Progress in QCD Amplitude Calculations

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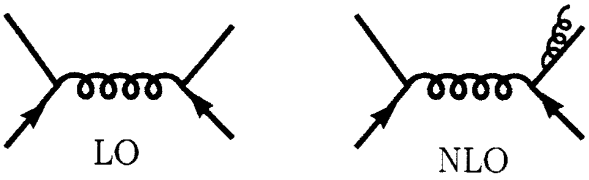
- Need for NLO QCD computations.
- Difficulty with Feynman diagrams.
- Overview of non-traditional approaches.
- Some basic tools: helicity and color decompositions.
- What superstring theory can teach us about QCD amplitudes.
- Analytic properties of amplitudes.
- Amplitudes with an arbitrary number of particles.
- Applications to jet physics.
    - (a) NLO 3 jet production at Fermilab
    - (b) NLO 4 jet production at CERN and SLAC
- Prospects for future: NNLO

NLO calculations in Jet Physics

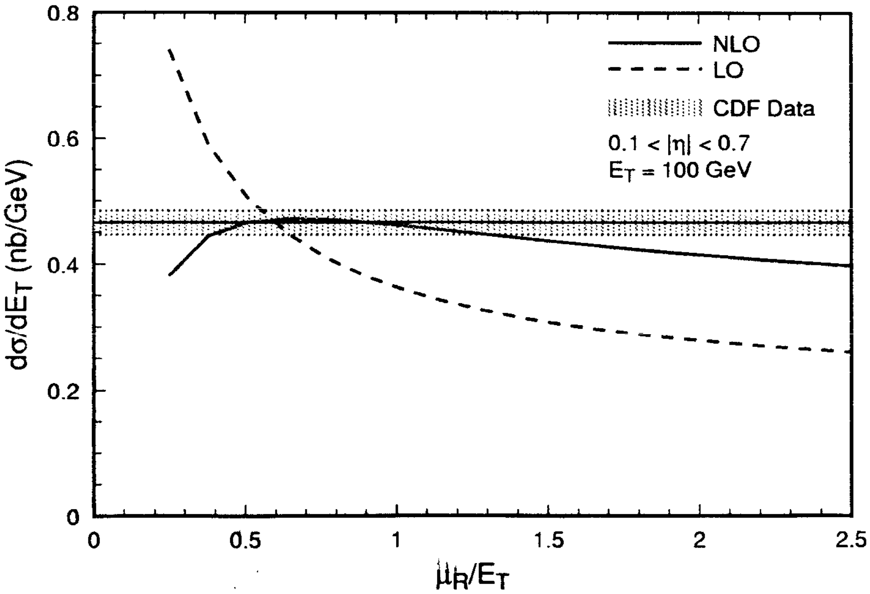
At LO there is no prediction as to the structure of jets – are they fat or skinny?



At NLO we obtain information on jet structure.

In many processes at colliders, the dominant theoretical uncertainties are due to uncalculated higher order terms in the perturbative expansion.

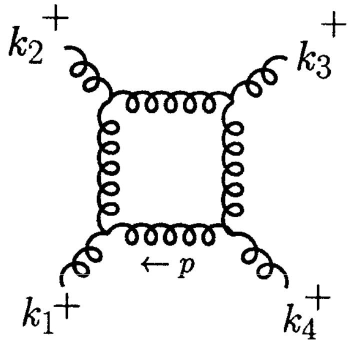
At LO renormalization scale dependence is large. The renormalization scale, μ, is purely a calculational artifact.



Large theoretical uncertainty at LO.

Difficulty with Feynman Diagrams

Consider the scattering of four positive helicity gluons at one-loop.



We encounter the following integral:

$$\int \frac{d^4p}{(2\pi)^4} \frac{p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}$$

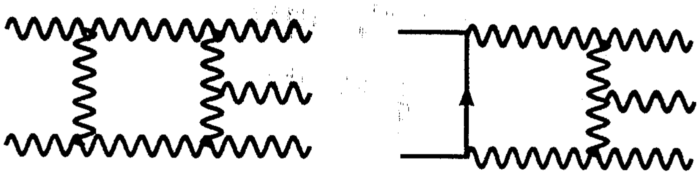
Result

Combining this integral with a few hundred others yields:

$$|A(1^+, 2^+, 3^+, 4^+)| = \frac{1}{48\pi^2}$$

Techniques developed over last few years allow us to obtain this in just a few lines of calculation.

Consider a brute force calculation for 3 jets.



Roughly 6^5 or 10^4 terms at the start of the calculation.

$$V_3^{\mu\nu\rho} = (p_1 - p_2)\eta_{\mu\nu} + (p_2 - p_3)_\mu \eta_{\nu\rho} + (p_3 - p_1)_\nu \eta_{\rho\mu}$$

Structure of result using conventional methods:

$$\frac{\mathrm{Num}_1}{\mathrm{Den}_1} + \frac{\mathrm{Num}_2}{\mathrm{Den}_2} + \cdots$$

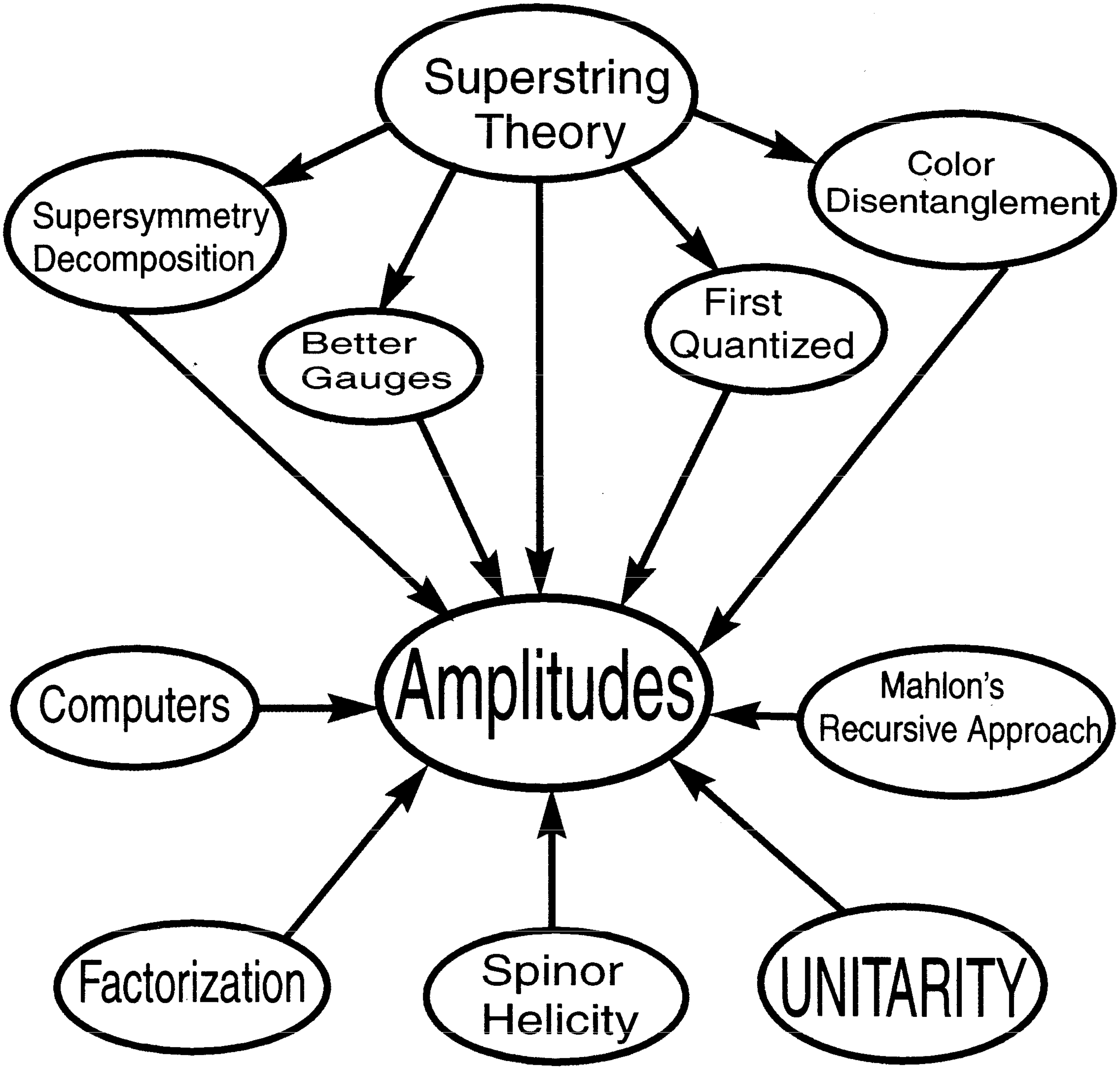
The denominator contains spurious singularities such as the Gram determinant

