Mesons as Bound States

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Dyson Schwinger Equations

- Nonperturbative, continuum approach to QCD
- Weak coupling expansion $\Rightarrow$ Perturbation theory
- Dynamical Chiral Symmetry Breaking (DχSB)
- Quark and gluon confinement
- Bound state structure: Identifiable signatures of quarks and gluons in mesons and baryons
  - Mesons: Bethe–Salpeter Eqn
  - Baryons: Fadde'ev Eqn — Axel Bender
- Finite temperature and density — David Blaschke
  - Phase transition to a quark-gluon plasma: Deconfinement & χ symmetry restoration
  - Response of bound state properties
  - Thermodynamics
- Applications to meson phenomenology
  - $F_{\pi,\,K^{\pm},\,K^0}(Q^2)$ — $F_{K\ell 3,\,\pi e3}(Q^2)$ — $\gamma^*\,\pi \rightarrow \gamma$
  - $\gamma\,\pi^* \rightarrow \pi\,\pi$ — $\gamma^*\,p \rightarrow V\,p$ — $B^0 \rightarrow D^-\ell^+\nu$

Peter Tandy, Mike Pichowsky

Recent Collaborators and Contributors

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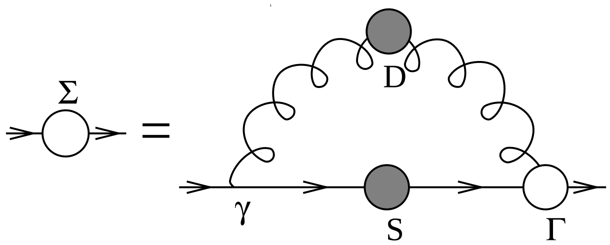
Papers

P. Maris, C.D. Roberts, and P.C. Tandy,
"Pion Mass and Decay Constant", nucl-th/9707003

P. Maris and C.D. Roberts, "π- and K-meson Bethe–Salpeter Amplitudes", nucl-th/9708029

- QUARK DYSON–SCHWINGER EQN

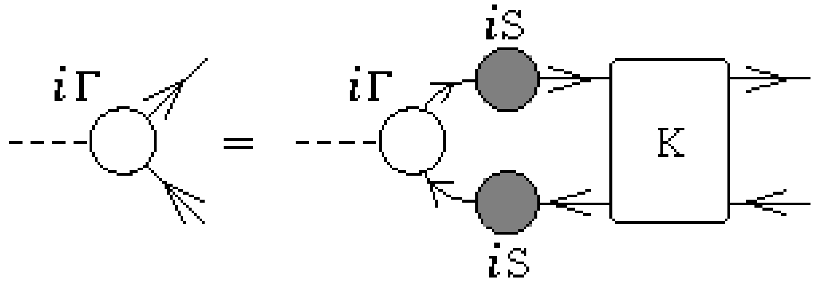
$$
\begin{aligned}
S(p,\mu)^{-1} &= i\gamma\cdot p A(p^2,\mu^2) + B(p^2, mu^2) \\
&= Z_2 i\gamma\cdot p + Z_4 m(\mu) + \Sigma'(p,\Lambda) \\
\Sigma'(p,\Lambda) &= Z_1 \int^{\Lambda} \frac{d^4k}{(2\pi)^4}\, g^2 D_{\mu\nu}(q)\gamma_\mu \tfrac{\lambda^a}{2} S(k)\Gamma_\nu(k,p)\tfrac{\lambda^a}{2}
\end{aligned}
$$



- MESON BETHE–SALPETER EQN

$$\Gamma_H(p;P) = -\int^{\Lambda} \frac{d^4q}{(2\pi)^4} \mathrm{K}(p,q)[S(q_+)\Gamma_H(q;P)S(q_-)]$$

