is restored.

- Tune ω and c_A by demanding the (unrenormalized) mass m defined by matrix elements between states to be **independent of states and kinematic parameters** (x, BCs, volume, etc).

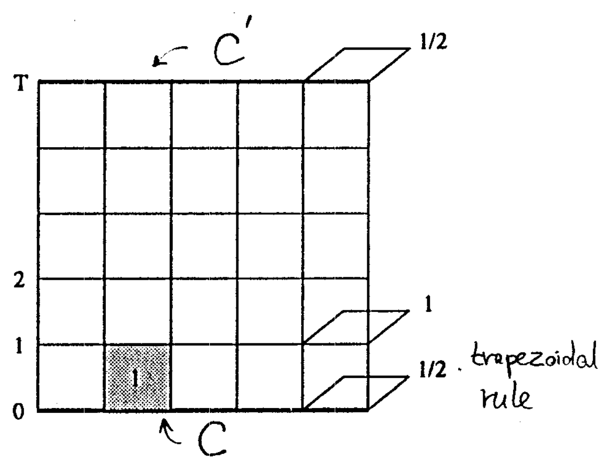
- ω and c_A have $O(a)$ ambiguity. Instead of assigning error to tuned parameters due to different improvement conditions, choose a **specific**, **"reasonable"** one: Errors are **guaranteed** to extrapolate away in the continuum limit!

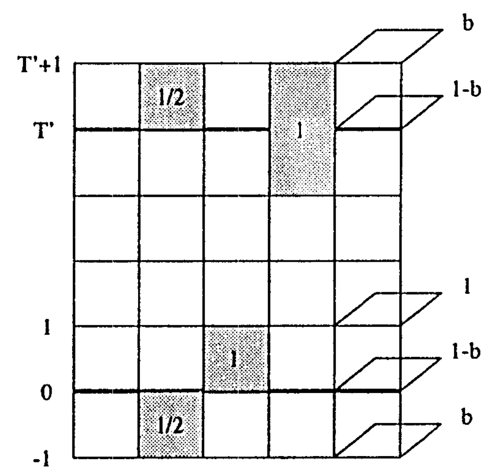
Why we need the Schrödinger Functional

- Need to work at zero (small) quark masses.
- Impossible in practice with periodic boundary conditions (BCs), due to zero modes.
- Impose **fixed** BCs in time direction (say) $\Rightarrow$ no zero modes!
- Furthermore, fixed BCs allow us to generate a **background** color(-electric) **field** that couples to $\sigma \cdot F$ (even classically).
- SF = quantum/lattice field theory with Dirichlet BCs.

The Schrödinger Functional on the Lattice

- Have to figure out how to impose fixed BCs on the lattice, in particular for improved actions. For details see `hep-lat/9705025`; here we just give a flavor of the issues:
- **Gauge fields**: Modify coefficients of loops in action at boundary:





- **Fermions**: Can define **upper** and **lower boundary fields** as functional derivatives (within the path integral) with respect to the boundary values of the fermion fields.

Matrix Elements of the PCAC Relation

- Boundary fields are very useful as **"test states"**.
- Using suitable matrix elements with **upper** and **lower** boundary fields (and averaging over space), one obtains **two different** estimates of the current mass, $M(x_0)$ and $M(x_0')$, respectively, in addition to an estimate of c_A.
- Remember: Due to the background field there is an **asymmetry** between the upper and lower parts of the "universe" — which is of order a^2 (classical/quantum for Wilson/improved) if we have properly improved the action and axial current.
- Impose the **improvement condition**

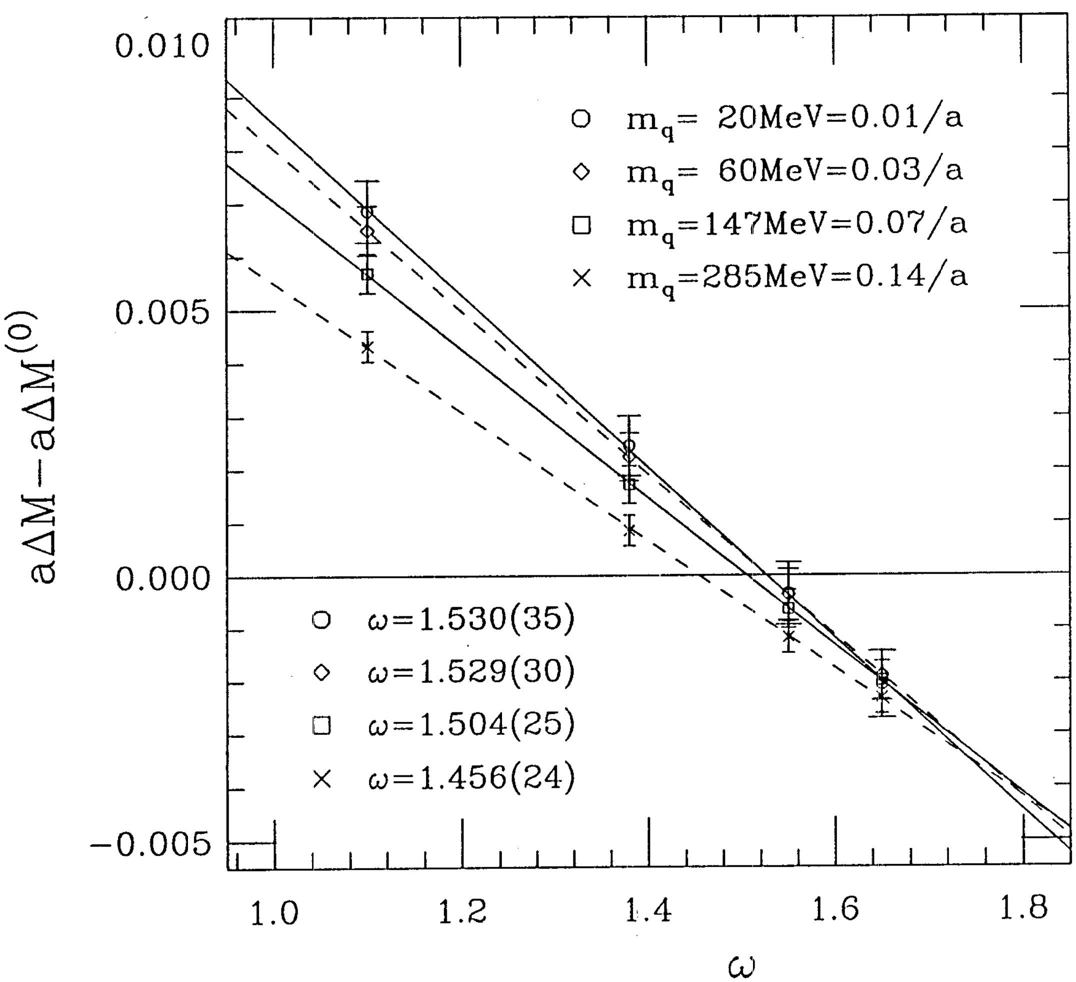
$$\Delta M(x_0) \equiv M(x_0) - M'(x_0) \stackrel{!}{=} \text{tree level}$$

for suitable x_0 to determine ω.