$$\frac{1}{\det(k_i \cdot k_j)^5}$$

Must put on a common denominator to cancel the 'Grim' determinants.

Need $15^5 \sim 10^5$ terms in numerator to remove leading spurious singularity.

Causes serious trouble with brute force approaches.


Superstring Theory
Supersymmetry Decomposition
Color Disentanglement
Better Gauges
First Quantized
Amplitudes
Computers
Mahlon's Recursive Approach
Factorization
Spinor Helicity
UNITARITY

Helicity

Xu, Zhang and Chang

& many others

Gluon amplitudes like to be expressed in terms of circular polarization.

$$\varepsilon_\mu^\pm(k) = (1, 0, 0, \pm i)$$

Better to use Lorentz covariant formulation

$$\varepsilon_\mu^+(k; q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\, \langle q\, k \rangle}, \qquad \varepsilon_\mu^-(k, q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\, [k\, q]}$$

All required properties of polarization vectors satisfied:

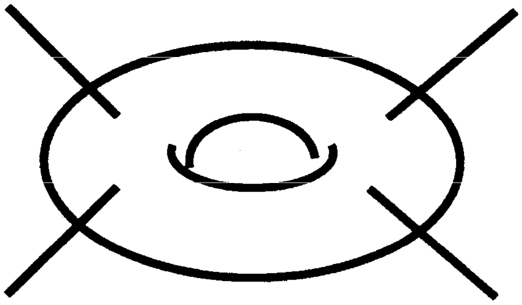
$$\varepsilon_i^2 = 0\,, \qquad k \cdot \varepsilon(k, q) = 0\,, \qquad \varepsilon^+ \cdot \varepsilon^- = -1$$

Useful because we can adjust the reference momentum q to make terms vanish. (Changes in q are equivalent to gauge transformations.)

Notation

$$\langle j\, l \rangle = \langle k_j{}^- | k_l{}^+ \rangle = \sqrt{2 k_j \cdot k_l}\; e^{i\phi}$$
$$[j\, l] = \langle k_j{}^+ | k_l{}^- \rangle = -\sqrt{2 k_j \cdot k_l}\; e^{-i\phi}$$

Supersymmetry Relations



The string contains all particles in a unified picture.

Both fermions and bosons are unified into a single object: supersymmetry.

Expect susy relations between bosonic and fermionic contributions to n-gluon amplitudes

$$A^{\text{fermion}} = -A^{\text{scalar}} + A^{N=1 \text{ susy}}$$

$$A^{\text{gluon}} = A^{\text{scalar}} - 4A^{N=1 \text{ susy}} + A^{N=4 \text{ susy}}$$

Holds in integrands of each string diagram.

We can decompose QCD amplitudes into non-supersymmetric and supersymmetric pieces.

Color Decomposition

For example for one-loop four-gluon amplitudes

$$\mathcal{A}_4^{1-\mathrm{loop}} = g^4 \sum_{\mathrm{non-cyclic}} N_c \,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_{4;1}(1,2,3,4) + \sum \mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4}) A_{4;3}(1,2;3,4)\ .$$

The partial amplitudes have been stripped of all color.

String theory suggests:

$$A_{n;c>1} = \sum_{\mathrm{perms}} A_{n;1}$$

Can also prove this in field theory using color ordered Feynman rules.

Leading color amplitudes give everything.

We only need to calculate $A_{n;1}$.

The same type of decomposition works for external fundamental representation fermions.

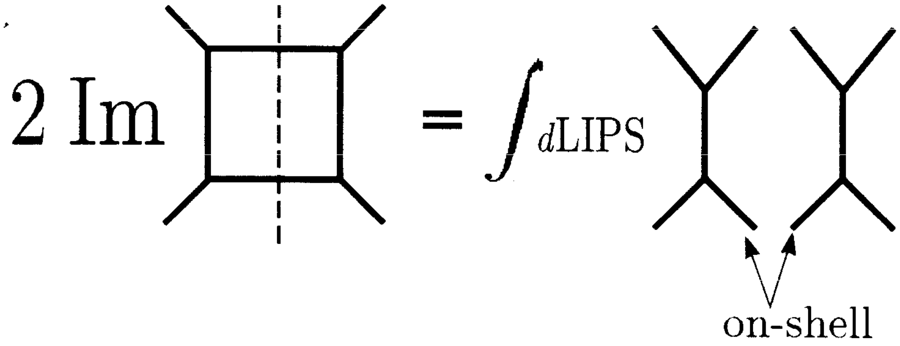
Unitarity

The scattering matrix is unitary.

$$S^\dagger S = 1$$

Take $S = 1 + iT$

$$2 \text{ Im } T = T^\dagger T$$



Harder to calculate = Easier to calculate

From unitarity we can obtain the imaginary parts of one-loop amplitudes from tree amplitudes.

To be a practical method of calculation we also need real parts, especially polys.

Generic form of an amplitude:

$$\begin{aligned} A &\sim \ln(-s - i\epsilon) + \text{rational} + \text{other logs} \\ &\sim \ln(s) - i\pi + \text{rational} + \text{other logs} \end{aligned}$$

The $i\pi$ term is fixed by unitarity and the $\ln(s)$ can be reconstructed from this.

However rational terms seemingly cannot be reconstructed.

Problem seems basic. Consider complex function

$$a(\ln(s) - i\pi) + b$$

You can get a from imaginary part but not b.

In this talk will discuss how to get around this problem.

