Lutsen 97

Coupled Cluster Formulation
of
Lattice Hamiltonian QCD
and
Maximum Momentum Eigenstates

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Collaboration:

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Many-body structure of Lattice-Hamiltonian QCD

Maximum momentum eigenstates:

motivation: Kröger + Schen

Survey :

- Standard many-body theory
- Expectation for quantum field theory
- Lattice regularization and rest symmetry
- Coupled cluster approach
- Application to Lattice Hamiltonian QCD
- challenges
  including maximum momentum eig. st.

Structure of non-rel. many-body theory

$$\psi(\vec{x}_1 \ldots \vec{x}_N)$$

$$H = \sum \frac{p_k^2}{2m_k} + \sum V_{kl}(|x_k - x_l|)$$

Galilei-invariance

$$\Rightarrow \quad H \; \vec{P} \; \vec{L} \; \vec{B} \qquad \text{ray-rep } \Omega \text{ of Gal-gr.}$$

Structure of theory $\Leftrightarrow$ Spectrum $H, \vec{P}, \vec{L}$

($\vec{B}$ automatic)

Bound states

$$\mathcal{H}^{M\delta j} = \{\psi_{\vec{k},m}\} \qquad \begin{array}{l} \vec{k} \in \mathbb{R}^3 \\ -j \le m \le j \end{array}$$

$$H \psi_{\vec{k}m} = \left(\frac{k^2}{2M} + \delta\right) \psi_{\vec{k}m} = E \psi_{\vec{k}m}$$

$$\vec{p}\, \psi_{\vec{k}m} = \vec{k}\, \psi_{\vec{k}m}$$

$$\Omega(R)\psi_{\vec{k}m} = D^j_{mm'}(R)\, \psi_{R\vec{k},m'}$$

$$\psi_{\vec{k}m} = e^{i\vec{k}\vec{x}_{cm}} \varphi_m(\vec{x}_{rel})$$

finite range in $\vec{x}_{rel}$

$$\Omega\big|_{\mathcal{H}^{M\delta j}} = \Omega_{M\delta j} \qquad \text{irred. rep.}$$

(Mass, Binding, Spin)

2 - channel scattering states

ψ^1, ψ^2 bound. states

$$\Rightarrow \quad \Psi^{\pm} = \lim_{\varepsilon \to 0^+} \frac{\pm i \varepsilon}{E - H \pm i\varepsilon} \psi^1 \otimes \psi^2$$

in Fockspace

$$H \Psi^{\pm} = E \psi^{\pm} = (E_1 + E_2) \psi^{\pm}$$

$$\vec{p} \psi^{\pm} = (\vec{k}_1 + \vec{k}_2) \psi^{\pm}$$

S - matrix $\Leftrightarrow$ $\langle \psi^- | \psi^+ \rangle$

$$\Omega \Big|_{\{\psi^+\}} = \Omega \Big|_{\{\psi^-\}} = \Omega_{M^1 \delta^1_{j^1}} \times \Omega_{M^2 \delta^2_{j^2}}$$

Expected structure for QFT

Poincaré-invariance $\Rightarrow H, \vec{P}, \vec{L}, \vec{B}$ ray-rep of $\mathcal{P}$

structured by spectrum $H, \vec{P}, \vec{L}$

- φ = ground state (vacuum) $\quad \mathcal{U}(g)\varphi = \varphi$

  $\mathcal{U}|_\varphi$ = trivial

- bound states $\quad M, j \to \psi_{\vec{k}m}$

$$H\psi_{\vec{k}m} = \sqrt{\vec{k}^2 + M^2}\,\psi_{\vec{k}m}$$

$$\vec{P}\psi_{\vec{k}m} = \vec{k}\psi_{\vec{k}m} \quad \text{or} \quad e^{i\vec{P}\vec{a}}\psi = e^{i\vec{k}\vec{a}}\psi$$

$$\mathcal{U}(R)\psi_{\vec{k}m} = D^j_{mm'}(R)\,\psi_{R\vec{k},m'}$$

$\mathcal{U}|_{\{\psi_{\vec{k}m}\}} = \mathcal{U}_{Mj}$ irred. rep. Mass, Spin

- 2 channel scattering states

$$\psi^{\pm} = \lim \frac{\pm i\varepsilon}{E - H \pm i\varepsilon}\,\psi^1 \otimes \psi^2$$

$\quad$ ↑ suitably defined

$$\mathcal{U}|_{\{\psi^+\}} = \mathcal{U}_{\{\psi^-\}} = \mathcal{U}_{M_1 j_1} \times \mathcal{U}_{M_2 j_2}$$

Problem: $H, \vec{B}$ pathological

- combine with renormalization

Lattice regularisation

makes spectrum problem well-defined

<u>But</u>: Euclidean symmetry $(\vec{P}, \vec{L})$ broken

Remnant: Lattice-Euclid. symm. E_{Lat}

$\vec{x} \to \vec{x} + \vec{a}$ $\qquad\qquad$ $\vec{x} \to R\vec{x}$

$\vec{x}, \vec{a} \in \mathbb{Z}^3 \cdot \varepsilon$ $\qquad\qquad$ $R \in$ cubic group