$O(a)$ Tuning of ω in Practice

- At SCRI (Edwards/Heller/TK) investigated so far: **SW quarks** on isotropic Wilson or improved glue.
- Choose appropriate BCs, volume, and x_0, y_0 in improvement condition $\Delta M(x_0, y_0) \stackrel{!}{=}$ tree level. $= O(a^2) =$ tiny
- For various fixed ω measure ΔM.
- Due to problems with **exceptional configurations** at small quark masses and large ω (for coarse lattices), it is important to study:
  - How large a quark mass can be used?
  - How linear is ΔM as a function of ω?

O(a) tuning of SW on β=8.4 LW glue



0.1 fm

Now to actual results...

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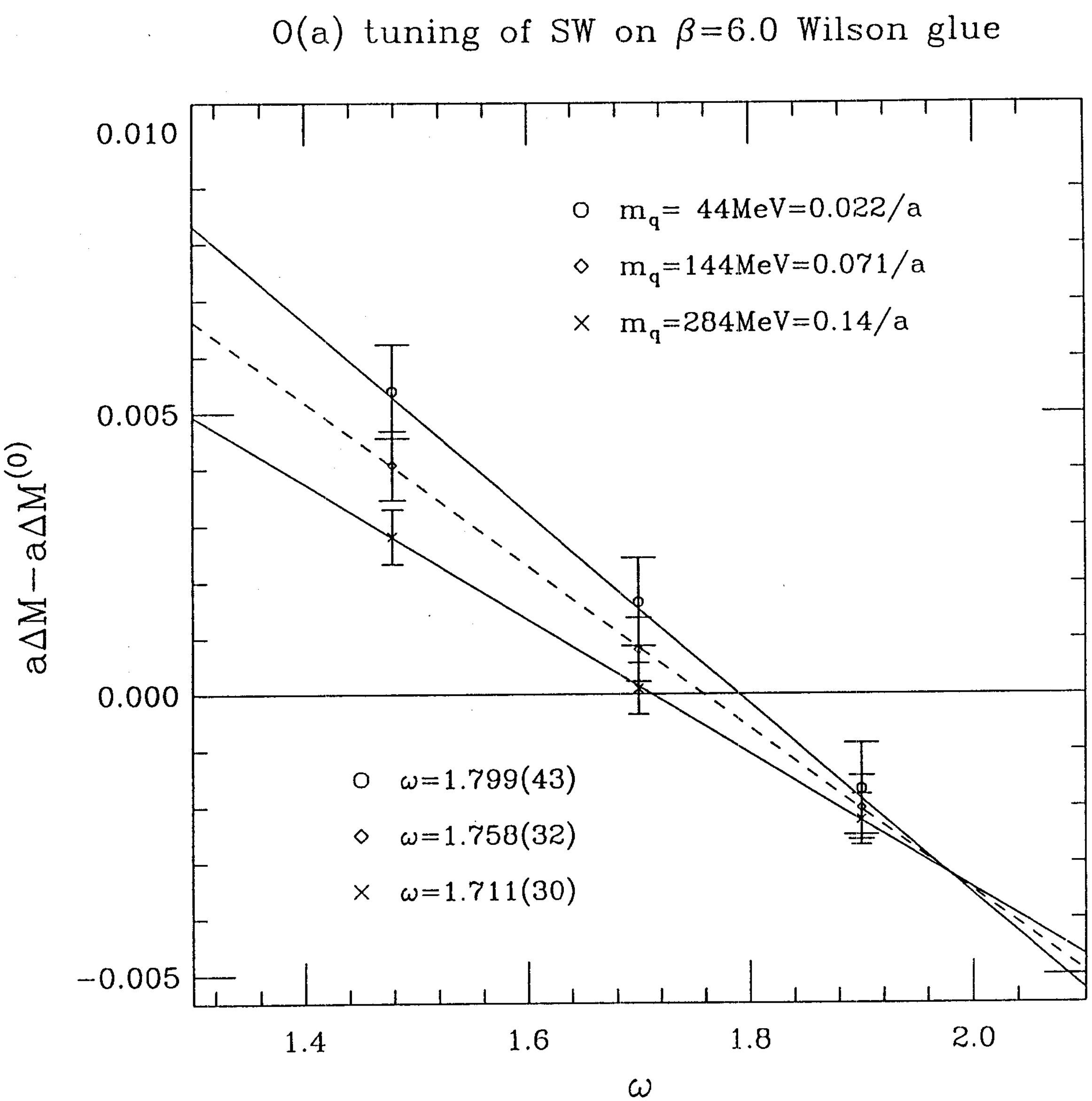
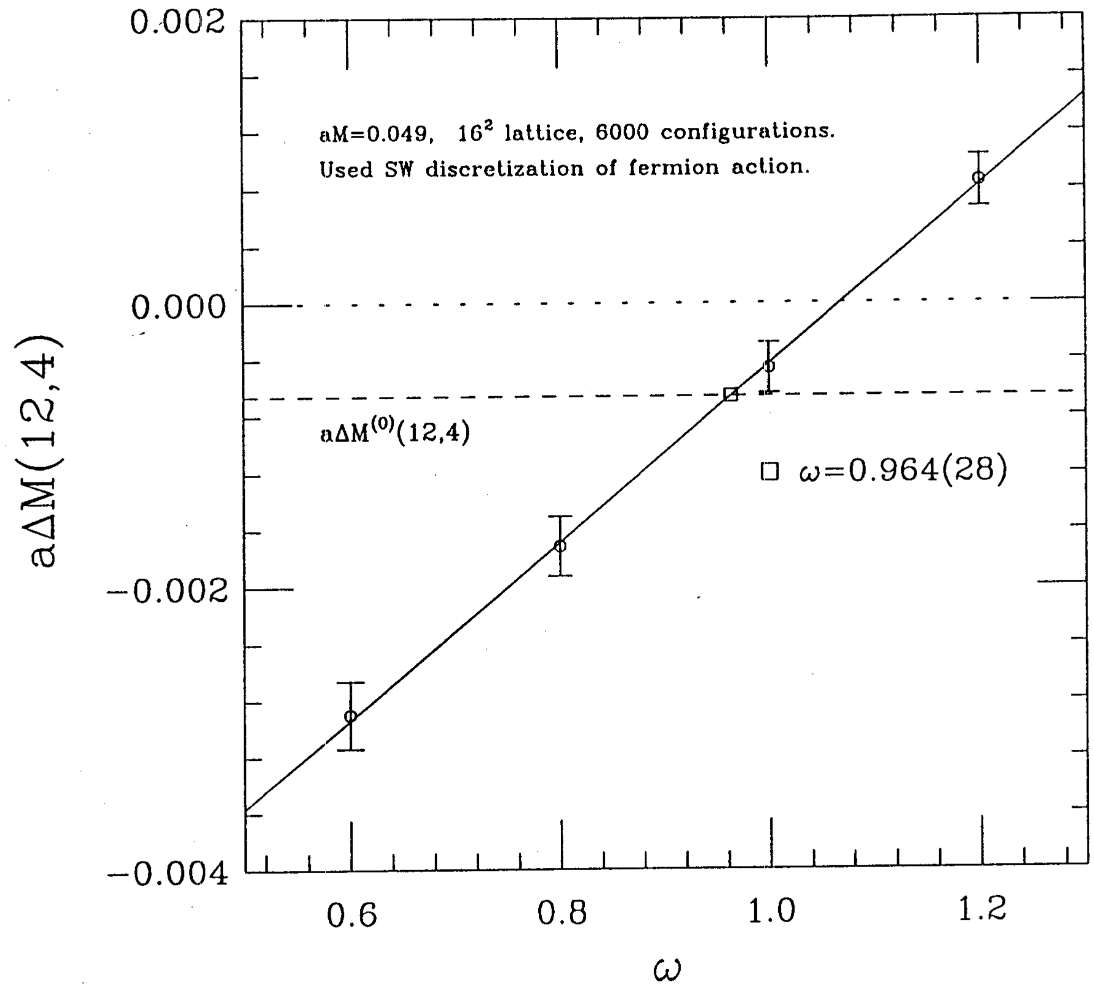
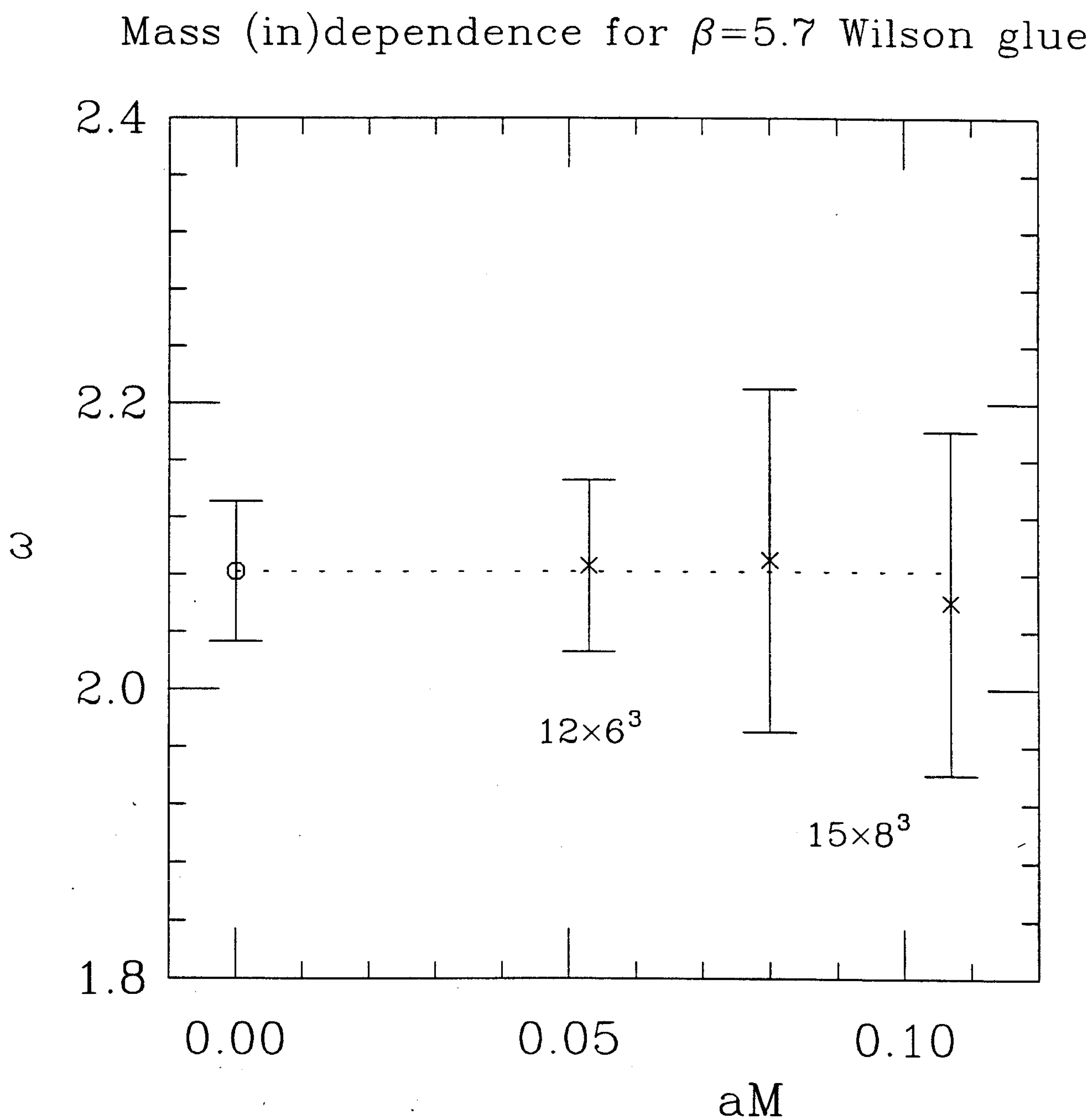
SCRI, Edwards/Heller/TK: to appear!



