Precision Predictions and Radiative Effects in the Heavy Quark Expansion

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Applications of Heavy Quark Expansion results in a few cases in precize predictions even when strong interaction effects intervene.

$$|V_{cb}| \simeq 0.041$$

Requires accurate control of perturbative and nonperturbative effects simultaneously.

Naive way:

Perturbative $\longrightarrow \quad 1 + a_1 \frac{\alpha_s}{\pi} + a_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$

Nonperturbative $\longrightarrow \quad 1 + \frac{C_1}{m_Q} + \frac{C_2}{m_Q^2} + \cdots$

Full result: 'perturbative' + 'nonperturbative' corrections

Typical situation: higher-order $(\alpha_s/\pi)^2 \ldots$ corrections appear too large, casting doubts on reliability of perturbative predictions.

But when perturbative effects are included everywhere to a given order, final results changes insignificantly

Wilson OPE: separation of short-distance and long-distance effects

Novikov
Shifman 1985
Vainshtein
Zaharov

Re-emerged practically recently, in particular, in HQE

- High precision

- Power corrections in the OPE start with $1/m_Q^2$ but m_Q^{Pole} has IR effects $1/m_Q$ ($\delta m_Q \sim \Lambda_{QCD}$) similar to M_{H_Q} ; m_Q^{Pole} is not defined perturbatively to $1/m_Q$ accuracy

OPE : use $m_Q(\mu)$ with $\Lambda_{QCD} \ll \mu \ll m_Q$

Practical reason : Typical opinion: Performing Feynman integration with explicit cutoff is ~~almost~~ impossible, at least beyond one loop, or for X-section - type quantities.
Problem with gauge invariance.

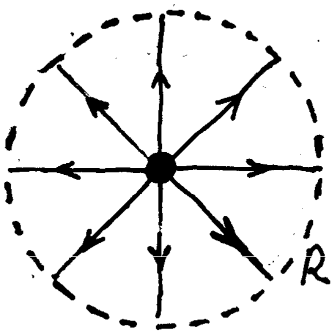
Not a real problem for quantities where OPE applies
Explicit calculations in two loops

m_Q in QCD

$\delta m_Q^{Pole} \sim$ few $\times \Lambda_{QCD}$ Counter - intuitive ?

$$m_e^{Pole} \equiv m_e = 0.510999\ldots \text{ MeV}$$

Significant 'numerical' uncertainty in m_Q led to concentrating on mass - independent dimensionless formfactors less reliable theoretically.
m_Q is a basic parameter of the HQE...

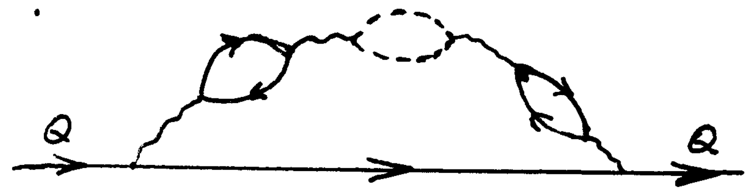


$$\delta E_{coul}(R) \sim \int_{1/m_b}^{R} \vec{E}^{\,2}_{coul}\, d^3x \sim const - \frac{\alpha_s(R)}{\pi}\,\frac{1}{R}$$

Pole mass: $R \to \infty$

problems at $R \lesssim \Lambda_{QCD}^{-1}$, $\delta m_Q \sim \Lambda_{QCD}$

It is seen in the PT:



$k_0 \simeq 0$

$$\delta m_Q \sim \frac{4}{3}\, 2\pi \int \frac{d^3\vec{k}}{(2\pi)^3}\, \frac{\alpha_s(\vec{k}^2)}{|\vec{k}|^2}$$

$$\frac{\delta m_Q}{m_Q} \sim \frac{4}{3} \sum_{\ell=0}^{\infty} \frac{\alpha_s}{\pi} \cdot \left(\frac{\beta_0 \alpha_s}{2\pi}\right)^{\ell} \cdot \ell!$$

Irreducible uncertainty $\sim e^{-\frac{2\pi}{\beta_0 \alpha_s(m_Q)}} = \frac{\Lambda_{QCD}}{m_Q}$

$$m_b^{pole} = \underbrace{\underset{tree}{4.55\,\text{GeV}} + \underset{\alpha_s}{0.25\,\text{GeV}}}_{\substack{4.8\,\text{GeV} \\ \text{"one-loop" pole mass}}} + 0.22\,\text{GeV} + 0.38\,\text{GeV} + 1\,\text{GeV} + 3.3\,\text{GeV} + 14\,\text{GeV}$$

IR part of pole mass is not related to any local operator $\bar{Q} \ldots Q$.