8 (48) Elements $(d = 2(3))$

Spectrum $\quad H, E_{Latt}$

Eigenstates $\quad \psi_{\vec{k},m}$

$$H \psi_{\vec{k}m} = E(\vec{k}) \psi_{\vec{k}m}$$

$$\Omega(\vec{a}) \psi_{\vec{k}m} = e^{i\vec{k}\vec{a}} \psi_{\vec{k}m} \qquad \vec{a} \in \mathbb{Z}^3, \quad \vec{k} \in \text{Brillouin}$$

$$\Omega(R) \psi_{\vec{k}m} = D^j_{mm'}(R) \psi_{R\vec{k},m}$$

$R \in$ cubic group

$D^j_{mm'}$ reps ~

φ: groundstate: $\quad \vec{k} = 0 \quad j = 0$ $\qquad$ what is special with φ

$$H\varphi = E_0 \varphi \quad \to \quad H \to H - E_0$$

boundstate $\quad (H - E_0)\psi_{\vec{k}m} = \sqrt{k^2 + M^2}\, \psi_{\vec{k}m}$ $\qquad$ how disting. from scatt.

$\varepsilon \to 0$

hope

Expected m.b. - structure

finite Volume V (thermodyn. formul.)

- ground state φ $\quad H\varphi = E_0 \varphi$ $\quad E_0 \sim V$
- excited state $\quad (H - E_0)\,\psi_{\kappa m} = E\,\psi_{\kappa m}$ $\quad E \sim 1$

guaranteed by (well establ.) many-body th.
(nuclear matter, solid state $(T=0)$)

ϕ = trial state (unpert. ground st.)

- $\Rightarrow$ $\varphi = e^S \phi$ $\quad S$ = correlation operator for φ

$S\phi \;\hat{=}\;$ bound state $\vec{k} = 0$ $\;\gamma = 0$ $\quad$ corr. have finite range

$|S\phi|^2 \sim V$

- $\vec{k}\;\gamma$ $\Rightarrow$ $\psi_{\vec{k} m} = F_{\vec{k} m}\,\varphi$ $\quad F$ = excitation op

$F_{\vec{k} m}\varphi \;\hat{=}\;$ bound state $\quad$ finite range in $\vec{x}_{rel}$

- 2 channel scattering states

$$\lim_{\varepsilon \to 0^+} \frac{\pm i\varepsilon}{E - H \pm i\varepsilon}\, F^1 F^2 \varphi$$

$\otimes \Rightarrow$ product of operators

e.g. Atom - Atom - collision
in the background of solid state

Systematic approach to S, F via coupled-cluster formulation

rewriting $H\psi = E\psi$

- groundstate

$$\varphi = e^{S}\phi$$

$$H\varphi = E_0\varphi \iff \boxed{H_S\,\phi = E_0\,\phi}$$

$$H_S = e^{-S} H e^{S}$$

$$= H + [S, H] + \frac{1}{2}[S,[S,H]] + \dots$$

- excited state

$$\psi = F\varphi \qquad [F, S] = 0$$

$$(H - E_0)\psi = E\psi \iff \boxed{[H_S, F]\,\phi = E\,\phi}$$

non-linear in S

linear in F

$V \to \infty$ possible

Structure for Lattice YM

3 dim Lattice • N links
(finite Volume)

Hilbert space $\psi(U_1 \ldots U_N) \to \mathbb{C}$ $U_\alpha \in SU(n)$

$$\langle \psi | \psi \rangle = \int dU_1 \ldots dU_n \, |\psi|^2$$

dU = Haar-m. $\boxed{\int dU = 1}$

$$H_{KS} = \frac{g^2}{2\varepsilon} H \qquad H = \sum \operatorname{tr} E_\alpha^2 + V(U)$$

$$V(U) = -x \sum_{\square} \operatorname{tr} P_\square \qquad x = \frac{2}{g^4}$$

$E_\alpha = E_{\alpha a} T^a$ conjugate to U_α

$$[E_{\alpha a}, U_\beta] = \delta_{\alpha\beta} T^a U_\beta$$

Gauge invariance $(SU(n))^M$ M = # sites

only trivial rep. are physical

$\hat{=}$ Lattice Gauss' Law

Coupled cluster formulation

$\int du = 1 \quad \Rightarrow \quad \phi = 1$ legitimate trial state

$\varphi = e^{S(u)}$ ground state

$\psi = F(u)\, e^{S(u)}$ exc.

$$e^{-S} H e^{S} = H + [S,H] + \frac{1}{2}[S[S,H]]$$

$$H\varphi = E_0\varphi \iff \sum p_{\alpha a}^2 S + (p_{\alpha a} S)(p_{\alpha a} S) + V = E_0$$

$$(H-E_0)\psi = E\psi \iff \sum p_{\alpha a}^2 F + 2(p_{\alpha a} S)(p_{\alpha a} F) = E F$$

$$p_{\alpha a} S = [E_{\alpha a}, S]$$

Structure

lim $N \to \infty$ ($V \to \infty$) can be made

$E_0 \sim V \qquad E \sim 1$ manifest

eq. for S non-linear

" " F linear, coupling only via S

$\varphi = e^{S}$ $\qquad \psi_{km} = F_{km}\,\varphi$

S, F bound states ??