Supersymmetric amplitudes

Observation: The rational terms in susy amplitudes are very simple.

$$A_5^{N=4\ \mathrm{susy}} = \ln\ \mathrm{terms} + A_5^{\mathrm{tree}} \frac{5}{6}\pi^2$$

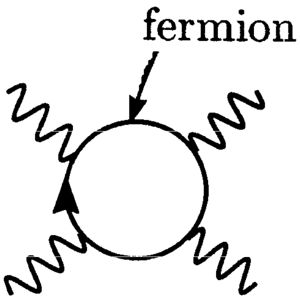
Compare to non-susy example:

$$A_5^{\mathrm{non-susy}} \sim \ln\mathrm{terms}$$
$$-\frac{1}{3}\frac{\langle 3\,5\rangle\,[3\,5]^3}{[1\,2]\,[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]} + \frac{1}{3}\frac{\langle 1\,2\rangle\,[3\,5]^2}{[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]}$$
$$+\frac{1}{6}\frac{\langle 1\,2\rangle\,[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]}{s_{23}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,s_{51}}$$

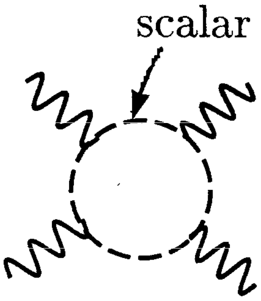
Idea: Link all rational terms in susy theory to logs.

- Only a limited set of integral functions enter into susy theories.
- Enough cut information exists to fix the coefficients of all integral functions which may enter.

Susy cancellations



$$\frac{1}{2}\ln\det{}_{[1/2]}[D^2 - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}]$$



$$\ln\det{}^{-1}_{[0]}[D^2]$$

Leading loop momentum cancels within susy multiplet

$$I_n[\ell^{\mu_1}\cdots\ell^{\mu_m}] = \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{\ell^{\mu_1}\ell^{\mu_2}\cdots\ell^{\mu_m}}{\ell^2(\ell-k_1)^2\cdots(\ell+k_n)^2}$$

This restricts the class of boxes, triangles and bubble functions that can appear in the amplitude.

e.g. No bubbles with two powers of loop momentum $I_2[\ell^\mu\ell^\nu]$

By systematically inspecting the integrals can prove that for this class of integral functions the cuts determine all coefficients so no rational function ambiguity for susy amplitudes.

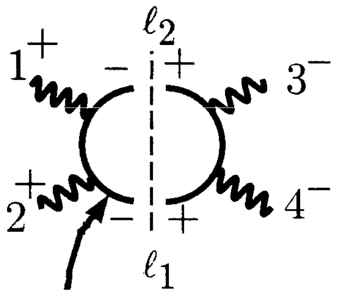
For non-susy theories proof breaks down since we can find rational functions not linked to any cuts, since a larger number of integral functions enter.

Will handle non-susy theories differently.

Example of Susy Cut Calculation

Consider $A_4^{N=4 \text{ susy}}(1^+, 2^+, 3^-, 4^-)$.

The $s = (k_1 + k_2)^2$ channel:



Only gluon loop contributes.

$$A_4^{\text{tree}}(\ell_1^-, 1^+, 2^+, \ell_2^-) = i\frac{\langle \ell_1\, \ell_2\rangle^4}{\langle \ell_1\, 1\rangle\, \langle 1\, 2\rangle\, \langle 2\, \ell_2\rangle\, \langle \ell_1\, \ell_2\rangle}$$

$$A_4^{\text{tree}}(\ell_2^+, 3^-, 4^-, \ell_2^+) = i\frac{\langle 3\, 4\rangle^4}{\langle \ell_2\, 3\rangle\, \langle 3\, 4\rangle\, \langle 4\, \ell_1\rangle\, \langle \ell_2\, \ell_1\rangle}$$

Must evaluate:

$$\int \frac{d^{4-2\epsilon}\ell_1}{(2\pi)^{4-2\epsilon}}\, \frac{1}{\ell_1^2}\, A_4^{\text{tree}}(\ell_1^-, 1^+, 2^+, \ell_2^-)\, \frac{1}{\ell_2^2}\, A_4^{\text{tree}}(\ell_2^+, 3^-, 4^-, \ell_2^+)$$

$$= A_4^{\text{tree}} \int \frac{d^{4-2\epsilon}\ell_1}{(2\pi)^{4-2\epsilon}}\, \frac{\text{tr}_+[\not\ell_1\ \not k_1\ \not k_4\ \not\ell_1\ \not\ell_2\ \not k_3\ \not k_2\ \not\ell_2]}{\ell_1^2(\ell_1 - k_1)^4(\ell_1 - k_1 - k_2)^2(\ell_1 + k_4)^4}$$

where

$$A_4^{\text{tree}} = \frac{\langle 3\, 4\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle\, \langle 3\, 4\rangle\, \langle 4\, 1\rangle}$$

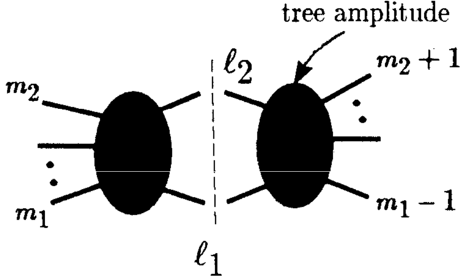
Used

$$\frac{1}{\langle 4\, \ell_1\rangle} = \frac{[\ell_1\, 4]}{2k_4\cdot \ell_1} = \frac{[\ell_1\, 4]}{(\ell_1 + k_4)^2},$$

$$[\ell_1\, 1]\, \langle 1\, 4\rangle\, [4\, \ell_1]\, \langle \ell_1\, \ell_2\rangle\, [\ell_2\, 3]\, \langle 3\, 2\rangle\, [2\, \ell_2]\, \langle \ell_2\, \ell_1\rangle$$

$$= \text{tr}_+[\not\ell_1\ \not k_1\ \not k_4\ \not\ell_1\ \not\ell_2\ \not k_3\ \not k_2\ \not\ell_2]$$

Cutting Procedure



1) Compress tree amplitudes using spinor helicity.

2) Apply Cutkosky rules to amplitudes not diagrams.

$$\int d\,\mathrm{LIPS}(-\ell_1,\ell_2)\; A^{\rm tree}(-\ell_1,m_1,\ldots,m_2,\ell_2) \\ \times\; A^{\rm tree}(-\ell_2,m_2+1,\ldots,m_1-1,\ell_1)$$

3) Replace phase space integral with unrestricted phase space integral. Avoids having to reconstruct logs.

$$\int \frac{d^D\ell_1}{(2\pi)^D}\; A^{\rm tree}(-\ell_1,m_1,\ldots,m_2,\ell_2)\frac{1}{\ell_2^2} \\ \times\; A^{\rm tree}(-\ell_2,m_2+1,\ldots,m_1-1,\ell_1)\frac{1}{\ell_1^2}\bigg|_{\rm cut}$$

4) Use $\ell_1^2=0$ and $\ell_2^2=0$ in $A^{\rm tree}$ to simplify.

5) Step through all cuts and write down function with correct cuts in all channels.

6) Write in terms of scalar bubble, triangle and box integral functions.

7) For susy amplitudes result automatically has correct rational terms.