$\Gamma_{sl}(B)$

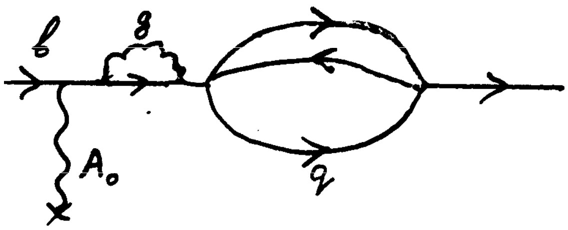
'Pole' masses do not allow to control already $1/m_Q$ effects which are as large as 20%

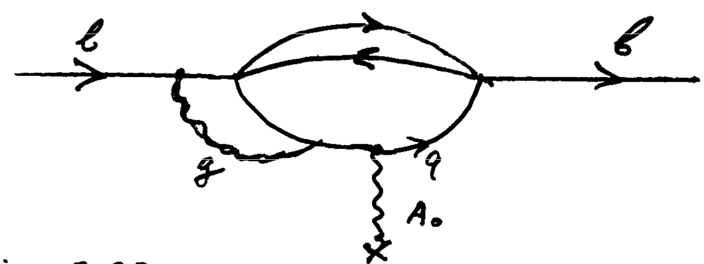
$$\Gamma^{pert}(b) \approx (m_b^{pole})^5 \left(1 + a_1^{pole}\frac{\alpha_s}{\pi} + a_2^{pole}\left(\frac{\alpha_s}{\pi}\right)^2 + \ldots\right)$$

	a_1^{pole}	a_2^{pole}		
	-1.7	$-(10 \div 20)$	...	$b \to c$
	-2.6	-30	...	$b \to u$

$$\Gamma^{pert}(b) \sim (m_b(\mu))^5 \left(1 + a_1(\mu)\frac{\alpha_s}{\pi} + a_2(\mu)\left(\frac{\alpha_s}{\pi}\right)^2 + \ldots\right)$$

$a_1(\mu)$: -1 $\quad$ $a_2(\mu)$: ~ 1





Conservation of color flow in QCD ensures cancelation between Coulomb binding in the initial state and Coulomb distortion of final-state wavefunctions

OPE statements hold whether A_0 is 'external' nonperturbative field or perturbative gluon dressing short-distance mass $m_Q(\mu)$ up to m^{pole}.

Numerical uncertainties were due to pole masses:

I. "Difficult to extract" m_b^{pole} $\quad$ ($\delta m \gtrsim 200$ MeV)

II. Large perturbative uncertainties in the widths > 10% due to attempts to relate it to m_b^{pole}, $\Gamma_{sl}/(m_b^{pole})^5$

$$\frac{1}{\Gamma_{sl}}\left(|\delta_I \Gamma_{sl}| + |\delta_{II}\Gamma_{sl}|\right) \gtrsim 20\% \longleftrightarrow 10\% \text{ in } |V_{cb}|$$

Peculiarity of heavy quarks: nonperturbative corrections start with $1/m_Q^2$ but the pole mass itself has $1/m_Q$ infrared uncertainty

Which mass to use?

m_Q^{Pole} does not exist...

$\overline{m}_Q(m_Q)$? – Yes, sometimes, but not for low energies $\ll m_Q$.

Toy Example:

Positronium mass, $E_{Coulomb}$

If use $\overline{MS}$ mass, need to know $\overline{m}_e(m_e)$, only $\alpha^2 m_e$ terms are known ...

'Theoretical uncertainty' is $\alpha^3 m_e \gtrsim 0.01\,eV$; without two-loop calculations would not know even with a few eV accuracy ?!

$$M_P = 2m_e(0) - \frac{\alpha^2 m_e}{4}$$

– without loops! (up to $\alpha^4 m_e$)

relevant $\mu \sim \frac{1}{r_B} \sim \alpha m_e$; physics above this scale reduces to renormalization of mass m_e and α. Below m_e only Coulomb exchanges survive.

$\Gamma(Z \to b\bar{b})$ $\qquad m_b(M_Z)$ is appropriate

$\Gamma(b \to q\ell\nu)$ $\qquad \mu \sim \frac{2}{n} \underset{\uparrow\, m_b - m_c}{m_b}, \; n = 5$

$m_Q(\mu)$ for $\mu < m_Q$ can be defined in the gauge-invariant way to any order in perturbation theory.

$\Gamma_{SL}(B)$, m_Q and $1/5$ expansion

$$\Gamma_{SL} \sim |V_{cb}|^2 \cdot m_b^n, \quad n = 5 \gg 1$$

$$\frac{\delta\Gamma}{\Gamma} \sim 5\, \frac{\delta m}{m}$$

$$m_b - m_c = \frac{M_B + 3M_{B^*}}{4} - \frac{M_D + 3M_{D^*}}{4} + \frac{\mu_\pi^2}{2}\left(\frac{1}{m_c} - \frac{1}{m_b}\right) + \cdots \;-$$

– well constrained

Vices $\longrightarrow$ virtues: $1/n$ expansion

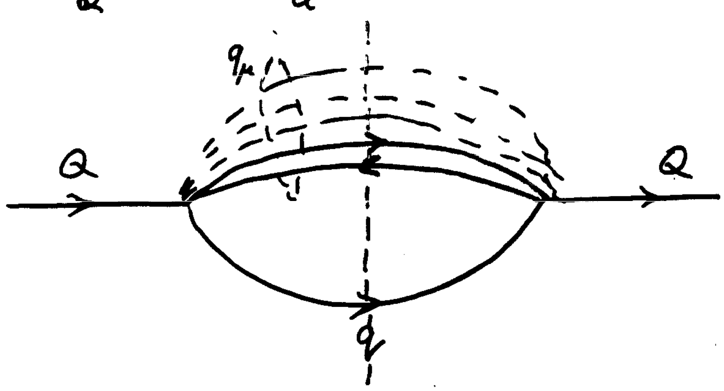
Bigi
Shifman
N.U.
Vainshtein

Large power of m_Q comes from lepton phase space, i.e. kinematics rather than strong dynamics

