NULL SURFACES, INITIAL VALUES AND EVOLUTION OPERATORS FOR SPINOR FIELDS

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IN J MATH. PHYS. 37, 1091 (1996)

AUGUST, 1997
8TH LCW

- INTRODUCTION AND MOTIVATION

- DEVELOPMENT OF A SPINOR FUNCTION FROM A CONSTANT TIME SURFACE

- CHARACTERISTICS OF THE DIRAC EQUATION

- SOLUTION IN TERMS OF NULL COORDINATES

- INITIAL VALUES ON NULL SURFACES

- RELATION TO THE HAMILTONIAN(S)

- CONCLUSION

1. INTRODUCTION AND MOTIVATION

- USUAL CAUCHY PROBLEM FOR 1ST ORDER DIFF. EQ.: INITIAL DATA PLUS DIFF. EQ. CAN BE USED TO CONSTRUCT UNIQUE SOLUTION IN NEIGHBORHOOD OF SURFACE

BUT

IF INITIAL SURFACE IS WHERE SOLUTION HAS DISCONTINUITY, THIS PROCEDURE FAILS. SUCH SURFACES ARE CHARACTERISTICS.

IN THIS TALK WE PROPOSE TO CONSTRUCT UNIQUE SOLUTION FOR SUCH INITIAL SURFACES.

(PRE - QUANTIZATION)

SEVERAL REASON WHY WE WANT TO DO THIS:

- SHOW THE WAY TO CONSTRUCT SOLUTION FOR DATA SPECIFIED ON CHARACTERISTICS

- GIVE FIRMER FOUNDATION TO PREVIOUS WORK, AS REPORTED IN PHYS. REV. D50, 5289 (1994) AND PHYS. REV. D51, 3017 (1995).

- SHOW HOW TO CONSTRUCT UNIQUE PRE-QUANTIZED SOLUTION

DIRAC EQ.

2. SOLUTION FOR CONSTANT TIME SURFACE

$$i\,\Psi_{/t}(\vec{x},t) = [-i\,\vec{\alpha}\cdot\vec{\nabla} + \beta m]\,\Psi(\vec{x},t) = H\,\Psi(\vec{x},t)$$

WHERE $\quad /_t = \dfrac{\partial}{\partial t}$.

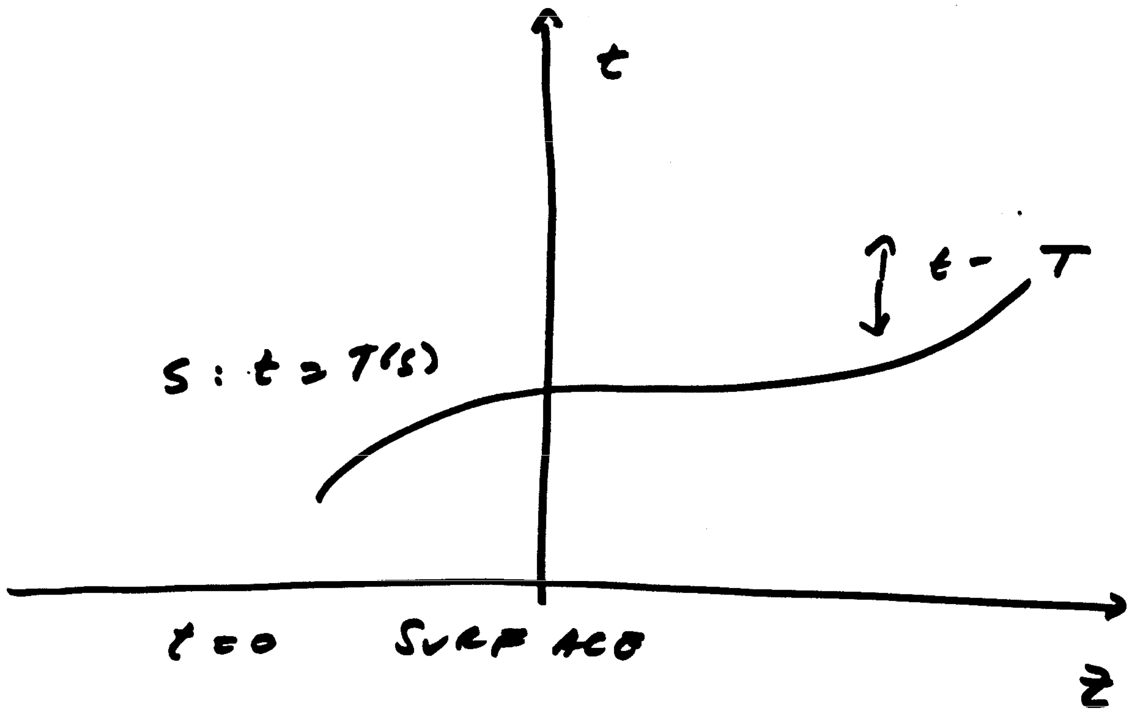
CAN EXPRESS SOLUTION AS A TAYLOR SERIES ABOUT $t=0$:

$$\Psi(\vec{x},t) = \Psi(\vec{x},t)\big|_{t=0} + \Psi_{/t}(\vec{x},t)\big|_{t=0}\,t + \Psi_{/t/t}(\vec{x},t)\,\frac{t^2}{2}\ldots$$

$$= \Psi(\vec{x},t)\big|_{t=0} + (-iH)\,\Psi(\vec{x},t)\big|_{t=0}\,t +$$

$$+ (-iH)^2\,\Psi(\vec{x},t)\big|_{t=0}\,t^2/2 + \cdots = e^{-iHt}\,h(\vec{x})$$

WHERE $\quad h(\vec{x}) = \Psi(\vec{x},t)\big|_{t=0}$.



INITIAL VALUE SURFACE FOR

THE DIRAC EQUATION

3. CHARACTERISTICS OF THE DIRAC EQUATIONS

LET US DEFINE AN ARBITRARY SURFACE S BY THE EQUATION $t = T(\vec{x})$, ON WHICH WE GIVE THE INITIAL VALUE:

$$\psi(\vec{x}, T(\vec{x})) = h(\vec{x}), \quad \text{SO THAT}$$

$$\psi(\vec{x}, t) = \psi(\vec{x}, t)\Big|_{t=T} + \psi_{/t}(\vec{x}, t)\Big|_{t=T}[t - T(\vec{x})] +$$

$$+ \psi_{/t/t}(\vec{x}, t)\Big|_{t=T}[t - T(\vec{x})]^2/2 + \cdots$$

NOW $$\vec{\nabla} h(\vec{x}) = \vec{\nabla}\psi(\vec{x}, t)\Big|_{t=T} + \psi_{/t}(\vec{x}, t)\Big|_{t=T}\vec{\nabla}T(\vec{x})$$

SO THIS PLUS DIRAC EQ. GIVES ON S

$$c\,[\,I - \vec{\alpha}\cdot\vec{\nabla}T(\vec{x})\,]\,\psi_{/t}(\vec{x}, t)\Big|_{t=T} = -c\,\vec{\alpha}\cdot\vec{\nabla}h(\vec{x}) +$$

$$+ \beta m\, h(\vec{x})$$

WE CAN OBTAIN $\psi_{/t}(\vec{x}, t)$ IF WE CAN INVERT $[\;\;]$.

S IS A CHARACTERISTIC IF

$$\det\left[I - \vec{\alpha}\cdot\vec{\nabla}T(\vec{x}) \right] = (1-\vec{\nabla}T(\vec{x}))^2 = 0$$