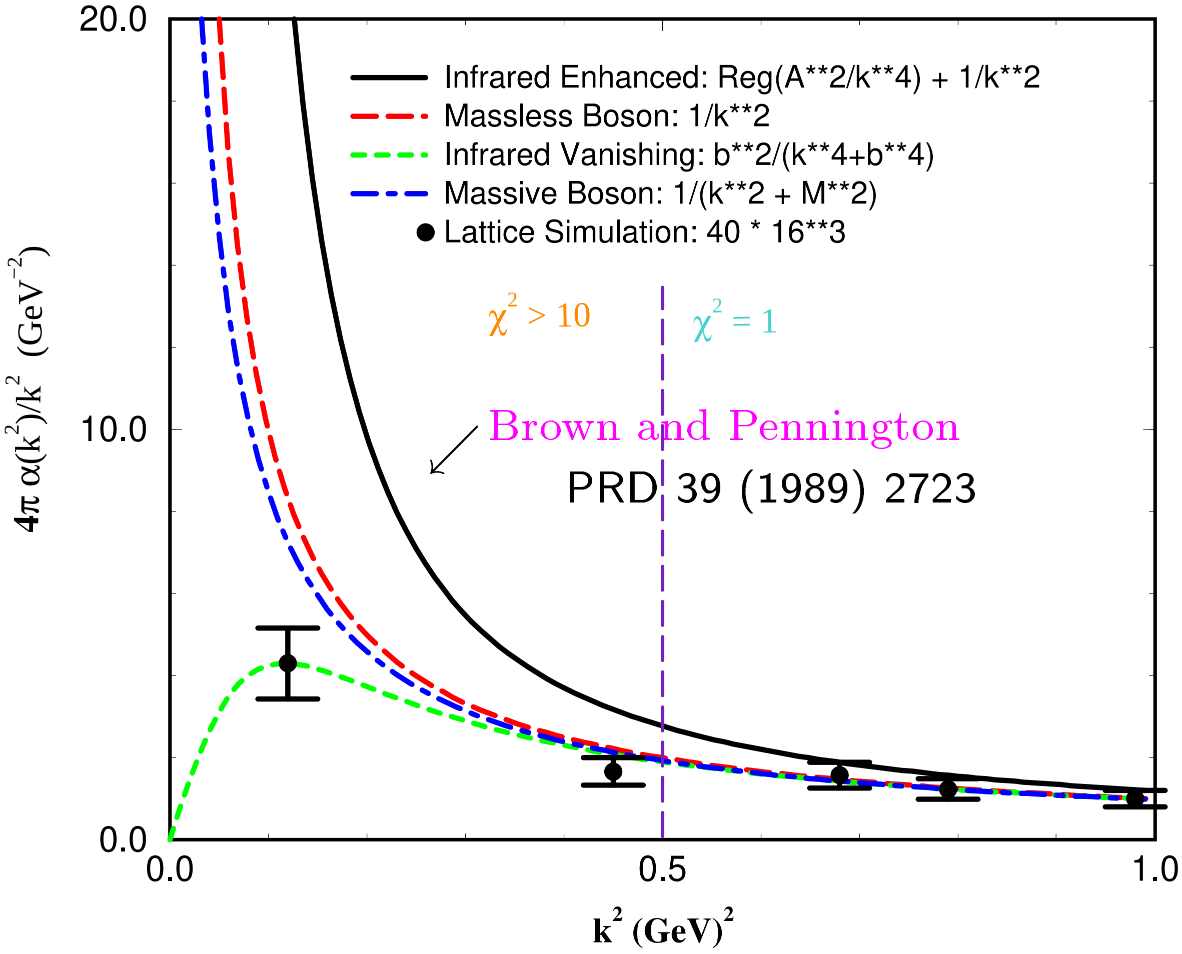
at mass pole $P^2 = -m^2_{\mathrm{meson}}$



- Need truncation consistent with axial-vector Ward–Takahashi identity: Ladder approximation

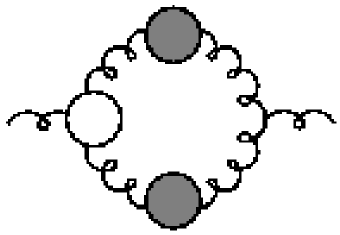
- Need Model Gluon Propagator

Gluon Propagator



- Lattice simulations cannot discriminate between IR vanishing and IR enhanced gluon propagators

- Self-consistent studies of gluon DSE show large enhancement in IR driven by gluonic contributions to $\Pi^g_{\mu\nu}$