Now simplify numerator:

$$\mathrm{tr}_+[\not{\ell}_1\ \not{k}_1\ \not{k}_4\ \not{\ell}_1\ \not{\ell}_2\ \not{k}_3\ \not{k}_2\ \not{\ell}_2] = -4\mathrm{tr}_+[\not{k}_4\ \not{k}_3\ \not{k}_2\ \not{k}_1]\ell_1\cdot k_4\ \ell_1\cdot k_1$$
$$= st(\ell_1 - k_1)^2(\ell_1 + k_4)^2$$

where we used $\not{\ell}_1^2 = 0, \qquad \not{\ell}_1\ \not{\ell}_2 = \not{\ell}_1\ \not{k}_3 + \not{\ell}_1\ \not{k}_4$

Cancels bad propagators!

Thus the s channel cut is:

$$A_4^{\text{tree}}\ st\ D(s,t)\Big|_{s-\text{cut}}$$

where $D(s,t)$ is the scalar box function.

Other cut is similar except fermion and scalar loops contribute. But answer is still of the same form:

$$A_4^{\text{tree}}\ st\ D(s,t)\Big|_{t-\text{cut}}$$

Putting together the cuts gives us

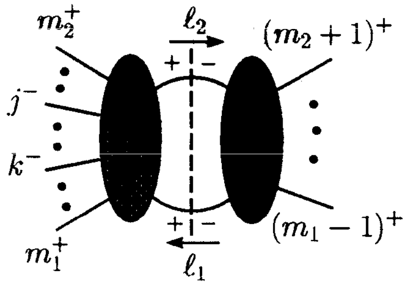
$$A_4^{N=4\ \text{susy}}(1^+,2^+,3^-,4^-) = A_4^{\text{tree}} st D(s,t)$$
$$= A_4^{\text{tree}}\frac{(4\pi)^\epsilon}{16\pi^2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{-\frac{2}{\epsilon^2}\left[(\frac{\mu^2}{-s_{12}})^\epsilon + (\frac{\mu^2}{-s_{23}})^\epsilon\right]\right.$$
$$\left. + \ln^2\left(\frac{-s_{12}}{-s_{23}}\right) + \pi^2\right\}$$

This result is guaranteed to have the correct rational term.

Agrees with string-based calculation.

Arbitrary number of legs

Consider maximally helicity violating amplitudes.

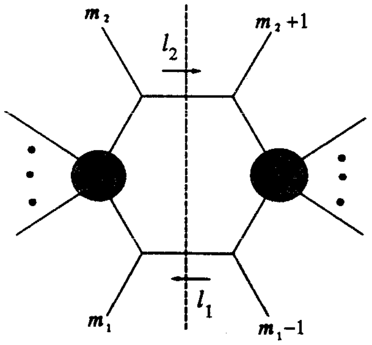


The tree-level Parke-Taylor amplitudes for n gluons have a remarkable property:

$$A^{\rm tree}(\ell_1^+, m_1^+, \cdots, k^-, \cdots, j^-, \cdots, m_2^+, \ell_2^+)$$
$$= \frac{\langle k\, j\rangle^4}{\langle \ell_1\, m_1\rangle \langle m_1,\, m_1+1\rangle \cdots \langle m_2-1,\, m_2\rangle \langle m_2\, \ell_2\rangle \langle \ell_1\, \ell_1\rangle}$$

Only 2 denominators in each tree have non-trivial dependence on loop momentum.

Together with 2 cut propagators the 4 denominators from the trees give at worst a hexagon integral.

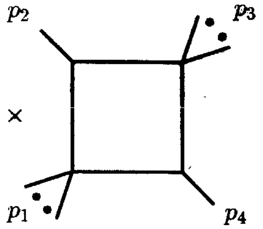


All-n MHV amplitudes can be calculated by doing hexagon integrals.

We have calculated:

- $A_{n;1}^{N=4\ \mathrm{MHV}}$

$$A_{n;1}^{N=4\ \mathrm{MHV}} = -\frac{1}{2} A_n^{\mathrm{tree}} \sum \left[(p_1+p_2)^2 (p_4+p_1)^2 - p_1^2 p_3^2 \right] \times$$



- $A_{n;1}^{N=1\ \mathrm{MHV}}$
- $A_{6;1}^{N=4}$ for all helicities.

Computers were not used to obtain all-n expressions: no large explosion of algebra.

Since QCD is not supersymmetric we would really like to calculate non-susy amplitudes.

$$A_n^{\mathrm{fermion\ loop}} = -A_n^{\mathrm{scalar\ loop}} + A_n^{N=1\ \mathrm{susy}},$$

$$A_n^{\mathrm{gluon\ loop}} = A_n^{\mathrm{scalar\ loop}} - 4A_n^{N=1\ \mathrm{susy}} + A_n^{N=4\ \mathrm{susy}}$$

We need another trick.

Non-Susy Amplitudes

Consider: $|A_{4;1}(1^+,2^+,3^+,4^+)| = \frac{1}{48\pi^2}$

Has no cuts! How do we construct real rational parts from nothing?

Favorite Magic Trick: Continue the amplitude to $D = 4 - 2\epsilon$ dimensions.

$$A^{D=4-2\epsilon} = \text{rational} - \epsilon \sum \text{rational}_i \times \ln s_i + \cdots$$

does have cuts at $\mathcal{O}(\epsilon)$.

From dimensional analysis in massless theories:

$$\begin{aligned} A^{D=4-2\epsilon} &\sim \int d^{4-2\epsilon}p \ \cdots \\ &\sim \sum_i (s_i)^{-\epsilon} \times \text{rational}_i + \cdots \\ &\sim \sum_i \text{rational}_i (1 - \epsilon \ln s_i) + \cdots \end{aligned}$$

Thus:

$$\text{rational} = \sum_i \text{rational}_i$$

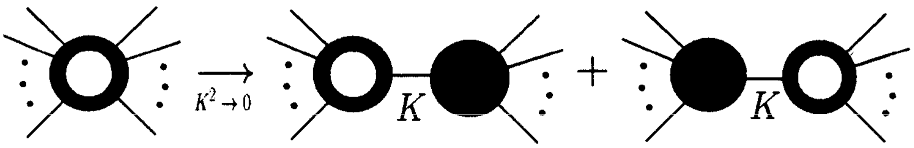
From $\mathcal{O}(\epsilon)$ cuts can reconstruct $\mathcal{O}(\epsilon^0)$ rational terms.

Ideas can also be used in massive theories. (Z.B and A.G. Morgan; J. Rozowsky).

Bern, Chalmers, Dixon, Kosower

Factorization

Amplitudes in quantum field theory should factorize on particle poles:



This is extremely useful:

- Can be used as strong check.
- Can be used to to construct new amplitudes from old ones – especially useful for the rational function parts.

$$A_n(1^+,2^+,\ldots,n^+) = \frac{1}{48\pi^2}\sum_{1\le i_1<i_2<i_3<i_4\le n}\frac{\mathrm{tr}[i_1 i_2 i_3 i_4]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

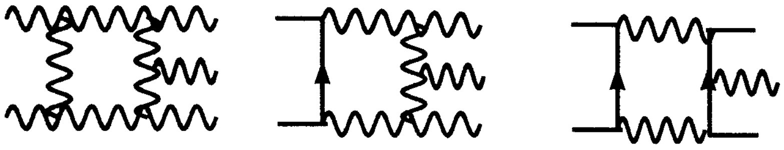
We have a proof of the universal factorization of one-loop amplitudes. This is non-trivial because of the IR divergences (Z.B. and G. Chalmers).