$$\mathcal{L}_W = \frac{G_\ell}{\sqrt{2}} V_{cb} \cdot \bar{c}\gamma_\mu(1-\gamma_5) b \cdot \bar{\ell}\gamma_\mu(1-\gamma_5)\nu \cdot \varphi^\ell \qquad m_\varphi = 0$$

$$[G_\ell] = [m]^{-2-\ell}$$

$$\Gamma_Q \sim m_Q^{5+2\ell} \qquad n = 5 + 2\ell$$



$$\sum k_{\ell_\mu} + \sum k_{\varphi_\mu} = q_\mu$$

$$d\,\mathrm{Ph.sp.} \sim d(q^2)^{1+\ell} \qquad \text{At } n \to \infty \quad \sqrt{q^2} \simeq (m_Q - m_q)\left(1 - \frac{c}{n}\right)$$

Heavy quark limit: major part of m_Q is 'eaten up' by lepton invariant mass

- powers of mass come from phase space,

hence seem to depend on M_B^n, M_D^n, physical hadron masses, at least at $n \gg 1$.

On the other hand, OPE states Γ depends on m_b, m_c without $\mathcal{O}\left(\frac{M_B - m_b}{m_b}\right)$ corrections at any n.

Not only $B \to D^{(*)} \ell \nu$ but also $B \to D^{**} \ldots \ell \nu$ where phase space is smaller than for $b \to c \ell \nu$

The least obvious for $b \to u$

SV sum rules ensure such a cancellation as long as $\frac{m_b - m_q}{n} \gg \Lambda_{QCD}$

-

$$\Gamma_{SL} \overset{pert}{\sim} 1 + n\left(\frac{\alpha_s}{\pi}\right) + n^2\left(\frac{\alpha_s}{\pi}\right)^2 + n^3\left(\frac{\alpha_s}{\pi}\right)^3 + \ldots$$

$a_k \sim n^k$ at $n \to \infty$ and k fixed

Even at $n = 5$ and $\frac{\alpha_s}{\pi} = 0.1$ higher-order terms are significant.

Leading-n series is summed up using the low-scale masses $m_Q(\mu)$ with $\mu \lesssim m_Q/n$

These terms are not captured by the BLM-type approximations, they are present even if $\beta(\alpha_s) = 0$

The scale-dependence is usually focused on the dependence on the scale for α_s. The primary question for $\Gamma_{SL} \ldots$ is the normalization scale for mass

- SV approximation works for inclusive decays even though $m_c^2/m_b^2 \approx 0.08 << 1$:
  'ESV' parameter is $\frac{(m_b - m_c)}{n\, m_c}$ rather than $\frac{m_b - m_c}{m_c}$

Why terms $\left(\frac{n\alpha_s}{\pi}\right)^k$ emerge ?

$$\Gamma \simeq d_n m_Q^n \left(1 + a_1 \frac{\alpha_s}{\pi} + a_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \cdots\right) = A_{pt}(\alpha_s)$$

if use a different $\tilde{m}_Q = m_Q\left(1 - c\frac{\alpha_s}{\pi}\right)$

$$\tilde{A}_{pt}(\alpha_s) = 1 + (\underline{nc} + a_1) + \left(\underline{\frac{n(n+1)}{2}c^2} + nca_1 + a_2\right)\left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$$

For $\overline{MS}$ $m_Q(m_Q)$ $c = \frac{4}{3}$

Why $a_1, a_2, \ldots$ are 'n-free' for low-scale mass and not for $\overline{MS}$?

In the SV limit it is obvious :

$$\Gamma = \frac{G_F^2}{60\pi^3}(\Delta m)^5\left(\underbrace{g_V^2}_{\sim 1} + 3\underbrace{g_A^2}_{\sim n_c}\right) + \mathcal{O}\left(\left(\frac{m_b - m_c}{m_b + m_c}\right)^2\right)$$

g_A, g_V are n-independent

$$\Delta m = M_B - M_C = m_b(\mu) - m_c(\mu) - \mathcal{O}\left(\frac{\mu^2 \alpha_s}{m_Q}\right) \cdots$$

m_Q^n comes from phase space which knows nothing about $\overline{m}_Q(m_Q)$

However,

IR strong-interaction effects which dress up short-distance mass from m_Q^{pole} or M_{H_Q} must cancel out at any n identically, while they affect only 'n-independent' hadronic part of the amplitude.

Contradiction? Cancel completely but only at $k \lesssim m_Q/n$:

$$M_B \longrightarrow M_B - 2 \cdot \sum_n \varepsilon_n \left(|\tau_{1/2}^{(n)}|^2 + 2 |\tau_{3/2}^{(n)}|^2 \right) e^{-\varepsilon/\mu}$$

$$\mu \sim m_b/n$$

$$\bar{\Lambda} \doteq M_B - m_b = 2 \sum_n \varepsilon_n \left(|\tau_{1/2}^{(n)}|^2 + 2 |\tau_{3/2}^{(n)}|^2 \right)$$ – 'optical' sum rule, M. Voloshin

Conflict of requirements:

summing up large-S terms requires using low-scale m_Q.