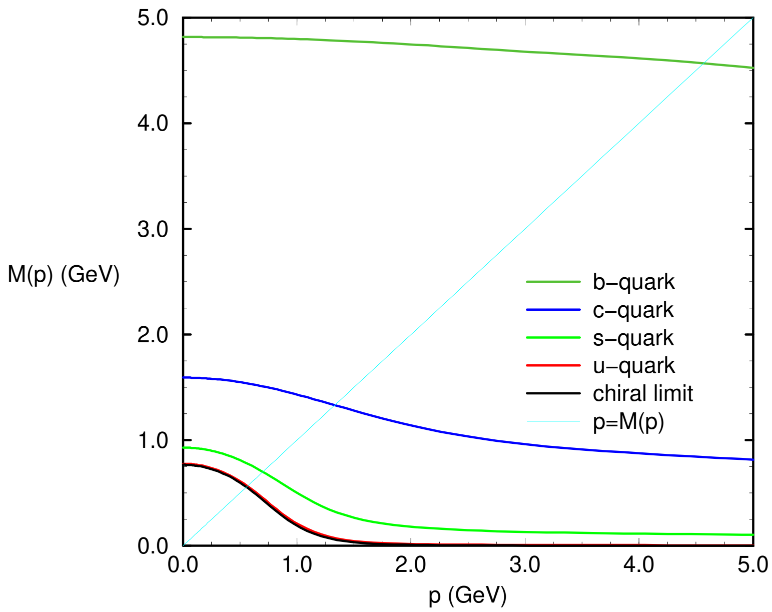
$\Longrightarrow$ Use IR enhanced gluon propagator

Typical results for Quark Propagator

- $M(p^2) \equiv \dfrac{B(p^2)}{A(p^2)}$
- Euclidean

constituent quark mass:

$$M_f^E\colon\ p^2 = M(p^2)^2$$



flavour	u/d	s	c	b	t
$\frac{M^E}{m_{\mu\sim 20\,\mathrm{GeV}}}$	150	10	2.3	1.1	$\to 1$

- IR Enhancement of gluon propagator

  $\Rightarrow$ IR Enhancement of quark mass function

- D χSB: More than just $\langle \bar{q}q\rangle_\mu \neq 0$

  Qualitative difference between light and heavy quarks

  Conrad Burden

- Confinement
  - Absence of quark/gluon production thresholds in colour-singlet-to-singlet $\mathcal{S}$-matrix amplitudes
  - Sufficient condition:

    Absence of Lehmann representation

    for dressed quark and gluon propagators

DχSB and Goldstone's Theorem

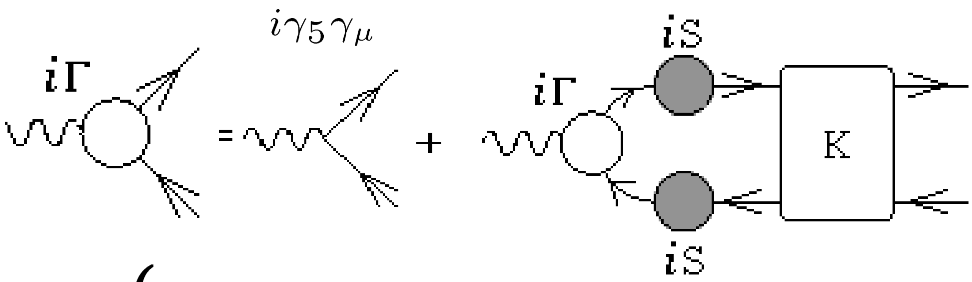
- Chiral-limit axial-vector Ward–Takahashi identity

$$-iP_\mu \Gamma_{5\mu}(k;P) = S^{-1}(k_+)\gamma_5 \frac{\tau}{2} + \gamma_5 \frac{\tau}{2} S^{-1}(k_-)$$

- Axial-vector vertex, chiral limit

$$\begin{aligned}\Gamma_{5\mu}(k;P) \quad &= \quad \frac{\tau}{2}\gamma_5 \left[\gamma_\mu F_R(k;P) + \gamma\cdot k k_\mu G_R(k;P)\right. \\ &\quad \left. -\sigma_{\mu\nu}\, k_\nu\, H_R(k;P)\right] + \tilde{\Gamma}_{5\mu}(k;P) + \; f_\pi \,\frac{P_\mu}{P^2}\,\Gamma_\pi(k;P)\end{aligned}$$

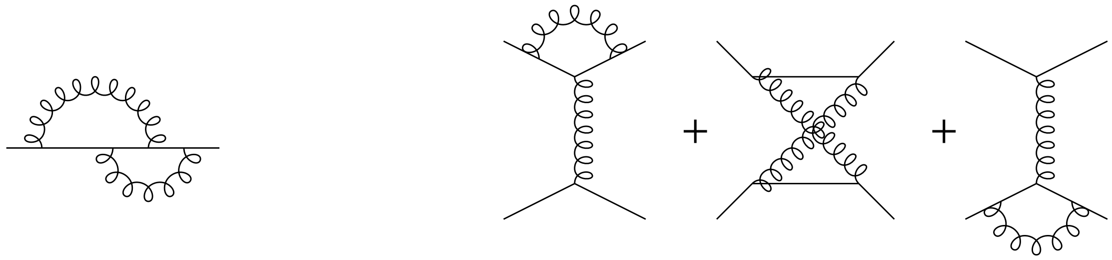
- Axial-vector vertex satisfies:



- Choice of $\left\{ \begin{array}{l} \Gamma_\mu \text{ in DSE}_q \\ \boxed{K} \text{ in BSE}_\pi \end{array} \right.$ tied together

  – Ladder approximation is consistent

  – can be improved systematically



(A. Bender, C.D. Roberts, L.v.Smekal, PLB380 (1996) 7)

- $P_\mu \Gamma_{5\mu}(k;P)$ is a pseudoscalar:
  Contains information about pseudoscalar mesons
- Pseudoscalar Bethe–Salpeter amplitude

$$\begin{aligned}\Gamma_\pi(k;P) &= \tau\gamma_5\Big[iE_\pi(k;P) + \gamma\cdot P F_\pi(k;P)\\ &\quad + \gamma\cdot k\, k\cdot P\, G_\pi(k;P) + \sigma_{\mu\nu}\, k_\mu P_\nu\, H_\pi(k;P)\Big]\end{aligned}$$

- Axial-vector Ward–Takahashi Identity (AV-WTI)

$$\begin{aligned} f_\pi E_\pi(k;P=0) &= B(k^2)\\ F_R(k;0) + 2\, f_\pi F_\pi(k;0) &= A(k^2)\\ G_R(k;0) + 2\, f_\pi G_\pi(k;0) &= 2A'(k^2)\\ H_R(k;0) + 2\, f_\pi H_\pi(k;0) &= 0\end{aligned}$$

- $B(k^2) \not\equiv 0$ solution of quark-DSE without
  an explicit current quark mass signals D χSB
- $B(k^2) \not\equiv 0$ implies $E_\pi(k;P=0) \not\equiv 0$:
  $\Rightarrow$ existence of a massless, pseudoscalar bound states: π's
  - F_π, G_π, H_π also necessarily nonzero
  - $\Gamma_{5\mu}(k;P)$ dominated by $P^2 = 0$ pole,
    associated with this bound state
    * Residue is leptonic decay constant: f_π

$$f_\pi P_\mu = Z_2 \int_q^\Lambda \mathrm{Tr}\big[\tfrac{\tau}{2}\gamma_5\gamma_\mu S(q_+)\Gamma_\pi(q;P)S(q_-)\big]$$

D χSB$\Longleftrightarrow$ existence of Goldstone bosons

Axial-Vector Ward-Takahashi Identity

with explicit current quark mass $m_q(\mu)$

$$-iP_\mu\Gamma_{5\mu}(k;P) = S^{-1}(k_+)\gamma_5\frac{\tau}{2} + \gamma_5\frac{\tau}{2}S^{-1}(k_-) - 2\,m_q(\mu)\,\Gamma_5(k;P)$$

relates poles in axial-vector and pseudoscalar vertices

$$\begin{aligned}\Gamma_{5\mu}(k;P) &= \frac{\tau}{2}\gamma_5\big[\gamma_\mu F_R(k;P) + \gamma\cdot k k_\mu G_R(k;P) \\ &\quad -\sigma_{\mu\nu}\,k_\nu\,H_R(k;P)\big] + \tilde{\Gamma}_{5\mu}(k;P) + \boxed{\frac{f_\pi\,P_\mu}{P^2+m_\pi^2}\,\Gamma_\pi(k;P)} \\ \Gamma_5(k;P) &= \frac{\tau}{2}\gamma_5\big[i\mathcal{E}_R^H(k;P) + \gamma\cdot P\,\mathcal{F}_R^H(k;P) \\ &\quad +\gamma\cdot k\,k\cdot P\,\mathcal{G}_R^H(k;P) + \sigma_{\mu\nu}\,k_\mu P_\nu\,\mathcal{H}_R^H(k;P)\big] \\ &\quad + \boxed{\frac{r_\pi}{P^2+m_\pi^2}\,\Gamma_\pi(k;P)}\end{aligned}$$

– $\mathcal{E}_R^H$, F_R^H, $\mathcal{F}_R^H$, G_R^H, $\mathcal{G}_R^H$, H_R^H, $\mathcal{H}_R^H$, $\tilde{\Gamma}_{5\mu}^H$ regular as $P^2\to -m_H^2$