OR IF $\nabla^2 T(\vec{x}) = 1$

THE CHARACTERISTICS WE WILL USE ARE

$$u = t - z \quad (x^-)$$
$$v = t + z \quad (x^+)$$

4. SOLUTION FOR NULL COORDINATES

DIRAC EQ. IN COVARIANT FORM IS

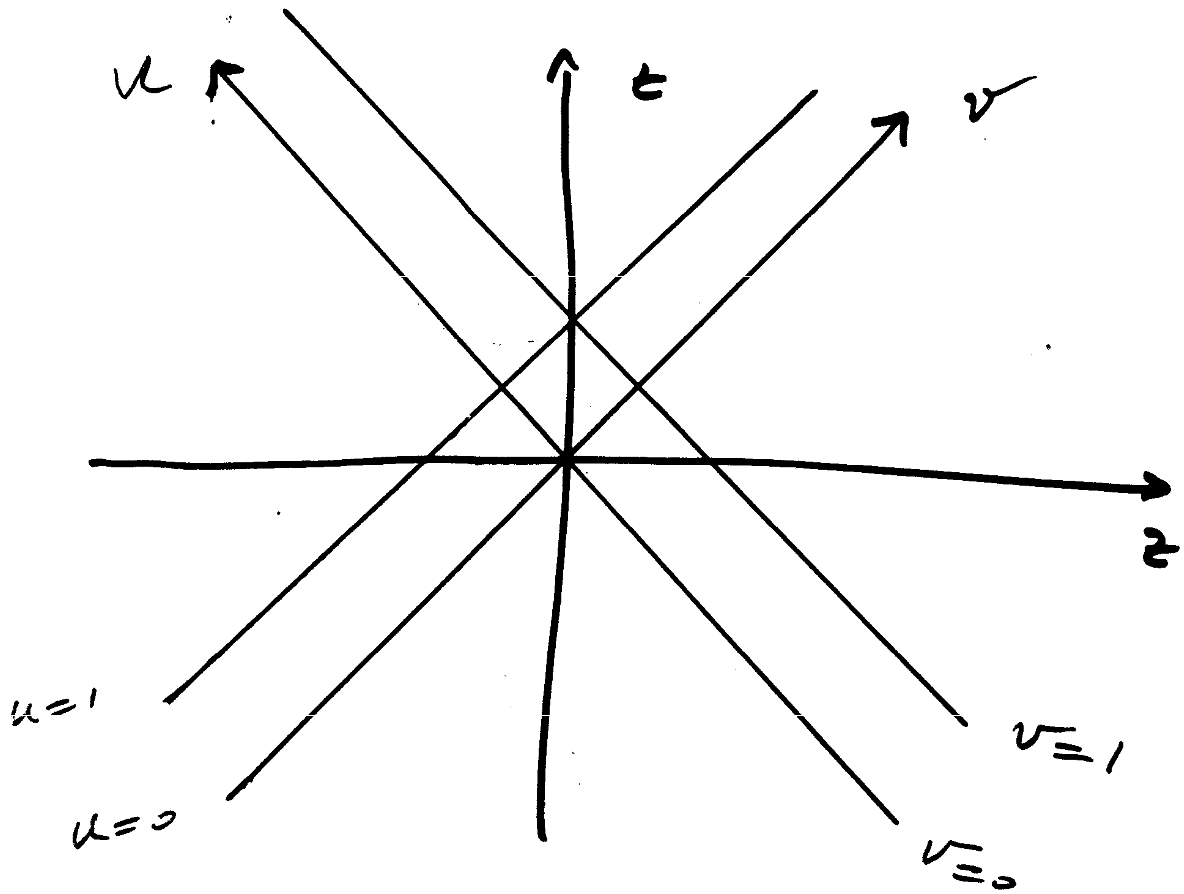
$$\gamma^\alpha i \frac{\partial}{\partial x^\alpha} \psi = m\psi$$

USING u AND v, $\gamma^u = \gamma^0 - \gamma^1$, $\gamma^v = \gamma^0 + \gamma^1$

AND SUPPRESSING PERPENDICULAR COMPONENTS

$$\left(\gamma^u i \frac{\partial}{\partial u} + \gamma^v i \frac{\partial}{\partial v}\right)\psi(u,v) = m\psi(u,v)$$

IT IS USEFUL TO DEFINE TWO PROJECTION



CARTESIAN COORDINATES t & z
AND NULL COORDINATES u & v

$$\Lambda_u = \frac{\gamma^u \gamma^v}{4}, \quad \Lambda_v = \frac{\gamma^v \gamma^u}{4}$$

SO THAT $\Lambda_u + \Lambda_v = 1, \quad \Lambda_u \Lambda_u = \Lambda_u,$

$$\Lambda_v \Lambda_v = \Lambda_v, \quad \Lambda_u \Lambda_v = \Lambda_v \Lambda_u = 0$$