Effects of running α_s leads to $1/m_Q$ IR renormalon, terms $\sim \frac{\alpha_s}{\pi} \cdot \left(\frac{\beta_0}{2} \frac{\alpha_s}{\pi} \right)^k \cdot k!$ which are removed if short-distance masses are used.

Both requirements are met if one uses low-scale short-distance mass $m_Q(\mu)$ with $\mu \ll m_Q$

In practice $\mu \sim 1 \div 2$ GeV

$$\frac{d\, m_Q(\mu)}{d\mu} = -\frac{16}{9}\frac{\alpha_s(\mu)}{\pi} - \frac{4}{3}\frac{\alpha_s(\mu)}{\pi}\frac{\mu}{m_Q} + O\left(\alpha_s^2, \frac{\mu^2}{m_Q^2}\right)$$

$m_Q^{Pole} = \lim_{\mu\to 0} m_Q(\mu)$ but the limit does not exist

$m_Q^{\overline{MS}}(\mu)$ is unphysical at $\mu < m_Q$:

$$m_Q^{\overline{MS}}(\mu) \simeq m_Q^{\overline{MS}}(m_Q)\left(1 + \frac{2\alpha_s}{\pi}\ln\frac{m_Q}{\mu}\right)$$

$$|V_{cb}| = 0.0419 \left(\frac{BR(B\to X_c \ell\nu)}{0.105}\right)^{\frac{1}{2}} \left(\frac{1.55\,ps}{\tau_B}\right)^{\frac{1}{2}} \left(1 - 0.012\,\frac{\mu_\pi^2 - 0.5\,GeV^2}{0.1\,GeV^2}\right) \times$$

$$\times\left(1 - 0.01\,\frac{\delta m_b(\mu)}{50\,MeV}\right)\cdot\left(1 + 0.006\,\frac{\alpha_s^{\overline{MS}}(1\,GeV) - 0.336}{0.02}\right)\cdot\left(1 + 0.007\,\frac{\bar\rho^3}{0.1\,GeV^3}\right)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\left(1 \pm 0.015_{pert} \pm 0.01_{m_b} \pm 0.012\right)$$

$a_2^{(0)}$ in $\Gamma(b\to c)$ is less than 1 — A. Czarnecki, K. Melnikov

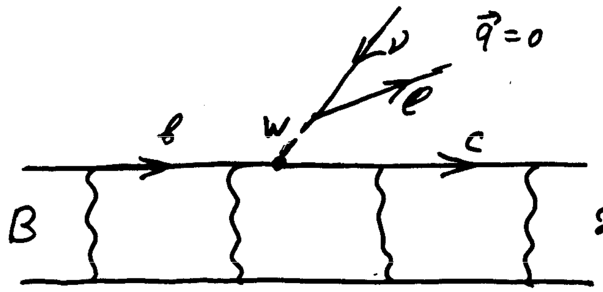
$$\delta m_b(\mu) \sim \left(\frac{\alpha_s}{\pi}\right)^2 \cdot \mu$$

$$|V_{ub}| \simeq 0.00415 \left(\frac{BR(B\to X_u \ell\nu)}{0.0016}\right)^{\frac{1}{2}} \left(\frac{1.55\,ps}{\tau_B}\right)^{\frac{1}{2}} \left(1 \pm 0.025_{pert} \pm 0.03\, m_b\right)$$

$$\mu_\pi^2 = \frac{1}{2M_B}\langle B|\,\bar b(i\vec D)^2 b\,|B\rangle$$

While $M_B^5/m_b^5 \sim 1.5 \div 2$, actual nonperturbative effects in Γ_{SL} are -5% increasing $|V_{cb}|$ by 2.5%

Zero-recoil sum rules:



$$\left.\frac{d\Gamma(B\to D^*\ell\nu)}{dq^2}\right|_{q^2\to(M_B-M_{D^*})^2} \sim |\vec{P}|\cdot|F_{D^*}(0)|^2$$

$$\langle D^*(\varepsilon)|\,\bar{c}\,\gamma_\mu\gamma_5\, b\,|B\rangle = 2\sqrt{M_B M_{D^*}}\;\varepsilon^*_\mu\cdot F_{D^*}$$

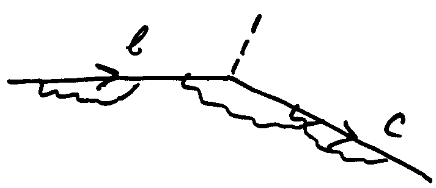
$F_{D^*} = 1 +$ corrections to HQS limit

$\mathcal{O}(\alpha_s)$, $1/m_c^2$, $1/m_b^2$, $1/m_c m_b$

No $1/m_Q$ corrections — M. Voloshin, M. Shifman, M. Luke

How to understand: IR effects from $k<\mu$ are suppressed by at least μ^2/m_Q^2

Pure PT:



$$\bar{c}\,\gamma_\mu\gamma_5\, b \longrightarrow \eta_A\cdot\bar{u}_c\,\gamma_\mu\gamma_5\, u_b$$

Not the whole story

$$F_{D^*} = \eta_A + \delta_{1/m^2} + \delta_{1/m^3} + \cdots$$

Since $\delta_{1/m^2} \sim \frac{\mu^2_{hadr}}{m_c^2}$, it is important to estimate it.