– $P_\mu\tilde{\Gamma}_{5\mu}^H(k;P)\sim \mathrm{O}(P^2)$

AV-WTI, independent of the magnitude of m_q, leads to

$$\begin{aligned}f_\pi\,m_\pi^2 &= 2\,r_\pi\,m_q(\mu) \\ &= -2\,i\,m_q(\mu)Z_4\int_q^\Lambda \mathrm{Tr}\Big[\frac{\tau}{2}\gamma_5 S(q_+)\Gamma_\pi(q;P)S(q_-)\Big] \\ &= Z_2\int_q^\Lambda \mathrm{Tr}\Big[\frac{\tau}{2}\gamma_5\gamma_\mu S(q_+)\Gamma_H(q;P)S(q_-)\,P_\mu\Big]\end{aligned}$$

Two independent integrals for f_π

Corollary: Gell-Mann–Oakes–Renner relation

- Chiral limit

$$r_0 = -\frac{1}{f_0}\,\langle\bar{q}q\rangle_\mu^0, \qquad -\langle\bar{q}q\rangle_\mu^0 \equiv N_c\,Z_4\int_q^\Lambda \mathrm{Tr}_{\mathrm{Dirac}}\left[S_{m_q=0}\right]$$

- Outside chiral limit: AV-WTI $f_H\,m_H^2 = r_H\,\mathcal{M}_H$ with

$$\mathcal{M}_H \equiv \mathrm{tr}_{\mathrm{flavour}}\left[M_{(\mu)}\,\left\{T^H,\left(T^H\right)^{\mathrm{t}}\right\}\right]$$

gives to leading order in current mass

$$\begin{aligned} {f_\pi}^2 m_\pi^2 &= -\left[m_u(\mu)+m_d(\mu)\right]\langle\bar{q}q\rangle_\mu^0 + \mathrm{O}\left(\hat{m}_q^2\right) \\ {f_{K^+}}^2 m_{K^+}^2 &= -\left[m_u(\mu)+m_s(\mu)\right]\langle\bar{q}q\rangle_\mu^0 + \mathrm{O}\left(\hat{m}_q^2\right) \end{aligned}$$

- involves the chiral limit vacuum quark condensate

General, exact, relation

$$f_H^2\,m_H^2 = -\langle\bar{q}q\rangle_\mu^H\,\mathcal{M}_H$$

with “In-Meson Condensate”

$$i\langle\bar{q}q\rangle_\mu^H = f_H\,Z_4\int_q^\Lambda \tfrac{1}{2}\mathrm{Tr}\left[\left(T^H\right)^{\mathrm{t}}\gamma_5 S(q_+)\Gamma_H(q;P)S(q_-)\right]$$