Claim: If we find a function that has all the correct poles, then this should be the amplitude. But no proof.

This can be a very powerful way of doing things since we avoid having to evaluate loop integrals.

$\bar{p}p \to$ Three Jets

All one-loop five amplitudes for three jet production at Fermilab or LHC.



Z.B., Dixon and Kosower
Kunszt, Signer and Trócsányi

Jet programs are currently being written. Giele and Kilgore
Trocsanyi

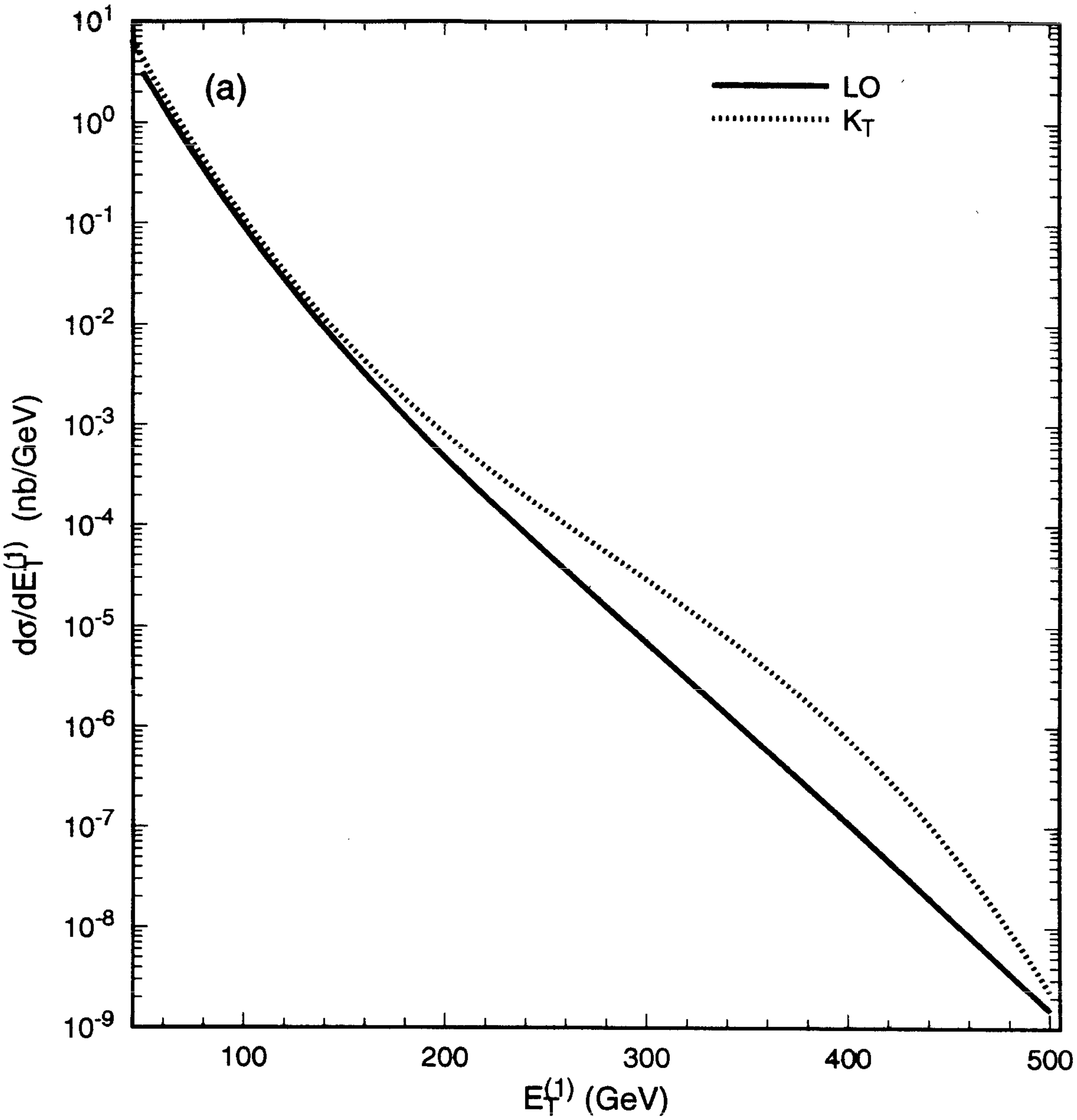
Need supercomputer ($>$ 10 Gigaflops) to run programs.

- Measurements of QCD coupling constant at highest possible energies.

$$\alpha_s \sim \frac{\sigma(\bar{p}p \to 3 \text{ jets})}{\sigma(\bar{p}p \to 2 \text{ jets})}$$

- Studies of jet structure.
- New physics searches.

Pure Glue Results



NLO corrections can be large

Z.B

L. Dixon

D.A. Kosower

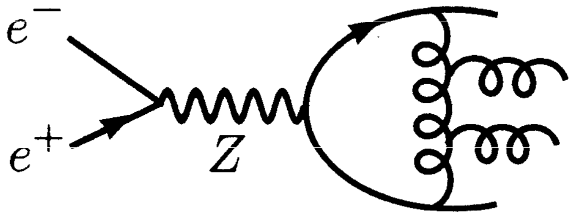
S. Weinzierl

NLO corrections to $e^+e^- \to 4$ Jets

- Improved measurements of QCD coupling constant.

- This is the lowest-order process in which the quark and gluon color charges can be measured independently.

- At LEP it is a background to threshold production of W pairs, when both Ws decay hadronically.

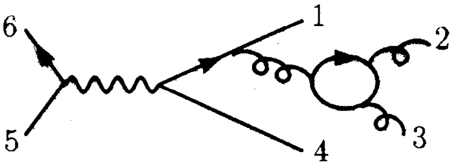


Six-point kinematics makes this rather technically involved.

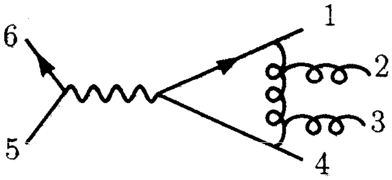
Amplitudes were obtained by finding functions with correct poles and cuts.

A. Signer has written a jet program which uses these amplitudes to make theoretical predictions for jets.

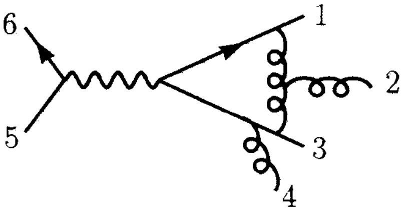
(a) $A_6^{s,f}(1_q, 2, 3, 4_{\bar{q}})$



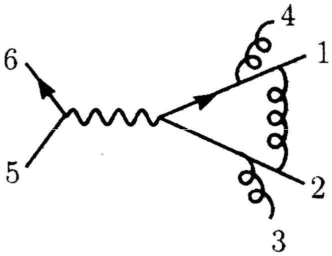
(b) $A_6(1_q, 2, 3, 4_{\bar{q}})$



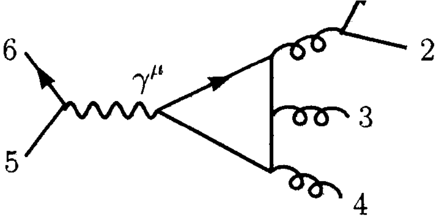
(c) $A_6(1_q, 2, 3_{\bar{q}}, 4)$



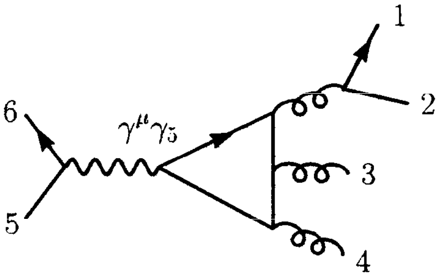
(d) $A_6(1_q, 2_{\bar{q}}, 3, 4)$



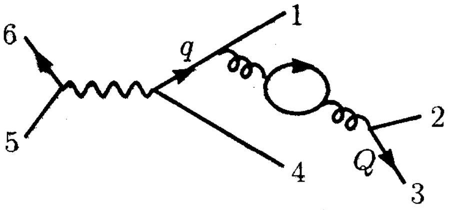
(e) $A_6^{\mathrm{v}\,s,f}(1_q, 2_{\bar{q}}, 3, 4)$