δ_{1/m^2} depends on the details of strong dynamics

$$|F_{D^*}|^2 + \sum_{i \neq D^*}^{\varepsilon<\mu} |F_{B\to i}|^2 = \zeta_A(\mu) - \frac{\mu_G^2}{3m_c^2} - \frac{\mu_\pi^2-\mu_G^2}{4}\left(\frac{1}{m_c^2}+\frac{1}{m_b^2}+\frac{2}{3}\frac{1}{m_c m_b}\right) + \\ + \mathcal{O}\left(\frac{1}{m^3}\right)$$

$$\sum |F_{B\to i}|^2 \longleftrightarrow \frac{1}{2\pi}\int w_1^A(\varepsilon, \vec{q}=0)\, d\varepsilon$$

$$\mu_G^2 = \frac{1}{2M_B}\langle B|\, \bar{b}\, \tfrac{i}{2}\sigma_{\mu\nu}G_{\mu\nu}\, b\, |B\rangle \simeq \frac{3}{4}\left(M_{B^*}^2 - M_B^2\right) \simeq 0.4\ \mathrm{GeV}^2$$

$$\mu_\pi^2 = \frac{1}{2M_B}\langle B|\, \bar{b}\, (i\vec{D})^2 b\, |B\rangle \quad > \mu_G^2$$

For hypothetical $\bar{c}\, i\gamma_5 b$ 'weak current'

$$\mu_\pi^2(\mu) - \mu_G^2(\mu) = \sum_i^{\varepsilon<\mu} |\tilde{F}_i|^2 > 0$$

Since $|F_i|^2, |\tilde{F}_i|^2 > 0$ one gets an upper bound on F_{D^*}; in this way one estimates F_{D^*} as well,

$F_{D^*} \approx 0.9$

$$\sum |F_{B\to i}|^2 \sim \frac{\alpha(\varepsilon)\cdot \varepsilon\, d\varepsilon}{m_Q^2} \sim \frac{\alpha(\varepsilon)}{m_Q^2}\cdot \mu^2$$

Likewise there is a "perturbative" piece $\sim \alpha\cdot\mu^2$ in $\mu_\pi^2(\mu)$

If one attempts to subtract these pieces then

$\zeta_A(\mu) \longrightarrow \eta_A^2$. However, positivity of $|F_i|^2$ is lost

$$-\lambda_1 = \mu_\pi^2(\mu) - 0.12\ \mathrm{GeV}^2 - \underbrace{0.3\ \mathrm{GeV}^2 + \cdots}_{\text{grow!}}$$

$$\zeta_A^{1/2}(\mu) \simeq \eta_A(\mu) + \underbrace{0.01 + \cdots}_{\text{blows out}} \qquad \mu \simeq 0.7 \div 1\ \mathrm{GeV}$$

QM interpretation of sum rules

Weak decay: instantaneous replacement $b \to c$

Probability to hadronize to some state is 1.

Why nonperturbative corrections are present?

Normalization of weak current $\bar{c} \ldots b$ is not exactly unity, depends on the gluon field:

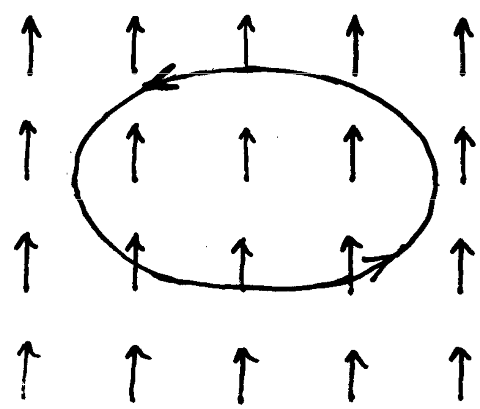
$$\bar{c}\,\gamma_5\gamma_k\, b \xleftrightarrow{QM} \varphi_c^+ \sigma_k \varphi_b - \varphi_c^+ \left\{ \frac{(\vec{\sigma}\vec{\pi})^2}{8m_c^2}\sigma_k + \sigma_k \frac{(\vec{\sigma}\vec{\pi})^2}{8m_b^2} - \frac{(\vec{\sigma}\vec{\pi})\,\sigma_k\,(\vec{\sigma}\vec{\pi})}{4m_c m_b} \right\} \varphi_b + \mathcal{O}\!\left(\tfrac{1}{m^3}\right)$$

$$\vec{\pi} = -i\vec{D}$$

$$\Psi_B \xrightarrow{\bar{c}\gamma_5\gamma_k b} \text{“}\Psi_{D^*}\text{”} = \sigma_k \Psi_B - \left\{ \frac{(\vec{\sigma}\vec{\pi})^2}{8m_c^2}\sigma_k + \sigma_k \frac{(\vec{\sigma}\vec{\pi})^2}{8m_b^2} - \frac{(\vec{\sigma}\vec{\pi})\sigma_k(\vec{\sigma}\vec{\pi})}{4m_c m_b} \right\} \Psi_B$$

$$\left\| \text{“}\Psi_{D^*}\text{”} \right\|^2 = \underset{\substack{\| \\ 1}}{\left\| \Psi_B \right\|^2} - \frac{\mu_G^2}{3m_c^2} - \frac{\mu_\pi^2 - \mu_G^2}{4} \cdot \left(\frac{1}{m_c^2} + \frac{1}{m_b^2} + \frac{2}{3m_c m_b} \right) + \ldots$$

$\mu_\pi^2 > \mu_G^2$ The Landau precession of a charged particle in the magnetic field



$$\vec{p}^{\,2} \geqslant |\vec{B}|$$

$$\mu_G^2 = \langle B| -\vec{\sigma}_l \cdot \vec{B}\, |B\rangle$$

$\vec{B} \sim \vec{S}_{\text{light cloud}}$ - non-classical !

$$|B_z| = \frac{\mu_G^2}{3}$$

$$\langle \vec{B}^2 \rangle \geqslant 3 \langle \vec{B} \rangle^2$$

This QM inequality literally holds in QFT (1994.