- Note that $\langle\bar{q}q\rangle_\mu^H\,\mathcal{M}_H$

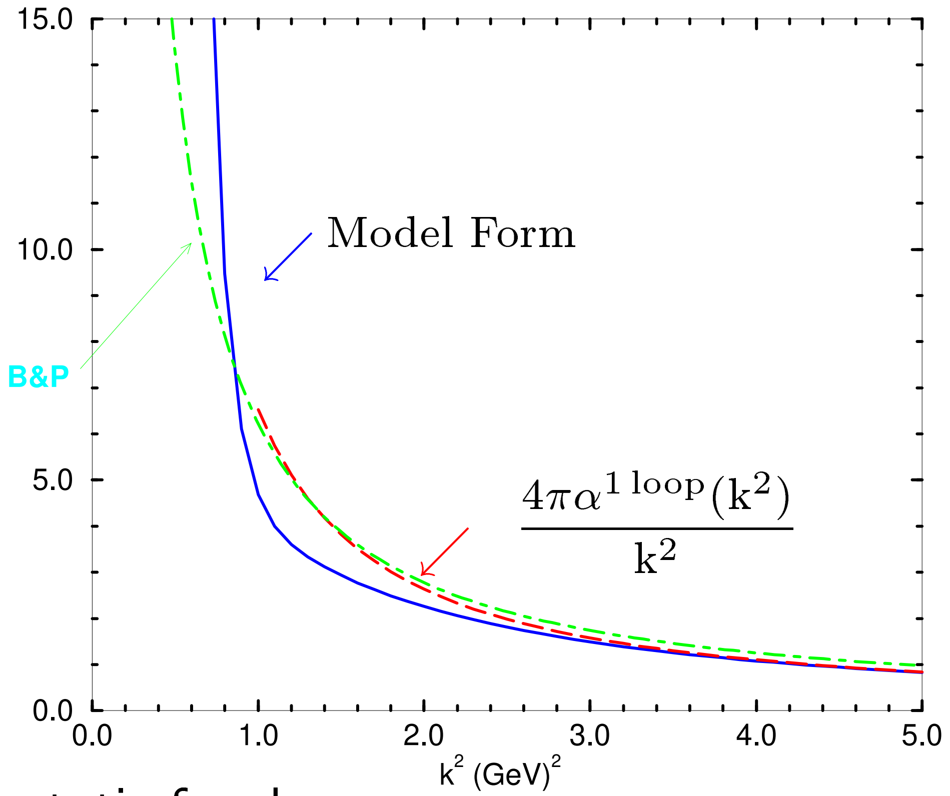
  is renormalisation point independent

- Does not involve

  a difference of massive-quark condensates

Explicit Model Calculation

$$S(p,\mu)^{-1} = Z_2 i\gamma\cdot p + Z_4 m_0 + \Sigma'(p,\Lambda)$$
$$\Sigma'(p,\Lambda) = \frac{4}{3}\int_k^\Lambda 4\pi\alpha((p-k)^2)\, D^{\rm free}_{\mu\nu}(p-k)\gamma_\mu S(k)\gamma_\nu$$

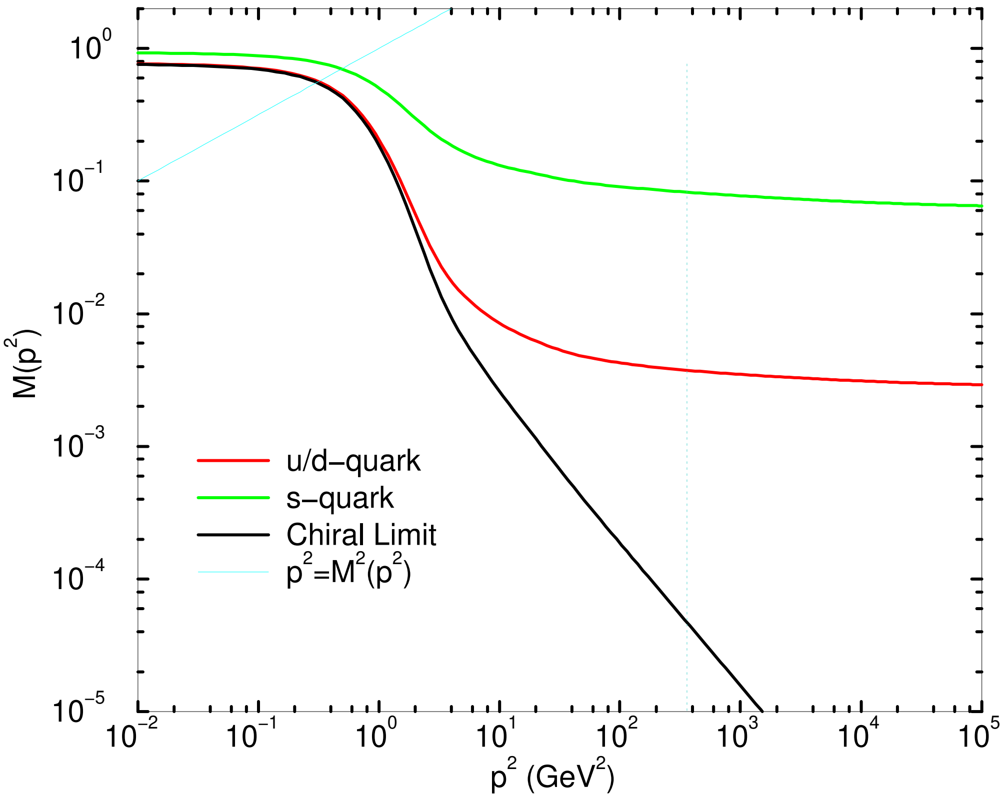


- Asymptotic freedom
- Renormalisation condition

$$S(p,\mu)^{-1}\Big|_{p^2=\mu^2} = i\gamma\cdot p + m(\mu)$$

- Chiral limit:
  - No perturbative contribution to $B(p^2,\mu^2)$
  - No divergence in $B'(p^2,\mu^2)$
  - No mass renormalisation

Quark Propagator



- Explicit Chiral Symmetry Breaking

$$M(p^2) \sim \frac{\hat{m}}{\left(\frac{1}{2}\ln\left[\frac{p^2}{\Lambda^2_{\rm QCD}}\right]\right)^{\gamma_m}}\,, \qquad \gamma_m = \frac{12}{33-2N_f}$$