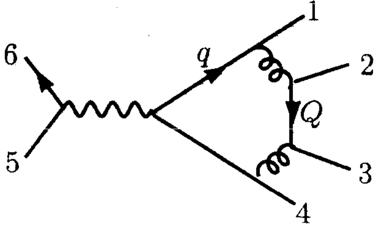
(f) $A_6^{\mathrm{ax}}(1_q, 2_{\bar{q}}, 3, 4)$



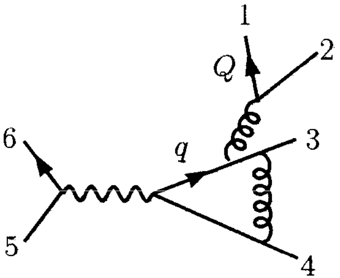
$(a)\quad A_6^{sf}(1_q, 2_{\bar{Q}}, 3_Q, 4_{\bar{q}})$



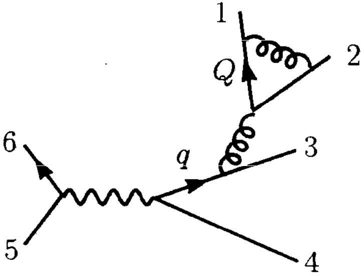
$(b)\quad A_6(1_q, 2_{\bar{Q}}, 3_Q, 4_{\bar{q}})$



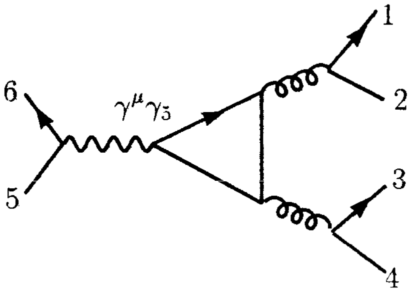
$(c)\quad A_6^{\mathrm{sl(a)}}(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}})$



$(d)\quad A_6^{\mathrm{sl(b)}}(1_Q, 2_{\bar{Q}}, 3_q, 4_{\bar{q}})$



$(e)\quad A_6^{\mathrm{ax}}(1_q, 2_{\bar{q}}, 3_Q, 4_{\bar{Q}})$



Sample Result: $e^+e^- \to q^+\, g^-\, g^+\, \bar{q}^-$

$$A_6^{\rm tree} = i\left[-\frac{[13]^2\,\langle 45\rangle\,\langle 2|(1+3)|6\rangle}{[12]\,s_{23}t_{123}s_{56}} + \frac{\langle 24\rangle^2\,[16]\,\langle 5|(2+4)|3\rangle}{\langle 34\rangle\,s_{23}t_{234}s_{56}} + \frac{[13]\,\langle 24\rangle\,[16]\,\langle 45\rangle}{[12]\,\langle 34\rangle\,s_{23}s_{56}}\right].$$

$$V^{sc} \;=\; \frac{1}{2\epsilon}\left(\frac{\mu^2}{-s_{56}}\right)^{\epsilon} + \frac{1}{2}\,,$$

$$\begin{aligned}
F^{sc} = {} & \frac{\langle 12\rangle\,\langle 23\rangle\,[13]^2\,\langle 1|(2+3)|6\rangle^2}{[56]\,\langle 13\rangle\,t_{123}\langle 1|(2+3)|4\rangle}\left[\frac{\mathrm{Ls}_1\left(\frac{-s_{12}}{-t_{123}},\frac{-s_{23}}{-t_{123}}\right)}{t_{123}^2} - \frac{1}{2}\,\frac{\mathrm{L}_1\left(\frac{-t_{123}}{-s_{23}}\right)}{s_{23}^2}\right] \\
& + \frac{\langle 23\rangle\,[46]^2\,t_{123}\langle 2|(1+3)|4\rangle}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^3}\,\mathrm{Ls}_{-1}^{2mh}(s_{34},t_{123};s_{12},s_{56}) \\
& + \frac{\langle 12\rangle\,[46]}{\langle 1|(2+3)|4\rangle\Delta_3}\Bigg[3\,[34]\,\langle 56\rangle\,\langle 2|(3+4)|1\rangle\big(\langle 3|5|6\rangle\delta_{56} - \langle 3|4|6\rangle\delta_{34}\big)\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3} \\
& \qquad + \big(3\,\langle 5|6|4\rangle\langle 2|3|1\rangle - \langle 5|3|4\rangle\langle 2|(3+4)|1\rangle\big)\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle} - [13]\,\langle 24\rangle\,[36]\,\langle 56\rangle \\
& \qquad + [14]\,\langle 23\rangle\,\langle 5|6|4\rangle(t_{123}-t_{124})\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle^2}\Bigg]\,I_3^{3m}(s_{12},s_{34},s_{56}) \\
& + \frac{\langle 24\rangle\,[46]^2\,\langle 2|(1+3)|4\rangle t_{123}}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\left(-\frac{1}{2}\frac{\langle 24\rangle}{\langle 23\rangle}\,\frac{\mathrm{L}_1\left(\frac{-s_{56}}{-t_{123}}\right)}{t_{123}^2} + \frac{1}{\langle 3|(1+2)|4\rangle}\,\frac{\mathrm{L}_0\left(\frac{-s_{56}}{-t_{123}}\right)}{t_{123}}\right) \\
& + \frac{\langle 2|13|2\rangle\,[46]^2\,t_{123}}{[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle^2}\,\frac{\mathrm{L}_0\left(\frac{-t_{123}}{-s_{12}}\right)}{s_{12}} + \frac{1}{2}\,\frac{\langle 2|1|3\rangle^2\langle 3|(1+2)|6\rangle^2}{[56]\,\langle 13\rangle\,t_{123}\langle 3|(1+2)|4\rangle}\,\frac{\mathrm{L}_1\left(\frac{-t_{123}}{-s_{12}}\right)}{s_{12}^2} \\
& + \frac{\langle 12\rangle\,[46]}{\langle 1|(2+3)|4\rangle}\Bigg[3\,\big(\langle 5|3|4\rangle\,\delta_{34} - \langle 5|6|4\rangle\,\delta_{56}\big)\,\frac{\langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle\,\Delta_3^2} \\
& \qquad + \frac{1}{2}\,\frac{1}{\langle 34\rangle\,[56]\,\langle 3|(1+2)|4\rangle\,\Delta_3}\Big(-\langle 24\rangle\,\delta_{12}\,\big([6|53|1] + [6|4(2+3)|1]\big) \\
& \qquad + \langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle\langle 3|(4-5)|6\rangle - \langle 2|(3+4)|1\rangle\langle 3|4|6\rangle\,\delta_{34}\,\frac{(t_{123}-t_{124})}{\langle 3|(1+2)|4\rangle}\Big) \\
& \qquad + \frac{1}{2}\,\frac{[46]\,\langle 2|(3+4)|1\rangle}{[56]\,\langle 3|(1+2)|4\rangle^2}\Bigg]\ln\left(\frac{-s_{12}}{-s_{56}}\right) \\
& + \frac{[46]}{\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle}\Bigg[-3\,\frac{\langle 12\rangle\,[34]\,\langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle\big(\langle 35\rangle\,\delta_{12} - 2\,\langle 3|46|5\rangle\big)}{\Delta_3^2} \\
& \qquad - \frac{1}{2}\,\frac{1}{[12]\,[56]\,\Delta_3}\Big(\langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle\big([4|3(1+2)|6] + [46]\,(\delta_{56} - 2s_{12})\big) \\
& \qquad - \delta_{34}\,\langle 2|4|3\rangle\big([6|53|1] + [6|4(2+3)|1]\big) - 2\,[46]\,t_{123}\Big(2\,\langle 2|(3-4)|1\rangle\langle 4|(1+2)|3\rangle \\
& \qquad + (t_{123}-t_{124})\Big([13]\,\langle 24\rangle + [14]\,\langle 23\rangle\,\frac{\langle 4|(1+2)|3\rangle}{\langle 3|(1+2)|4\rangle}\Big)\Big)\Big) - \frac{1}{2}\,\frac{[13]\,\langle 2|4|6\rangle}{[12]\,[56]}\Bigg]\ln\left(\frac{-s_{34}}{-s_{56}}\right) \\
& + \frac{1}{2}\,\frac{[46]\,\langle 2|(3+4)|1\rangle\langle 4|(1+2)|3\rangle\,\big(\langle 3|5|6\rangle\,\delta_{56} - \langle 3|4|6\rangle\,\delta_{34}\big)}{[12]\,\langle 34\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle\,\Delta_3} \\
& - \frac{1}{2}\,\frac{\langle 2|4|6\rangle\big([6|53|1] + [6|4(2+3)|1]\big)}{[12]\,\langle 34\rangle\,[56]\,\langle 1|(2+3)|4\rangle\langle 3|(1+2)|4\rangle} - \frac{1}{2}\,\frac{[13]^2\,\langle 1|(2+3)|6\rangle^2}{[12]\,[23]\,[56]\,\langle 13\rangle\,t_{123}\langle 1|(2+3)|4\rangle} \;+\; \mathrm{flip}_1\,.
\end{aligned}$$