One of the family of exact QCD inequalities:

$$\mu_\pi^2(\mu) = 3\Big(\sum_{\varepsilon_n<\mu} \varepsilon_n^2 \cdot |\tau_{1/2}^{(n)}|^2 + 2\sum_{\varepsilon_m<\mu} \varepsilon_m^2\, |\tau_{3/2}^{(m)}|^2\Big)$$

$$\mu_G^2(\mu) = 3\Big(-2\sum_{\varepsilon_n<\mu} \varepsilon_n^2\, |\tau_{1/2}^{(n)}|^2 + 2\sum_{\varepsilon_m<\mu} \varepsilon_m^2\, |\tau_{3/2}^{(m)}|^2\Big)$$

$$\mu_\pi^2(\mu) = \mu_G^2(\mu) + 9\sum_{\varepsilon_n<\mu} \varepsilon_n^2\, |\tau_{1/2}^{(n)}|^2$$

$$\varepsilon_n = M_n^{P_{1/2}} - M_B\,, \qquad \varepsilon_m = M_m^{P_{3/2}} - M_B$$

$$\tau^{(n)} \sim \langle n|\, A_\mu |B\rangle \sim \frac{\langle n|\, \bar{Q}\, \vec{\pi}\, Q|B\rangle}{\varepsilon_n}$$

$\mu_\pi^2 \approx 0.5 \div 0.6\ \mathrm{GeV}^2$

$\tau_{3/2}^{(1)} \approx$ 0.25 Colangelo et al. 92; 0.3 Isgur-Wise 91; 0.38 Wise et al. (exp) 97

QCD sum rule technique for μ_π^2:

$-1\ \mathrm{GeV}^2$ Neubert, 92

$0.7\ \mathrm{GeV}^2 \longrightarrow 0.6 \pm 0.15\ \mathrm{GeV}^2$ Ball, Braun, 94, 95

$0.2 \pm 0.05\ \mathrm{GeV}^2$ Neubert, 96

$\eta_A = 1 - 0.035 + \ldots$ factorially divergent, $1/m_Q^2$ IR renormalon

$\mathrm{Im}\, \eta_A^{Borel} \simeq 0.08$ $\quad\quad$ $\delta\eta_A = \frac{1}{\pi} \mathrm{Im}\, \eta_A^{Bor} \simeq 0.024$

And sum rules are 'next to useless' due to uncontrollable corrections to η_A^2 and to the $\mu_\pi^2 - \mu_G^2$ sum rule

$\xi_A^{1/2}(\mu) = $ 1 $\quad$ tree

0.975 $\quad$ $\mathcal{O}(\alpha_s)$

0.995 $\quad$ BLM - resummed $\quad\quad$ $\mu \simeq 0.7$ GeV

+ QM inequalities are preserved

Non-BLM $\mathcal{O}(\alpha_s^2)$ corrections to η_A turned out to be small $\quad$ Czarnecki Melnikov

A.C. K.M. + N.U.

Likewise, for $\xi_A(\mu)$ $\quad$ $a_2^0 \simeq -0.3 \div -0.8$ for reasonable μ/m_c

$\xi_A^{1/2} \simeq 0.99$

Exact $\mathcal{O}(\alpha_s^2)$ corrections to $\xi_A(\mu)$

$\mathcal{O}(\alpha_s)$ correction to coefficient for μ_π^2/m_Q^2

$\mathcal{O}(\alpha_s)$ corrections to coefficients for the vector current sum rule

$$\frac{d\mu_\pi^2(\mu)}{d\mu^2} = \frac{4}{3}\frac{\alpha_s(\mu)}{\pi} + \frac{4}{3}\left(\left(\frac{5}{3} - \ln 2\right)\frac{\beta_0}{2} - N_c\left(\frac{\pi^2}{6} - \frac{13}{12}\right)\right)\left(\frac{\alpha_s}{\pi}\right)^2$$

Estimate of F_{D^*}:

$$|F_{D^*}|^2 + \sum_{\varepsilon<\mu} |F_{exc}|^2 = \xi_A(\mu) - \xi_\pi(\mu)\frac{\mu_\pi^2}{m_c^2} - \xi_G(\mu)\frac{\mu_G^2}{m_c^2} + \mathcal{O}\left(\frac{1}{m_Q^3}\right)$$

$$|F_{D^*}| = \Big|\, \xi_A(\mu) \underbrace{- \xi_\pi(\mu)\frac{\mu_\pi^2}{m_c^2} - \xi_G(\mu)\frac{\mu_G^2}{m_c^2} - \sum_{\varepsilon<\mu} |F_{exc}|^2}_{2\,\delta_{1/m^2}} \Big|^{1/2} + \mathcal{O}\left(\frac{1}{m_Q^3}\right)$$

$\xi_A^{1/2}(\mu)$ is the short-distance renormalization of $\bar{b}\gamma_\mu\gamma_5 c$ current

Why not η_A? It does not depend on borderline μ below which is 'long-distance' and above-'short-distance'. It could be only if perturbative corrections were absent at the scale μ, and μ is arbitrary...