- Chiral Limit

$$M(p^2) \stackrel{\text{large}-p^2}{=} \frac{2\pi^2\gamma_m}{3}\,\frac{-\langle\bar{q}q\rangle^0}{p^2\left(\frac{1}{2}\ln\left[\frac{p^2}{\Lambda^2_{\rm QCD}}\right]\right)^{1-\gamma_m}}$$

- $\hat{m}$ and $\langle\bar{q}q\rangle^0$ are renormalisation-point-invariant
- multiplicative renormalisibility

$$S^{-1}(p,\bar{\mu}) = i\gamma\cdot p\, A(p^2,\bar{\mu}^2) + B(p^2,\bar{\mu}^2) = \frac{Z_2(\bar{\mu}^2,\Lambda^2)}{Z_2(\mu^2,\Lambda^2)} S^{-1}(p,\mu)$$

Quark Condensate

$$-\left.\langle \bar{q}q\rangle_\mu^0\right|_{\mu=19\,\mathsf{GeV}} = Z_4(\mu,\Lambda)\,N_c \int_q^\Lambda \mathrm{tr}\left[S_{\hat{m}=0}(q)\right] = (0.275\,\mathrm{GeV})^3$$

Explicitly established that

$$m(\mu)\,\langle \bar{q}q\rangle_\mu^0 = \text{constant} \equiv \hat{m}\,\langle \bar{q}q\rangle^0$$

is renormalisation-point-invariant

$$\hat{m}_{u/d} = 6.60\,\mathrm{MeV}\,, \quad \hat{m}_s = 147\,\mathrm{MeV}$$

Values at 1 GeV

$$-\left.\langle \bar{q}q\rangle_\mu^0\right|_{\mu=1\,\mathrm{GeV}} \equiv \left(\ln\left[1/\Lambda_{\mathrm{QCD}}\right]\right)^{\gamma_m}\,\langle \bar{q}q\rangle^0 = (0.241\,\mathrm{GeV})^3$$

$$m_{u/d}^{1\,\mathrm{GeV}} = 5.5\,\mathrm{MeV}\,,\; m_s^{1\,\mathrm{GeV}} = 130\,\mathrm{MeV}$$

Accuracy of GMOR

π: $(0.0924 \times 0.1385)^2 = (\,\underline{0.113\,\mathrm{GeV}})^4$

cf. $(\underline{0.111\,\mathrm{GeV}})^4 = 2 \times 0.0055 \times 0.24^3$

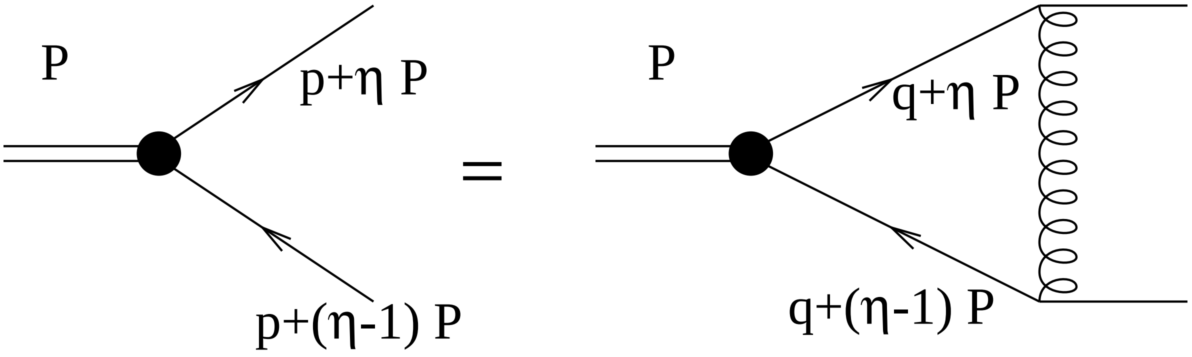
K: $(0.113 \times 0.495)^2 = (\,\underline{0.237\,\mathrm{GeV}})^4$

cf. $(\underline{0.206\,\mathrm{GeV}})^4 = (0.0055 + 0.13) \times 0.24^3$

O($\hat{m}^2$)-Corrections important important for $\hat{m} \sim \hat{m}_s$

Bethe-Salpeter Equation

$$\Gamma_H(p;P) = \\ -\frac{4}{3}\int_q^\Lambda 4\pi\alpha(k^2)\, D_{\mu\nu}^{\rm free}(k)\gamma_\mu S(q_+)\Gamma_H(q;P)S(q_-)\gamma_\nu$$