O(a) tuning of SW on β=6.0 Wilson glue
0.010
0.005
0.000
−0.005
aΔM−aΔM(0)
1.4
1.6
1.8
2.0
ω
○ m_q= 44MeV=0.022/a
◇ m_q=144MeV=0.071/a
× m_q=284MeV=0.14/a
○ ω=1.799(43)
◇ ω=1.758(32)
× ω=1.711(30)

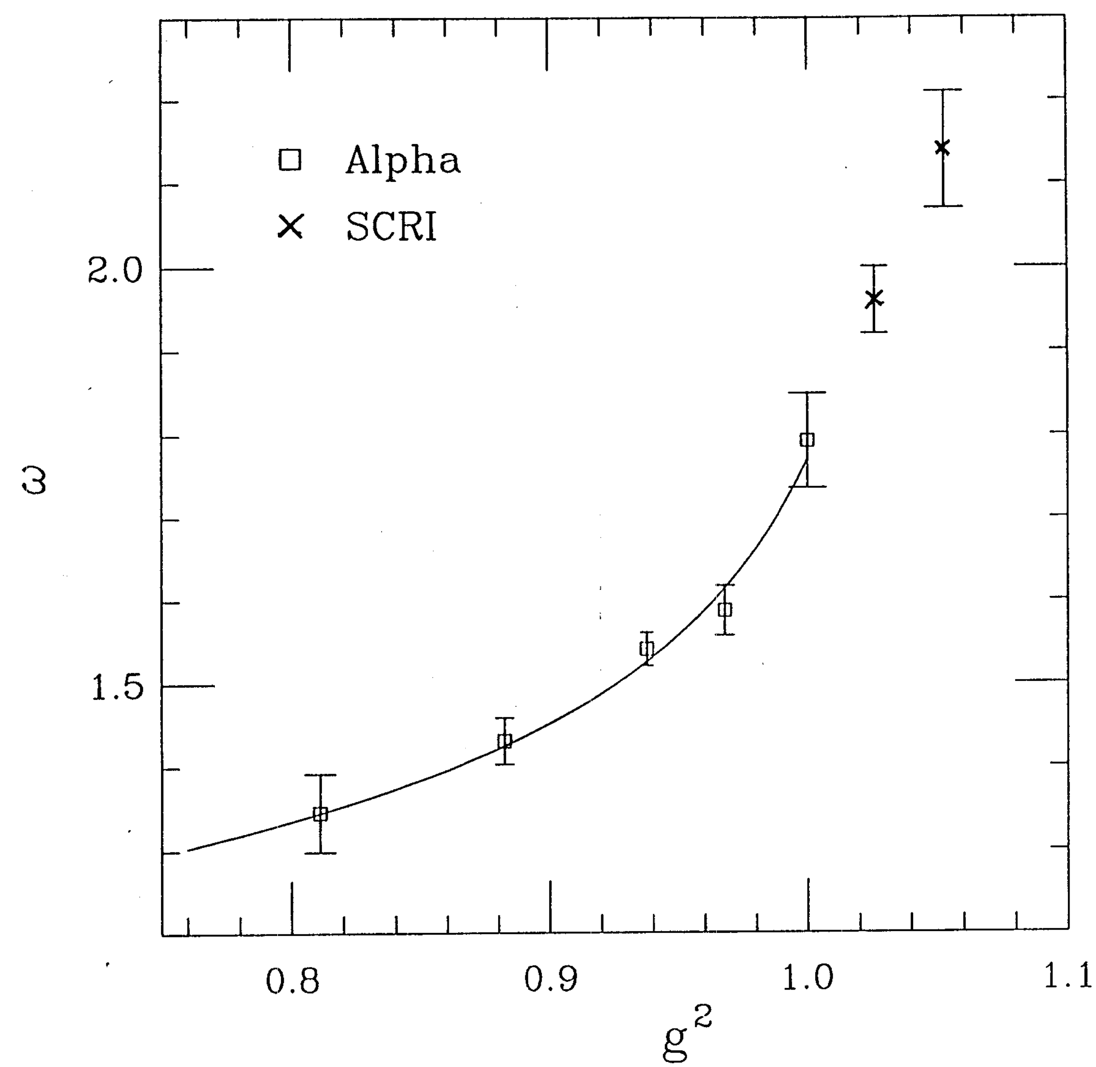
O(a) tuning of quenched QED_2 on $\beta=10$ Wilson glue



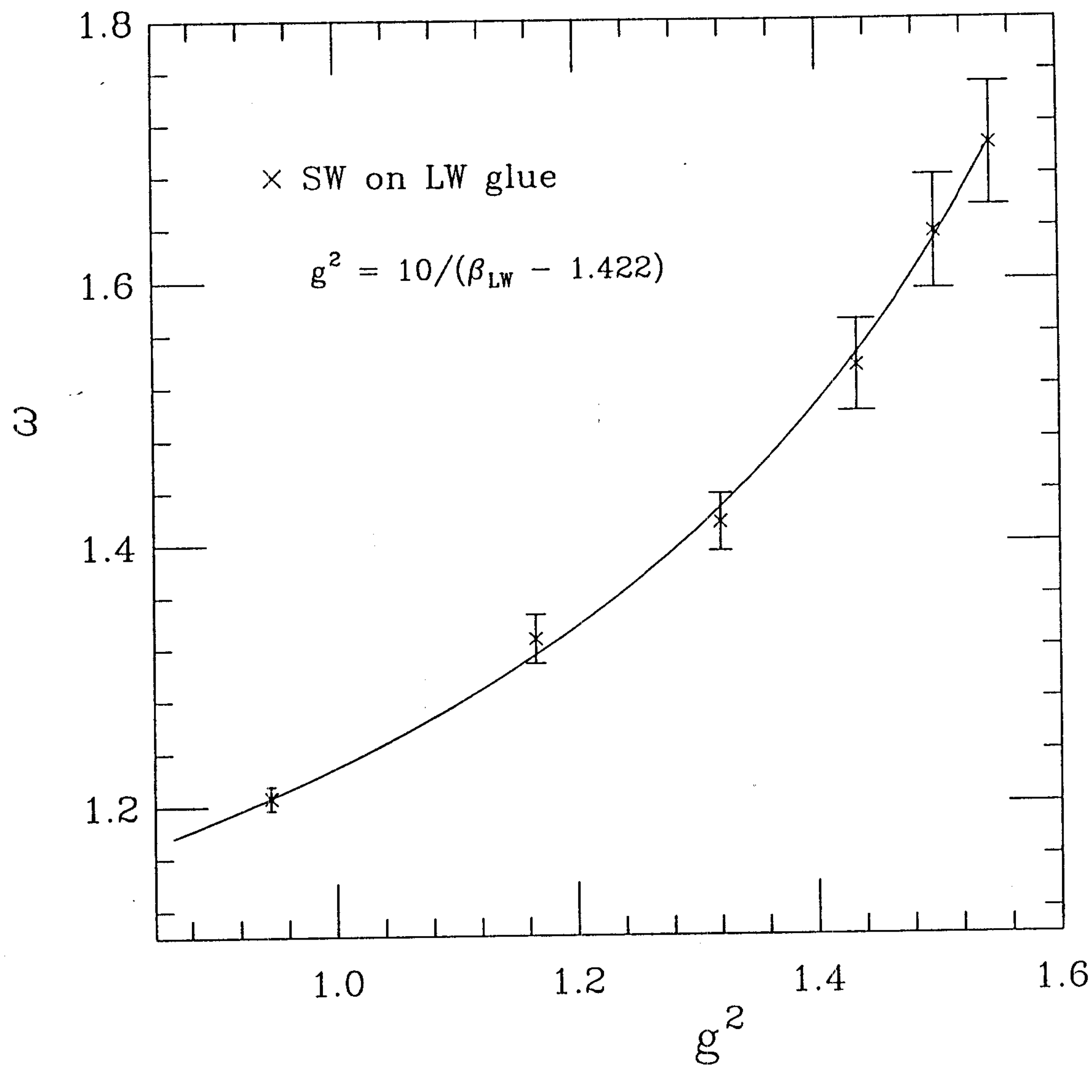


Mass (in)dependence for β=5.7 Wilson glue
2.4
2.2
2.0
1.8
3
12×6³
15×8³
0.00
0.05
0.10
aM