Understanding the spurious poles is essential for obtaining ‘compact’ expressions.

[illegible]

(1)

[illegible]

(2)

[illegible]

(3)

[illegible]

(4)

[illegible]

(5)

[illegible]

(6)

[illegible]

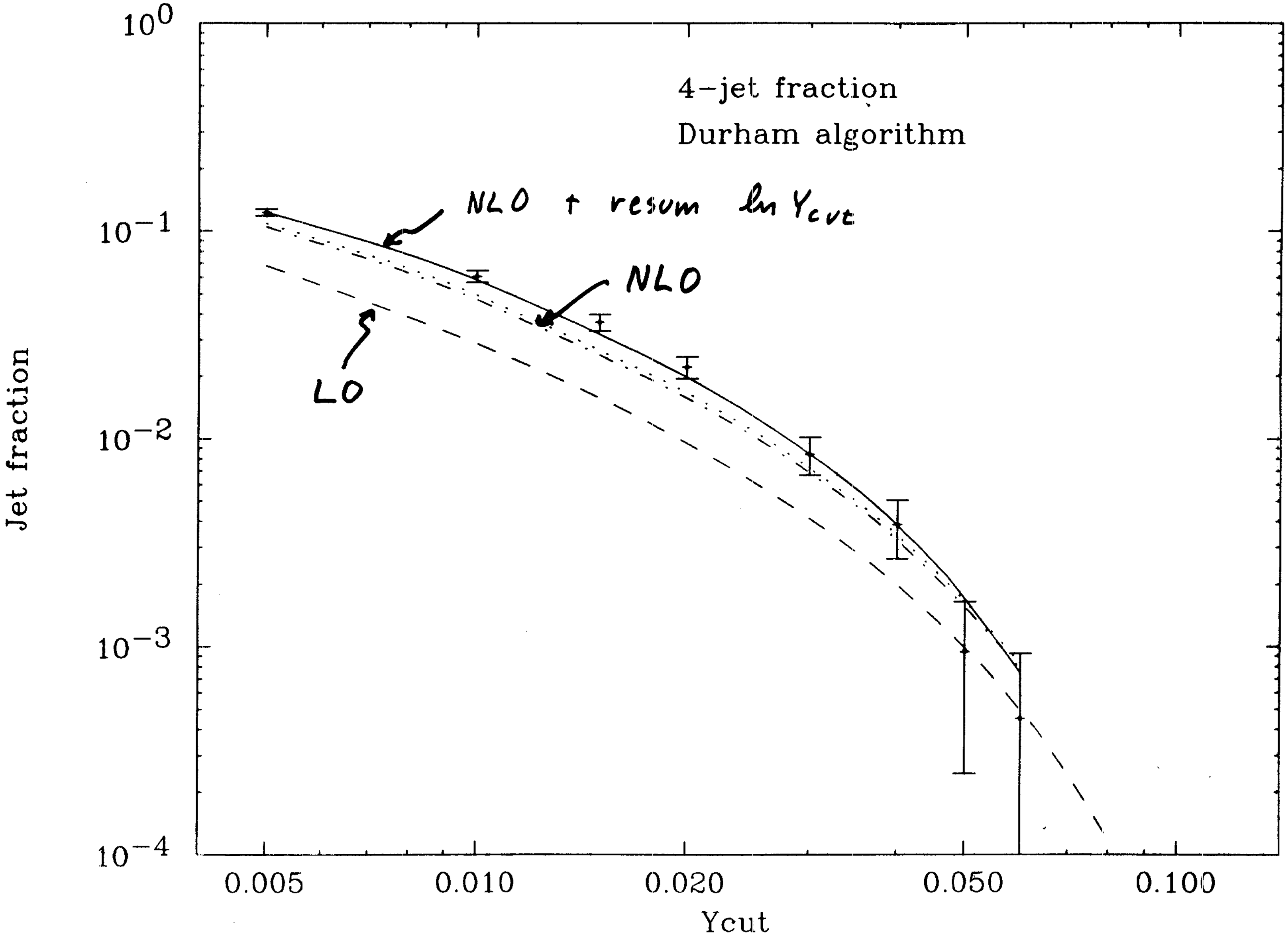
Correctness of Results?

How do we know results are correct?