WE EXPAND ABOUT THE SURFACES $u = 0$ AND $v = 0$ AND EXPAND IN POWERS OF m

$$\psi(u,v) = \psi^{(0)}(u,v) + m\,\psi^{(1)}(u,v) + m^2 \psi^{(2)}(u,v)$$

$$\psi^{(0)}(u,v) = \Lambda_u f(u) + \Lambda_v g(v)$$

PICK REPRESENTATION IN WHICH Λ'S ARE DIAGONAL, SO THAT

$$\Lambda_u = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda_v = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

I, 0 ARE 2×2 MATRICES,

$$\psi^{(0)}(u,v) = \begin{pmatrix} f_1(u) \\ f_2(u) \\ g_1(v) \\ g_2(v) \end{pmatrix}$$

AFTER SOME MANIPULATIONS, WE GOT

$$\psi^{(1)}(u,v) = \frac{-i}{4}\Big[\gamma^u \Lambda_v \int_{v_0}^{v} g(v')\,dv' + \gamma^v \Lambda_u \int_{u_0}^{u} f(u')\,du' \Big]$$

WE THEN GET

$$\psi^{(2)}(u,v) = -\frac{1}{4}\Big[(u-u_0)\Lambda_v \int_{v_0}^{v} g(v')\,dv' + (v-v_0)\Lambda_u \int_{u_0}^{u} f(u')\,du' \Big]$$

THE GENERAL SOLUTION IS

$$\psi(u,v) = \sum_{n=0}^{\infty} \frac{[-(m^2/4)(v-v_0)\,\Gamma_u]^n}{n!}\Big(1 - \frac{im}{4}\gamma^v \Gamma_u\Big)\Lambda_u f + \sum_{n=0}^{\infty} \frac{[-(m^2/4)(u-u_0)\,\Gamma_v]^n}{n!}\Big(1 - \frac{im}{4}\gamma^u \Gamma_v\Big)\Lambda_v g$$

AND
$$\Gamma_u^n = \int_{u_0}^{u} du' \int_{u_0}^{u'} du'' \cdots \int_{u_0}^{u^{n-1}} du^n f(u^n)$$

COMPACTLY, THIS IS WRITTEN AS

$$\Psi(u, v) = e^{+[-\frac{m^2}{4}(v-v_0)\Gamma_u]}\left(1 - \frac{im}{4}\gamma^v\Gamma_u\right)\Lambda_u f(u)$$
$$+ \quad e^{-\left(\frac{m^2}{4}\right)(u-u_0)\Gamma_v}\left(1 - \frac{im}{4}\gamma^u\Gamma_v\right)\Lambda_v g(v)$$

5. INITIAL VALUE ON NULL SURFACES

WE SET $u_0 = v_0 = 0$, SO

$$\Psi_0(u, 0) = \left(1 - \frac{im}{4}\gamma^v\Gamma_u\right)\Lambda_u f(u)$$
$$+ \quad \Lambda_v g(0)$$

$$\Psi_0(0, v) = \left(1 - \frac{im}{4}\gamma^u\Gamma_v\right)\Lambda_v g(v)$$
$$+ \quad \Lambda_u f(0)$$

ON THE SURFACE $u = v = 0$,

$$\Psi_0(0,0) = \Lambda_u f(0) + \Lambda_v g(0)$$

SO WE PICK $f(0) = g(0) = \Psi_0(0, 0)$

THE GENERAL SOLUTION THEN IS EXPRESSED THUS:

$$\Psi(u,v) = e^{-\frac{m^2}{4}(v-v_0)\Gamma_u}\left(\Psi_0(u,0) - \Lambda\Psi_0(0,0)\right)$$

$$+ e^{-\frac{m^2}{4}(u-u_0)\Gamma_v}\left(\Psi_0(0,v) - \Lambda_u\Psi_0(0,0)\right)$$

EXAMPLE:

PLANE WAVES

IF WE PUT INITIAL PLANE WAVES

IN OUR DIRAC SOLUTION, WE WILL PROPAGATE PLANE WAVES:

$$\psi(u,v) = e^{-i(Et-kx)}\,w(E,k)$$

$$= e^{-i(\lambda u + \tau v)}\,w(\lambda,\tau)$$

WHERE $\lambda = \frac{E+k}{2}$, $\tau = \frac{E-k}{2}$

$$\lambda\tau = \frac{E^2-k^2}{4} = \frac{m^2}{4}$$

THE INITIAL VALUES ARE

$$\psi_0(u,0) - \Lambda_v \psi_0(0,0) = (e^{-i\lambda u} - \Lambda_v)\,w$$

$$\psi_0(0,v) - \Lambda_u \psi_0(0,0) = (e^{-i\tau v} - \Lambda_u)\,w$$

THEN THE GENERAL SOLUTION IS

$$B = -i\lambda, \quad C = -i\tau, \quad BC = -m^2/4$$

$$\psi(u,v) = e^{BCv\Gamma_u}(e^{Bu} - \Lambda_v)\,w(\lambda,\tau) +$$
$$+ e^{BCu\Gamma_v}(e^{Cv} - \Lambda_u)\,w(\lambda,\tau)$$
$$= (e^{BCv\Gamma_u}e^{Bu} + e^{BCu\Gamma_v}e^{Cv})\,w(\lambda,\tau) -$$
$$- F_c(uv)\,w(\lambda,\tau)$$

AND

$$F_c(uv) = e^{-\frac{m^2}{4} v \Gamma_u} 1 = e^{-\frac{m^2}{4} u \Gamma_v} 1$$

$$= \sum_{n=0}^{\infty} \frac{\left(-\frac{m^2}{4} uv\right)^n}{n!}$$

AFTER SOME MANIPULATIONS WE HAVE

$$\left(e^{Bcv\Gamma_u} e^{Bu} + e^{Bcu\Gamma_v} e^{cv}\right) = e^{Bu + cv} + F_c(uv)$$

SO THAT THE COMPLETE SOLUTION IS

$$\psi(u, v) = e^{Bu + Cv} \omega(\lambda, \tau)$$

$$= e^{-i(\lambda u + \tau v)} \omega(\lambda, \tau)$$

OBSERVE THE CRUCIAL ROLE PLAYED BY THE CONSTANT TERM $F_c(uv)$; ON THE SURFACE $u = v = 0$.

6. RELATION TO THE HAMILTONIANS

THINK OF $\frac{\partial}{\partial t} = -iH$, SO THAT

$$\psi(\vec{x}, t) = e^{-(t-t_0)\partial/\partial t}\,\psi(\vec{x}, t_0) \leftarrow \text{SYMBOLIC TAYLOR SERIES}$$

$$= e^{-i(t-t_0)H}\,\psi(\vec{x}, t_0)$$

WE REWRITE OUR EQUATION FOR $\psi(u,v)$

$$\psi(u,v) = e^{-\frac{m^2}{4}(v-v_0)\frac{\partial}{\partial v}}\left(\psi_0(u,0) - \Lambda_v\psi_0(0,0)\right)$$

$$+\; e^{-\frac{m^2}{4}(u-u_0)\frac{\partial}{\partial u}}\left(\psi_0(v,0) - \Lambda_u\psi_0(0,0)\right)$$

SINCE $\frac{\partial^2}{\partial u\,\partial v}\psi(u,v) = -\frac{m^2}{4}\psi(u,v)$

AT LEAST FORMALLY,

$$\frac{\partial}{\partial v} = -\frac{m^2}{4}\left(\frac{\partial}{\partial u}\right)^{-1} = -\frac{m^2}{4}\Gamma_u$$

ALSO, WE NEED **TWO** HAMILTONIANS

Γ_u AND Γ_v.

CONCLUSIONS

- WE CONSTRUCTED SOLUTIONS TO DIRAC EQUATION AND WE FOUND THAT WE NEED DATA ON BOTH CHARACTERISTICS $u=0$ AND $v=0$ AS WELL AS $u=v=0$ TO OBTAIN A UNIQUE, CONSISTENT SOLUTION. IT IS THE UNIQUENESS I.E, LACK OF CONSTANTS, WHICH WILL GIVE AN UNAMBIGUOUS QUANTUM SOLUTION.

- NEED TWO HAMILTONIANS, ONE FOR EACH SURFACE TO EVOLVE THE INITIAL DATA.