Non–perturbative clover coefficient for Wilson glue



Non-perturbative clover coefficient for LW glue



Fitting Improvement Coefficients as Function of Coupling

- To parameterize numerical results as a smooth curve Pade fits of the form

$$c\,\frac{1 + c_1\, g^2 + \ldots}{1 - c_0\, g^2}$$

work very well.

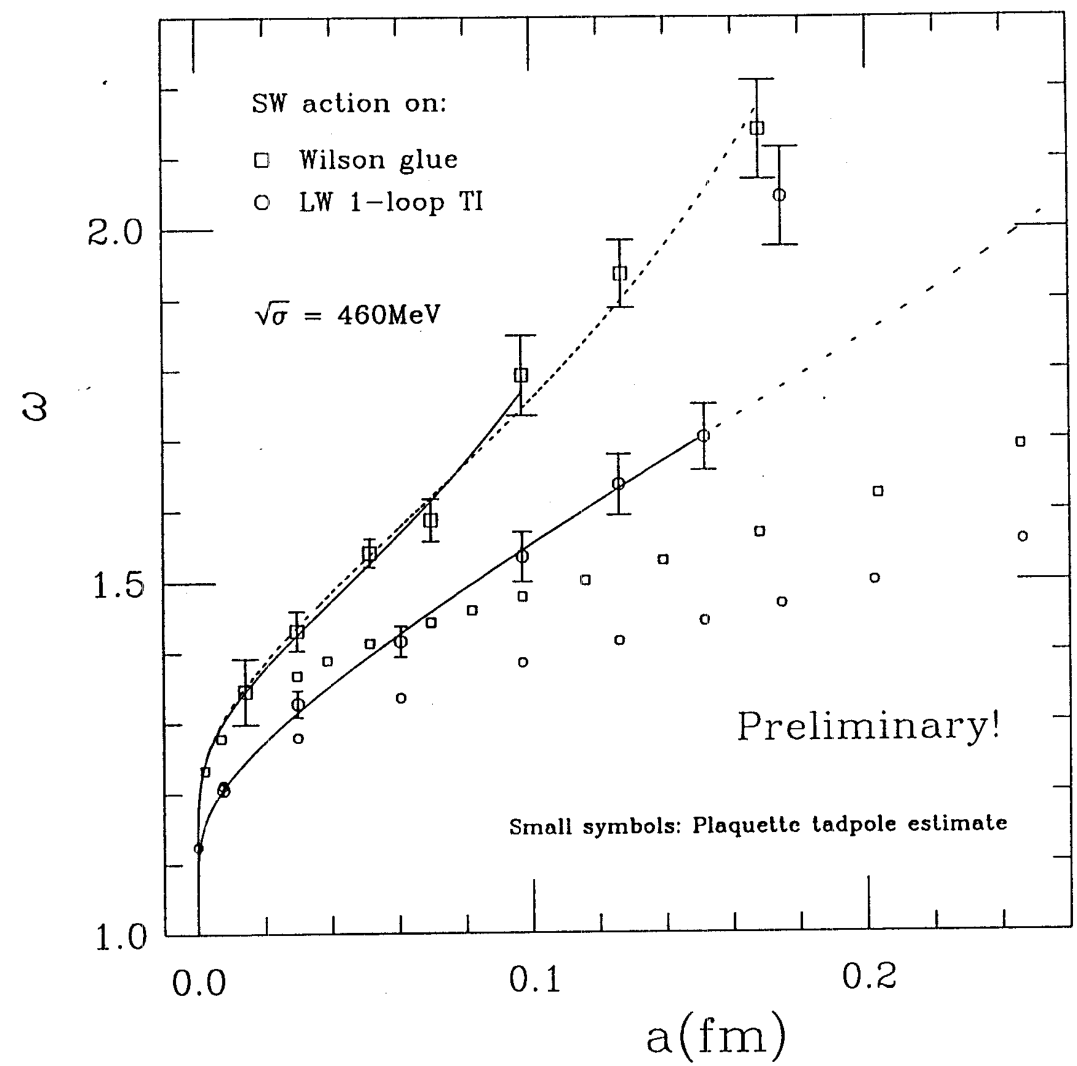
- Build in perturbative coefficients, if known (presently only for SW + Wilson glue).