$-\xi_\pi \cdot \frac{\mu_\pi^2}{m_c^2} - \xi_G \cdot \frac{\mu_G^2}{m_c^2}$ represent long-distance renormalizat.

$\sum |F_{exc}|^2 \longleftrightarrow$ deficit of $|\text{overlap}|^2$

$$F_{D^*} \simeq 0.91 - 0.013\,\frac{\mu_\pi^2 - 0.5\,\mathrm{GeV}^2}{0.1\,\mathrm{GeV}^2} \pm 0.025_{exc} \pm 0.01_{pert} \pm 0.025_{1/m^3}$$

$\xi_A^{1/2} \simeq 0.99$

Non-Abelian Dipole Radiation and the HQE

Czarnecki, Melnikov + N.U.
hep-ph/9708372

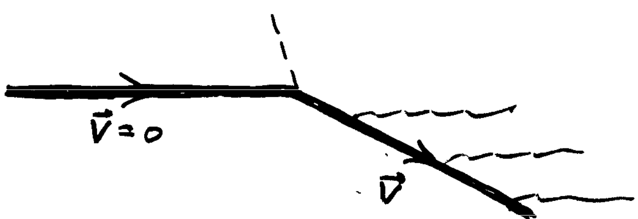
The idea to define $m_Q(\mu)$ with $\mu \ll m_Q$ as an "observable"; then it is gauge-invariant to any order in the α_s-expansion. Likewise $\mu_\pi^2(\mu)$, $\rho_D^3(\mu)$ etc. for operators which power-like mix with the unit operator ("have perturbative piece")

Use SV sum rules at $\vec{v} \neq 0$, $|\vec{v}| \ll 1$ but in the limit $m_Q \to \infty$.   BSUV

Effective theory for heavy quarks is subtle at $\vec{v} \neq 0$

Minkowskian nature!

Need to integrate high-momentum degrees of freedom to arrive at an effective low-energy theory



$$\frac{dW}{d\omega} \sim \vec{v}^2 \alpha_s \frac{d\omega}{\omega}$$ all the way up to $\omega \sim m_Q$

'Integrate out' real particles?

Gluon emission is almost classical effect, shaking off the Coulomb field

Classical electrodynamics:

$$\frac{1}{\omega}\frac{dI(\omega)}{d\omega} = \frac{\alpha}{\pi}\left(\frac{1}{|\vec{v}|}\ln\frac{1+\vec{v}}{1-\vec{v}} - 2\right)\frac{1}{\omega} = \frac{2}{3}\frac{\alpha}{\pi}\frac{\vec{v}^2}{\omega} + O(\vec{v}^4)$$

dipole radiation

QED: the same probability to radiate photons

Corrections only due to vacuum polarization effects (fermion loops), $\sim \omega^2/m_e^2$ at $\omega \to 0$, therefore are governed by $\alpha(0) = 1/137\ldots$

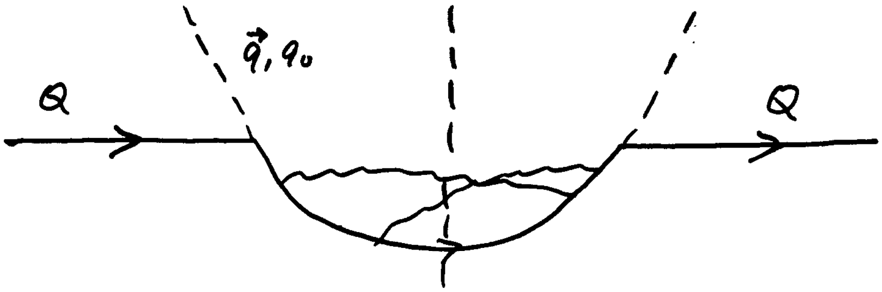
Different in QCD; nevertheless

$$\frac{dW}{d\omega} = \frac{4}{3}\cdot\frac{2}{3}\vec{v}^2\,\frac{\alpha_s^{(\omega)}(\omega)}{\pi}\,\frac{d\omega}{\omega} \qquad \text{when } \omega \ll m_Q \text{ and } |\vec{v}| \ll 1$$