We need strong checks:

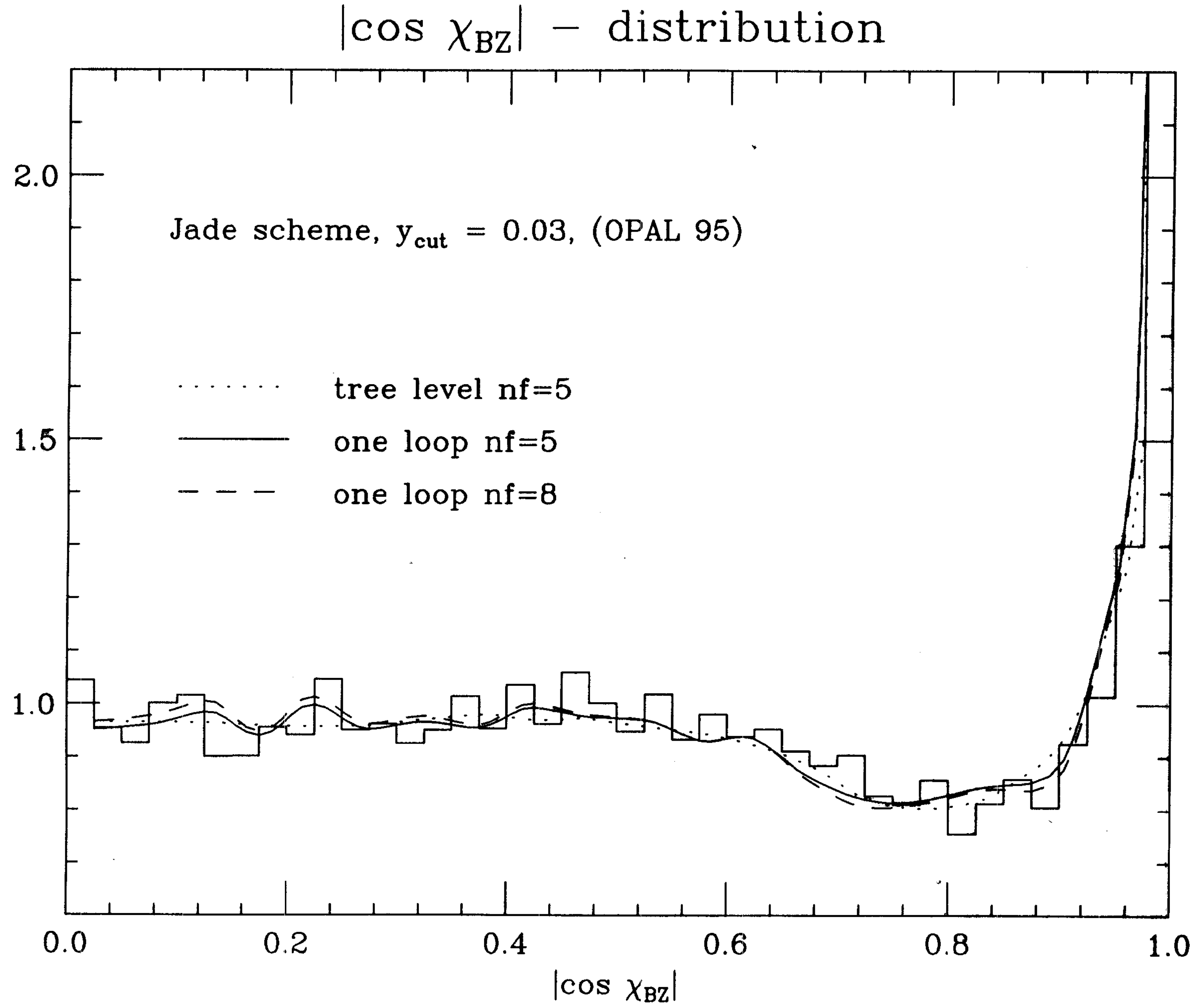
- Factorization
- Numerical comparison against Feynman diagrams.
- Gauge invariance
- Numerical comparison against independent calculation by Campbell, Glover and Miller.

Dixon + Signer



A. Signer's Program

Dixon & Signer


$|\cos \chi_{BZ}|$ – distribution

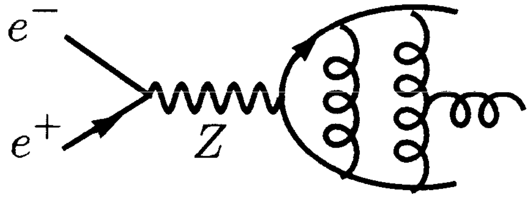
Future: Higher Loops

Some of the higher loop computations that have been performed:

- Gyromagnetic ratio of the electron, $g-2$, 4 loops, Kinoshita, etc.
- $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$, $\mathcal{O}(\alpha_s^3)$ Gorishny, Kataev and Larin, etc
- Four-Loop QCD β function, Ritbergen, Vermaseren, Larin.
- Two-loop form factors, van Neerven, etc

Examples of calculations which remain to be done :

No two- or higher-loop amplitudes have been computed which involves more than 1 kinematic variable.



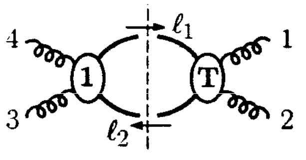
Important for improved measurements of α_s at LEP.

Another example is NNLO DGLAP splitting functions for parton evolution.

Provides motivation for investigating new calculational methods.

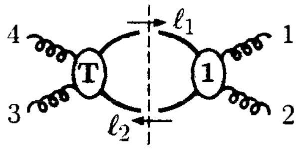
N=4 Susy as Two-Loop Test

Two-loop cuts:



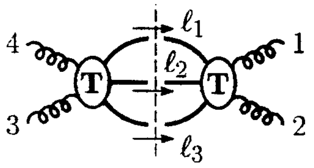
$\ell_1^2 = \ell_2^2 = 0$

(a)



$\ell_1^2 = \ell_2^2 = 0$

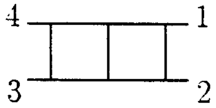
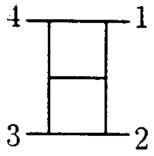
(b)



$\ell_1^2 = \ell_2^2 = \ell_3^2 = 0$

To test whether unitarity methods can lead to efficient calculations we evaluated two-loop $N = 4$ susy amplitudes. Results are amazingly simple:

For leading color

$A^{\rm LC}_{4;1;1}(1^-, 2^-, 3^+, 4^+) = -st\, A_4^{\rm tree} \Big\{ s$  $+\ t$  $\Big\}$

Similar results for all subleading in color terms.

Conjecture to all loop orders.

Much work remains:

- QCD calculations.
- Evaluation of basic integrals. (Davydychev and Ussyukina)
- IR cancellations for numerical jet programs.

Analytic approaches look promising.

Summary

1. New physics searches at SLAC, CERN and Fermilab require that we perform one- and higher-loop Feynman diagram calculations.

2. Feynman diagrams are clumsy beyond leading order.

3. Powerful tools:

   - String theory ideas
   - Helicity or circular polarization
   - Analytic properties

4. Amplitudes with an arbitrary number of external legs – unimaginable via Feynman diagrams.

5. New amplitudes for use in theoretical predictions for jets.

6. Comparisons of theory and experiment to search for new physics.

7. Future: NNLO calculations.

Powerful tools are now available for performing difficult Feynman diagram calculations required by experiments.