- For improved glue we find excellent representations with

$$\omega(g^2) = \frac{1 - c_1\, g^2}{1 - c_0\, g^2}$$

$$c_A(g^2) = \frac{c\, g^2}{1 - c_0\, g^2}\,.$$

Non-perturbative and tadpole clover coefficients



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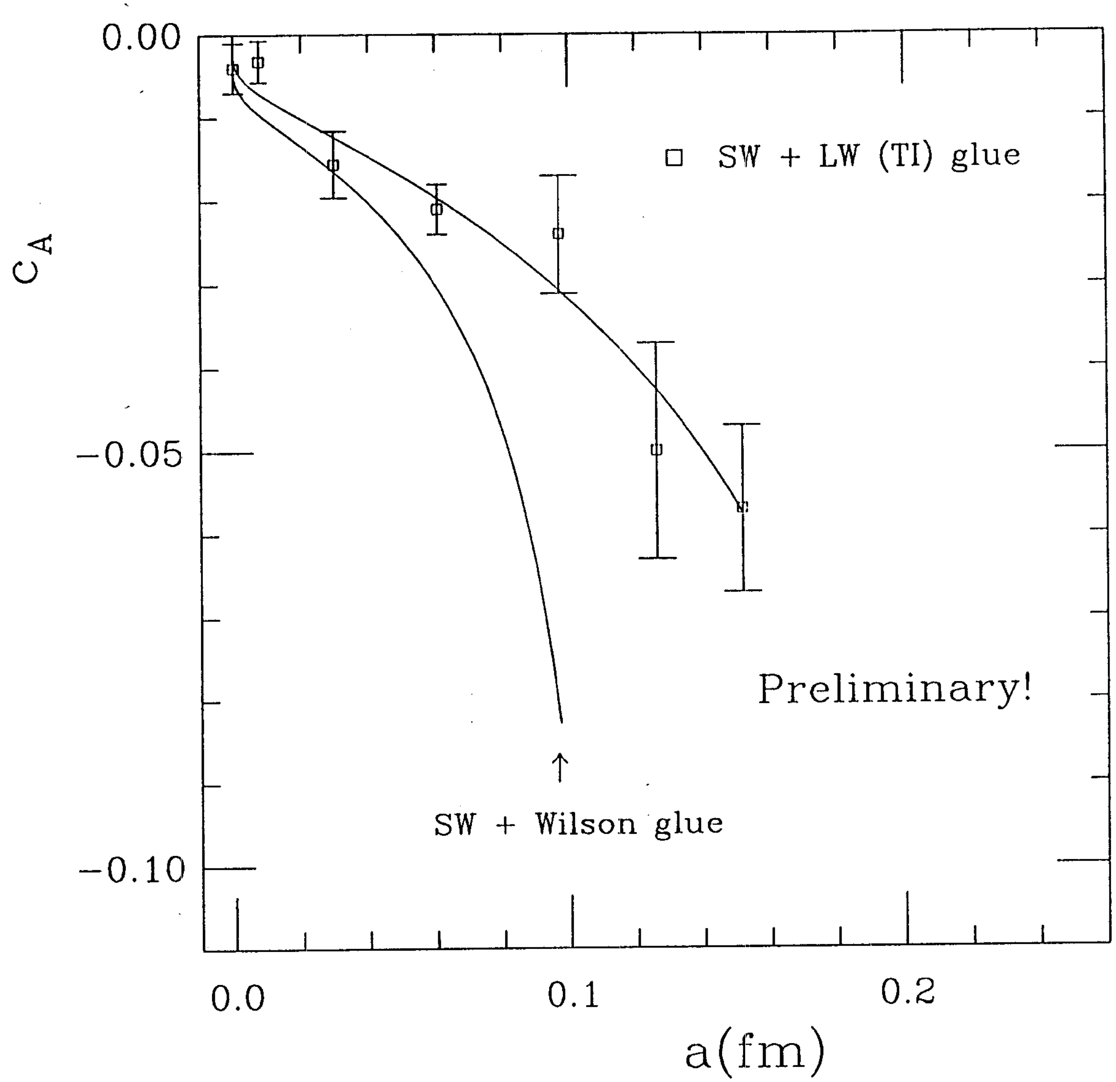
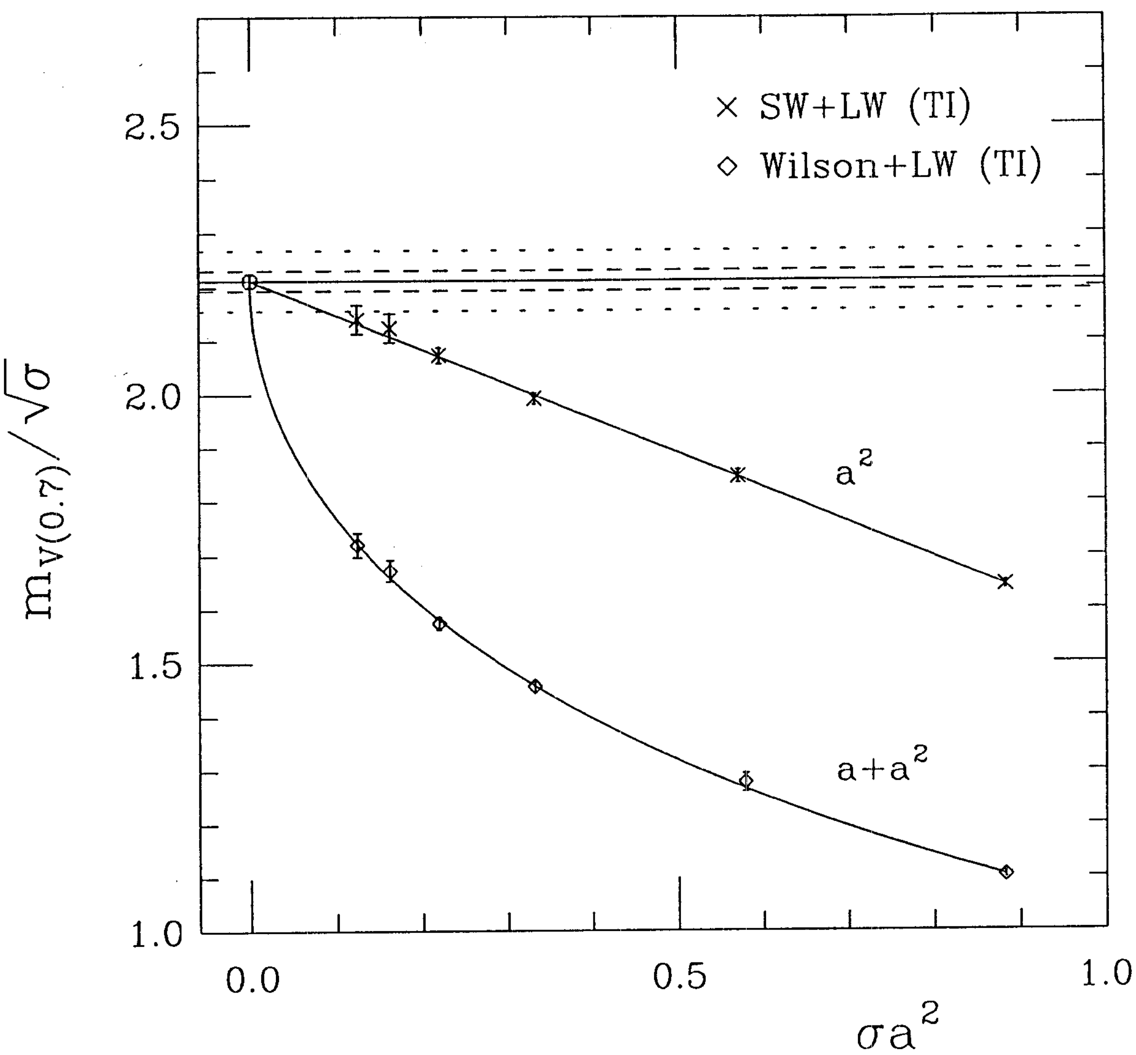
Axial current improvement coefficient c_A