$\alpha_s^{(\omega)}(\omega)$ – effective coupling

$$\omega\frac{d}{d\omega}\frac{\alpha_s^{(\omega)}(\omega)}{\pi} = -\beta\left(\frac{\alpha_s^{(\omega)}(\omega)}{\pi}\right) = -\frac{\beta_0}{2}\left(\frac{\alpha_s^{(\omega)}(\omega)}{\pi}\right)^2 - \frac{\beta_1}{8}\left(\frac{\alpha_s^{(\omega)}(\omega)}{\pi}\right)^3 - \ldots$$

$$\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f \,, \qquad \beta_1 = 102 - \frac{38}{3}n_f$$



$$T(q_0,\vec q) = \frac{1}{2M_Q}\int d^4x\, e^{-iqx}\langle Q|\, iT\, J(x)\, J(0)\,|Q\rangle$$

$$W(q_0,\vec q) = 2\,\mathrm{Im}\, T(q_0,\vec q)$$

Threshold $q_0^{min} \simeq \sqrt{\vec q^{\,2}+m_Q^2} - m_Q^2 \simeq m_Q \frac{\vec v^{\,2}}{2}$, $\vec q = m_Q \vec v$

$$\omega = q_0 - q_0^{min}$$

$$w(\omega,\vec v) = N\,\delta(\omega) + \vec v^{\,2}\,\frac{d(\omega)}{\omega} + O(\vec v^{\,4})$$

$$\frac{4}{3}\,\frac{\alpha_s^{(\omega)}(\omega)}{\pi}\,\frac{1}{\omega} = \lim_{\vec v\to 0}\ \lim_{m_Q\to\infty} \frac{3}{2\vec v^{\,2}}\ \frac{w(\omega,\vec v)}{\int_0^\omega w(\varepsilon,\vec v)\,d\varepsilon}$$

- $\alpha_s^{(\omega)}$ does not depend on J if it does not vanish at $\vec v = 0$ ($\bar Q \gamma_0 Q$, $\bar Q Q$ etc), on masses etc.

- does not depend on the type of hadron above the onset of duality

$$\frac{\alpha_s^{(\omega)}(\mu)}{\pi} = \frac{\alpha^{\overline{MS}}(e^{-\frac{5}{3}+\ln 2}\mu)}{\pi} - C_A\left(\frac{\pi^2}{6} - \frac{13}{12}\right)\left(\frac{\alpha_s}{\pi}\right)^2 + O(\alpha_s^3)$$

-1.685

$$O(\alpha_s^3) \overset{BLM}{=\!=} -\underbrace{\left(\frac{\pi^2}{6} - \frac{31}{36}\right)}_{0.78}\left(\frac{\beta_0}{2}\right)^2\left(\frac{\alpha_s}{\pi}\right)^3$$

$$\bar{\Lambda}(\mu) = \lim_{\vec{v}\to 0} \lim_{m_Q\to\infty} \frac{2}{\vec{v}^2} \frac{\int_0^\mu \omega \cdot w(\omega,\vec{v})\, d\omega}{\int_0^\mu w(\omega,\vec{v})\, d\omega}$$

$$\mu_\pi^2(\mu) = \lim_{\vec{v}\to 0} \lim_{m_Q\to\infty} \frac{3}{\vec{v}^2} \frac{\int_0^\mu \omega^2 w(\omega,\vec{v})\, d\omega}{\int_0^\mu w(\omega,\vec{v})\, d\omega} \qquad \text{etc}$$

No perturbative correction to these relations

$$\frac{d\bar{\Lambda}(\mu)}{d\mu} = \frac{4}{3} C_F \frac{\alpha_s^{(\omega)}(\mu)}{\pi}$$

$$\frac{d\mu_\pi^2(\mu)}{d\mu^2} = C_F \frac{\alpha_s^{(\omega)}(\mu)}{\pi} \qquad \ldots$$

OPE guarantees that the same evolution as at zero recoil where there is no dipole radiation

$$m_Q(\mu) = \left[m_Q^{\text{Pole}}\right]_{\text{pert}} - \left[\bar{\Lambda}(\mu)\right]_{\text{pert}} - \frac{1}{2m_Q(\mu)}\left[\mu_\pi^2(\mu)\right]_{\text{pert}}$$

Two-loop relation between $m_Q(\mu)$ and $\bar{m}_Q(m_Q)$ is at hand (hep-ph/9708372)

BLM series for $\alpha_s^{(\omega)}(\omega)$ in terms of $\alpha_s^{\overline{MS}}$ has a finite radius of convergence.

However, as an 'effective charge', $\alpha_s^{(\omega)}(\mu)$ has some power-suppressed piece from infrared. E.g., it would differ if extracted from B or Λ_b, etc.

How much?

Here OPE applies, and corrections are given by

$$c_K \cdot \frac{\langle \bar{Q}\, O_K\, Q \rangle}{\omega^K}$$

First candidate is $\bar{Q}(i\vec{D})^2 Q$ with $K=2$. However, its coefficient vanishes identically.

$K=3$: $\quad O_D = \bar{Q}\,(i\vec{D})\,(iD_0)\,(i\vec{D})\,Q$

$$\frac{\delta \alpha_s^{(\omega)}(\omega)}{\alpha_s^{(\omega)}(\omega)} \simeq \left(\frac{\alpha_s(\mu)}{\alpha_s(\omega)}\right)^{\frac{\gamma_D}{\beta_0}} \frac{3\gamma_D}{16}\,\frac{\rho_D^3(\mu)}{\omega^3} \approx -\left(\frac{0.55\,\mathrm{GeV}}{\omega}\right)^3$$

$\gamma_D = -\frac{13}{2}$ is anomalous dimension of O_D (SV, 86)

$$\rho_D^3 \simeq \frac{2\pi\alpha_s}{9}\,\tilde{f}_B^2\, M_B \approx 0.1\ \mathrm{GeV}^3$$

There are other advantages of $\alpha_s^{(\omega)}$ as an effective coupling. It must be useful for heavy quark decays.