non-rel $\qquad \psi_{km} = e^{i\vec{k}\vec{x}_{cm}}\,\psi_m^{int}(x_{rel})$

finite size

$$= \Pi_{\vec{k}}\;\psi_m^{int}$$

Peierls - Yoccoz - Projection

Lattice QCD

$$\Pi_{\vec{k}} = \sum_{\vec{a}\in\mathbb{Z}^3} e^{i\vec{k}\vec{a}}\,\mathcal{R}(\vec{a}) \qquad \vec{k}\in \text{Brillouin zone}$$

$\Rightarrow$ Ansatz

$$S = \Pi_0\, S^{int}$$

$$F_{\vec{k}m} = \Pi_{\vec{k}}\,F_m^{int}$$

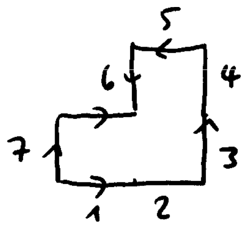
S^{int} F^{int} have finite range

CCE reproduce finite range ansatz

$\Rightarrow$ ansatz consistent

Expansion - basis for finite range ansatz

"Loop-space"



$$\chi(u) = \mathrm{tr}\; U_7^+ U_6^+ U_5^+ U_4 U_3 U_2 U_1$$

$$\Pi_{jm} = \sum_R D^j_{mm}(R)\;\Omega(R)$$

projector on lattice ang. mom.


etc

$\Longrightarrow$ linked cluster basis

$$\chi^r_{jm} \qquad\qquad \sum_{\alpha a} p^2_{\alpha a}\,\chi^r_{jm} = \varepsilon(j,r)\,\chi^r_{jm}$$

strong coupl. E.F.

structure of CCE

$$p^2 S + (pS)(pS) + V = E_0$$

$$p^2 F + 2(pF)(pS) = E F$$

$$2(pF)(pS) = p^2(FS) - (p^2 F)S - F(p^2 S)$$

Structure of CCE

$$F_{\vec{k}m} = \sum_r F^r_{\vec{k}m} \chi^r_{\vec{k}m}$$

$$S = \sum S^r \chi^r_{00}$$

$$p^2 = \sum_{\alpha a} p^2_{\alpha a}$$

$$p^2 S + \frac{1}{2} p^2 S^2 - (S p^2 S)^2 + V = E_0$$

$\Rightarrow$ non linear eq. for S^α

$$p^2 F + P^2 (FS) - S P^2 F - F p^2 S = E F$$

$\Rightarrow$ linear EVP for F

$\Rightarrow$ $E_0 \sim N \sim V$ manifest

$\chi = 1$ is not localized

$E \sim 1$ manifest

$\lim V \to \infty$ in CCE possible

finite volume restriction
due to truncation

Challenges

- handle χ^{r}_{jm} independent orthogonal strong coupling eig.st.

- work out decomposition of coupling term

$$\left(\Pi_{\vec{k}}\ \chi^{r}_{jm}\right)\left(\Pi_{0}\ \chi^{s}_{00}\right) = \sum_{t} C^{rs}_{t}(\vec{k},j,m)\ \Pi_{\vec{k}}\ \chi^{t}_{jm}$$

- Find „good" truncation

Ll. Smith ? Coester-Kümmel ? Guo ?

- Is calculation for maximum momentum eigenstates easier ¿

Structure of coupling coefficients

$C_t^{rs}(\vec{k}, jm)$ $\qquad \vec{k} \in$ Brillouin-

two steps in computation

- determine $\tilde{C}_t^{rs}(\vec{a}, j, m)$ $\qquad \vec{a} \in$ Lattice

$$\sum_{\vec{a}\text{, linking}} \chi^r_{jm}\, \Omega(\vec{a})\, \chi^s_0 = \sum_{t, \vec{a}>0} \tilde{C}_t^{rs}(\vec{a}, jm) \sum_R \Omega(R\vec{a})\, \chi^t_{jm}$$

- $\Rightarrow \quad C_t^{rs}(\vec{k}, j, m) = \sum_{\vec{a}>0} C_t^{rs}(\vec{a}, j, m) \sum_R e^{i(\vec{k} R \vec{a})}$

$\vec{k}$ maximal $\rightarrow \quad \vec{k} = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$

$\vec{a} = (n_1\, n_2\, n_3)\ (n_j \geq 0)$

$\Rightarrow \sum_R e^{i\vec{k} R \vec{a}} \neq 0 \iff n_j$ even

More challenges

- work out improved Hamiltonian
  (à la Lepage)

- include Fermions

quenched approx: analogous to glueballs

non-quenched: Coester - Kümmel procedure