The vector meson mass at fixed $m_P/m_V=0.7$

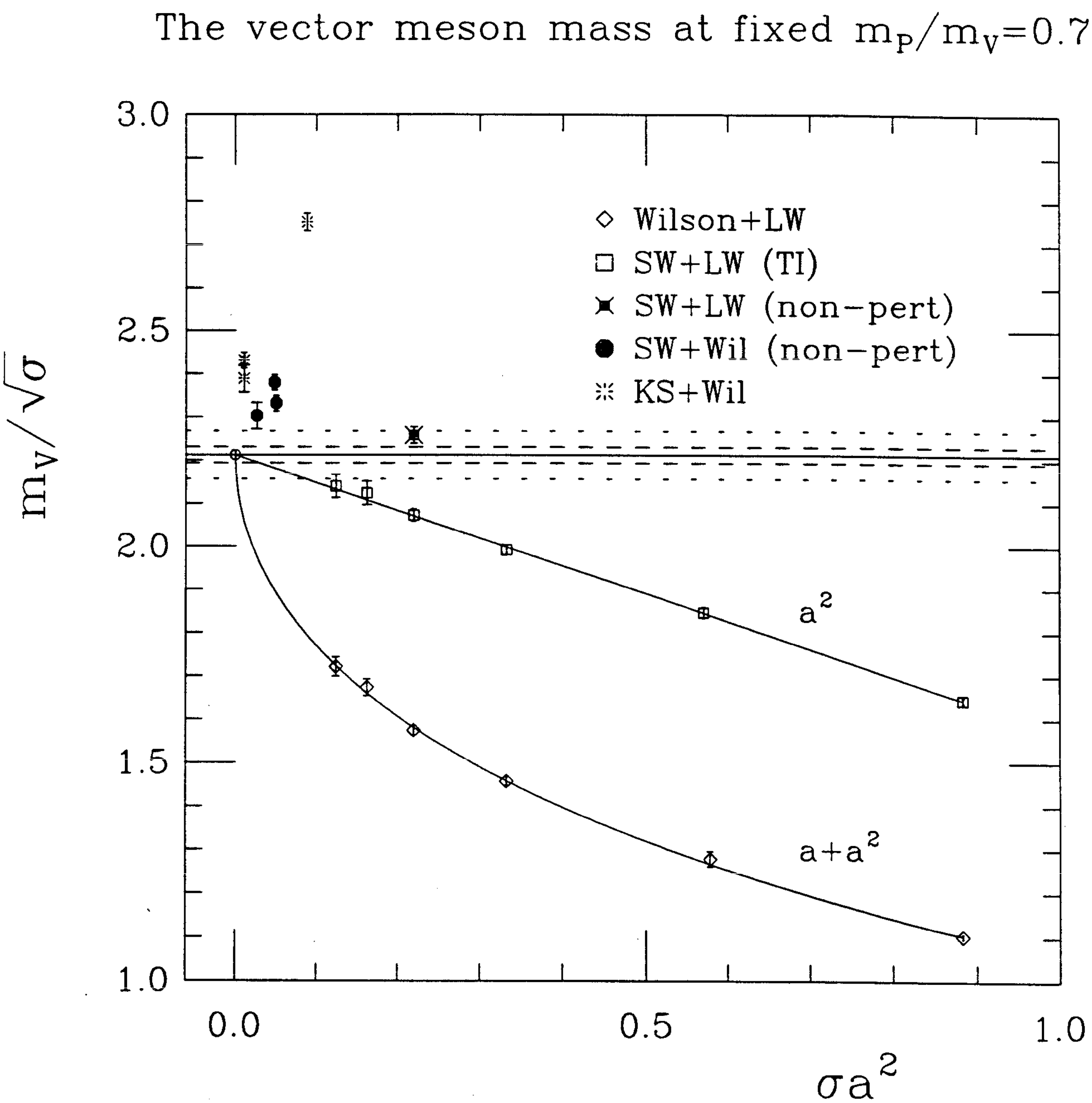


Collins
Edwards
Heller
Sloan

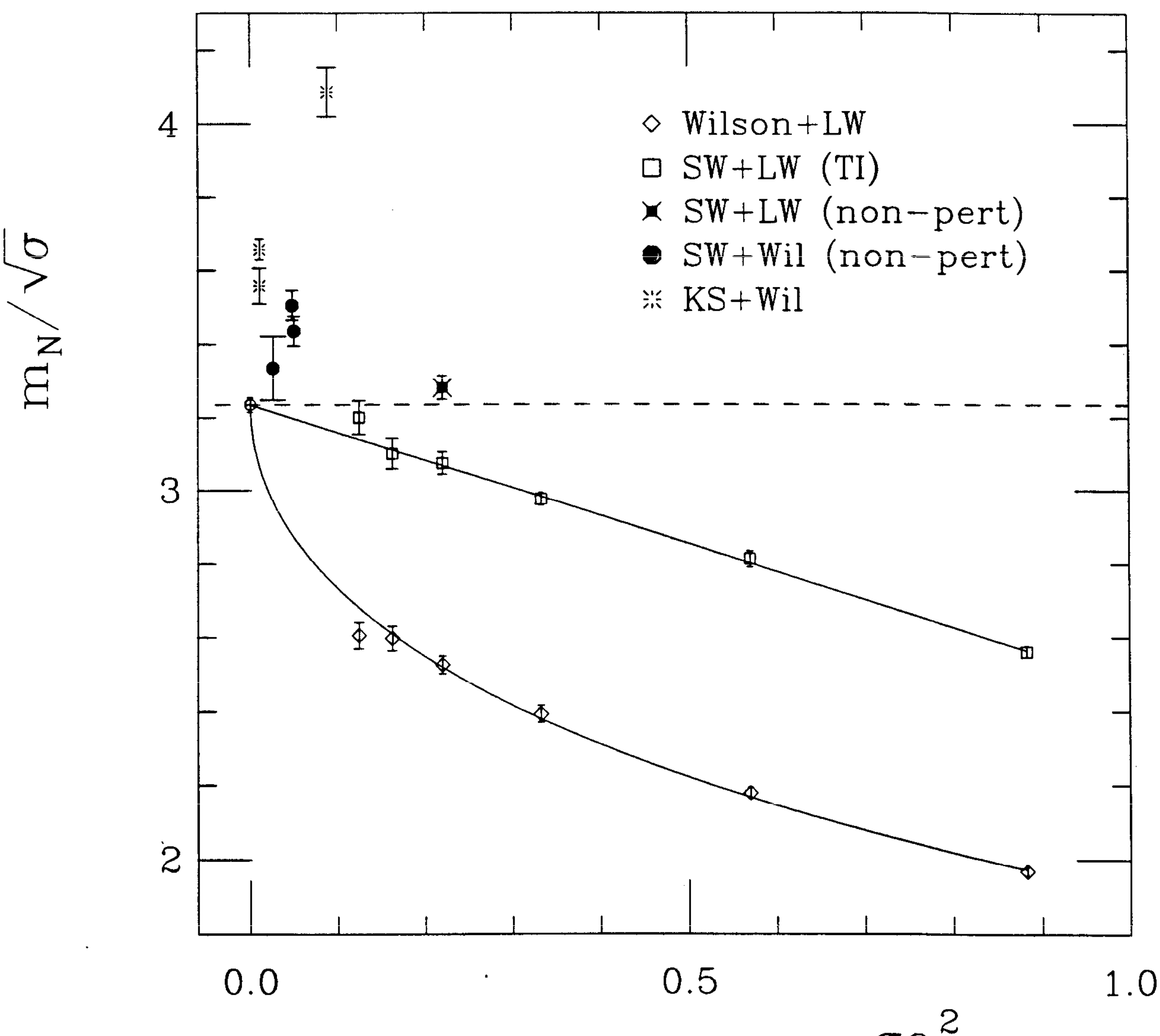
Tim Klassen

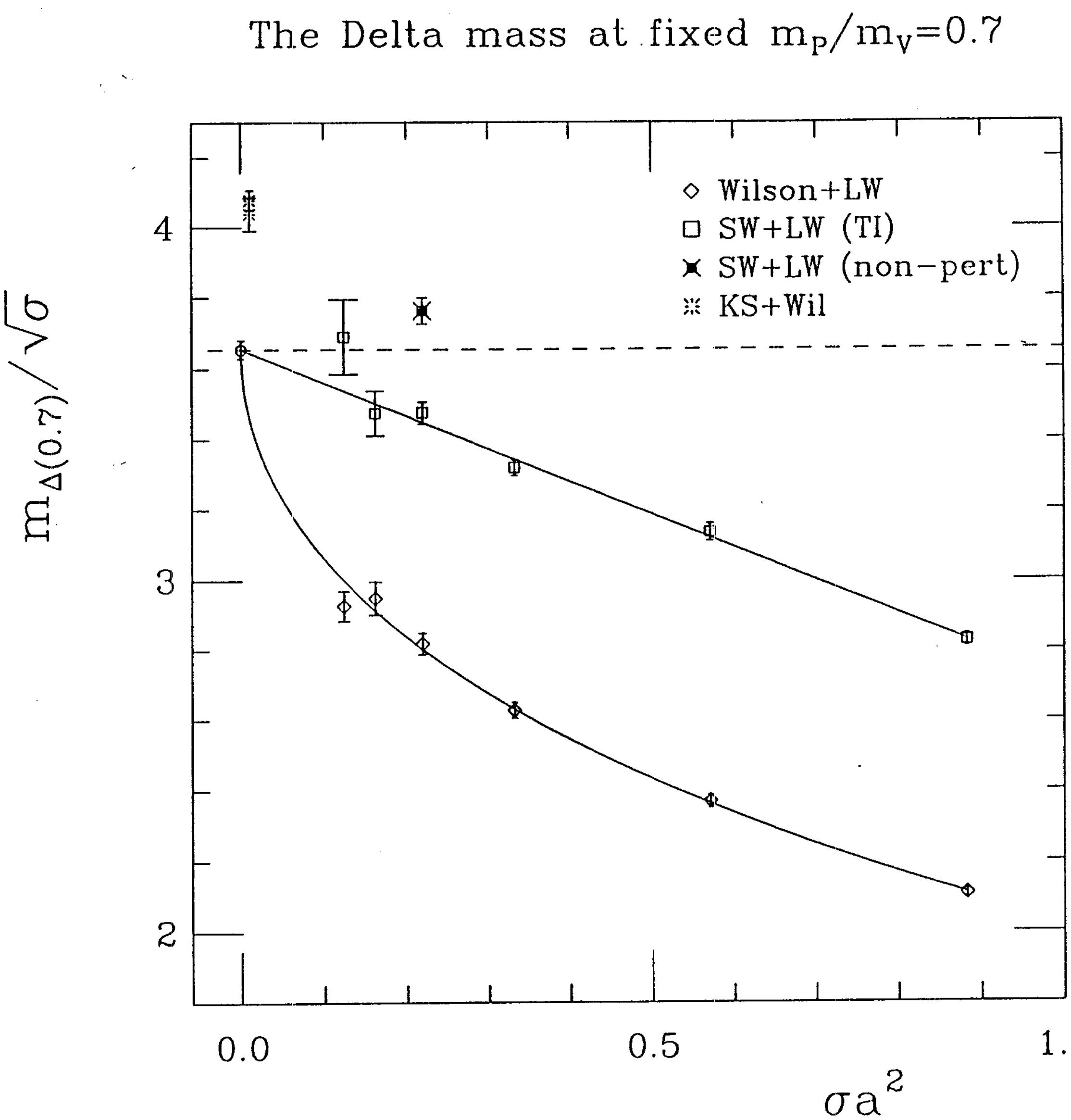
amazing!

SCRI


The vector meson mass at fixed m_P/m_V=0.7
3.0
2.5
2.0
1.5
1.0
0.0
0.5
1.0
$m_V/\sqrt{\sigma}$
σa^2
Wilson+LW
SW+LW (TI)
SW+LW (non-pert)
SW+Wil (non-pert)
KS+Wil
a^2
$a+a^2$

The nucleon mass at fixed $m_P/m_V=0.7$




The Delta mass at fixed m_P/m_V=0.7
$m_{\Delta(0.7)}/\sqrt{\sigma}$
σa^2
4
3
2
0.0
0.5
1.0
Wilson+LW
SW+LW (TI)
SW+LW (non−pert)
KS+Wil

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Conclusions and Outlook

Several major developments have revitalized lattice QCD:

- Rotational errors at $O(a^2)$ are relatively easy to eliminate.
  - Were leading errors for gauge fields; can now use "coarse glue".
- For Wilson-type fermion actions, the chiral symmetry violations at $O(a)$ can now be eliminated non-perturbatively, using the SF.
  - Done so far for SW on Wilson and improved glue (almost finished).
  - Improved glue will allow the use of coarser lattices (smaller ω, c_A).
- Major parts of quenched QCD should be **"solved"** within a year.
  - Calculate Z_A, Z_V, c_V etc yet for improved glue.
  - Further work on 4-fermi currents is needed for weak matrix elements.
- Can finally attack **"big"** problems:
  - Full QCD (Wilson started, do so for improved glue, tune ω, c_A, ...).
  - Heavy quarks, glueballs, hybrids (use **anisotropic** lattices, D234).