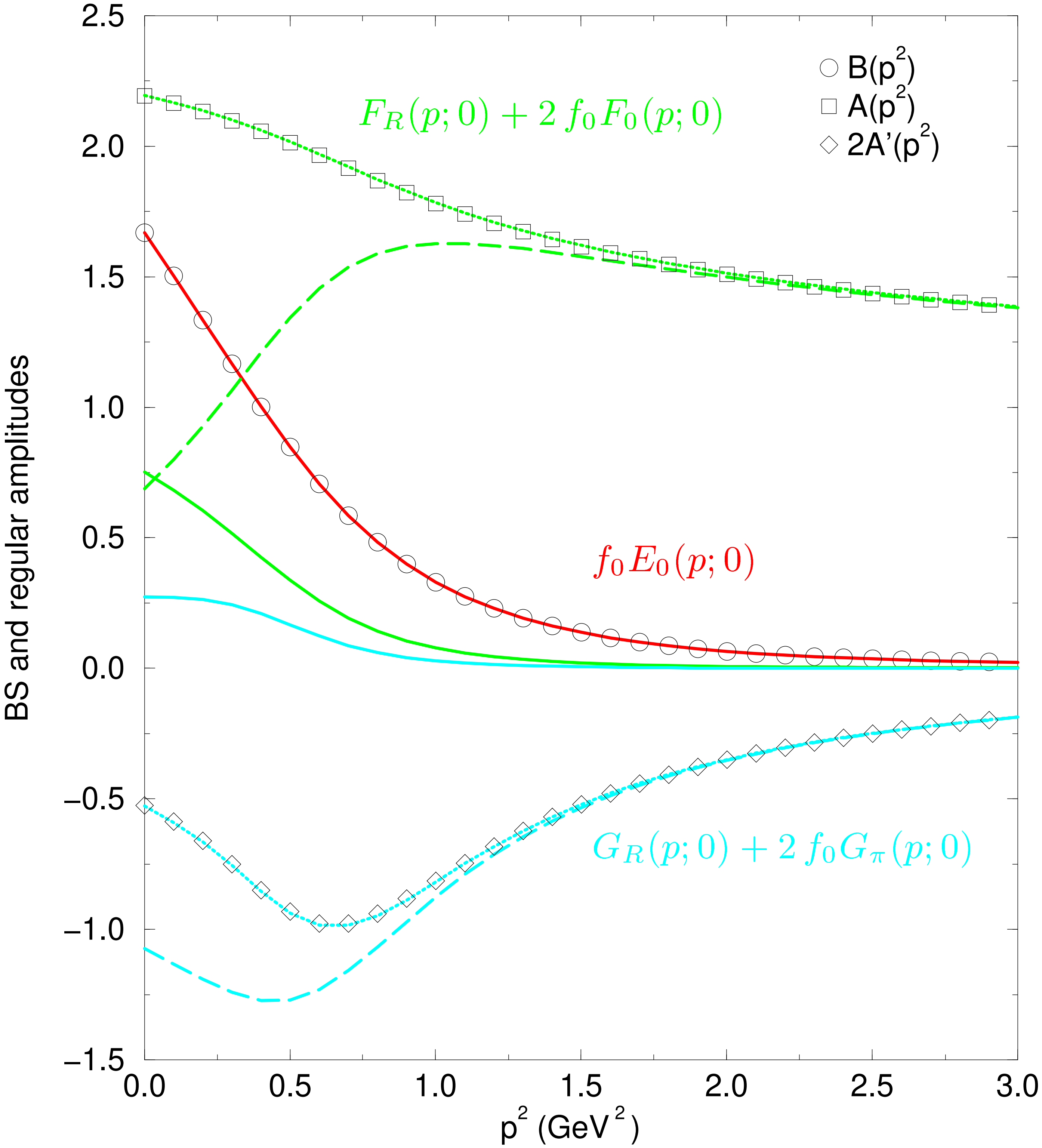


- Ladder approximation: consistent with AV-WTI
- Solve numerically at $P^2 = -m^2_{\rm meson}$
  - need quark propagator outside Euclidean reqion
  - 4 coupled 2-dimensional integral equations for $E(p;P)$, $F(p;P)$, $G(p;P)$, and $H(p;P)$
  - or use Chebyshev decomposition

$$E(k^2, k\cdot P; P^2) \approx \sum_{i=0}^{N_{\rm max}} {}^iE(k^2;P^2)\, U_i(\cos\beta)$$

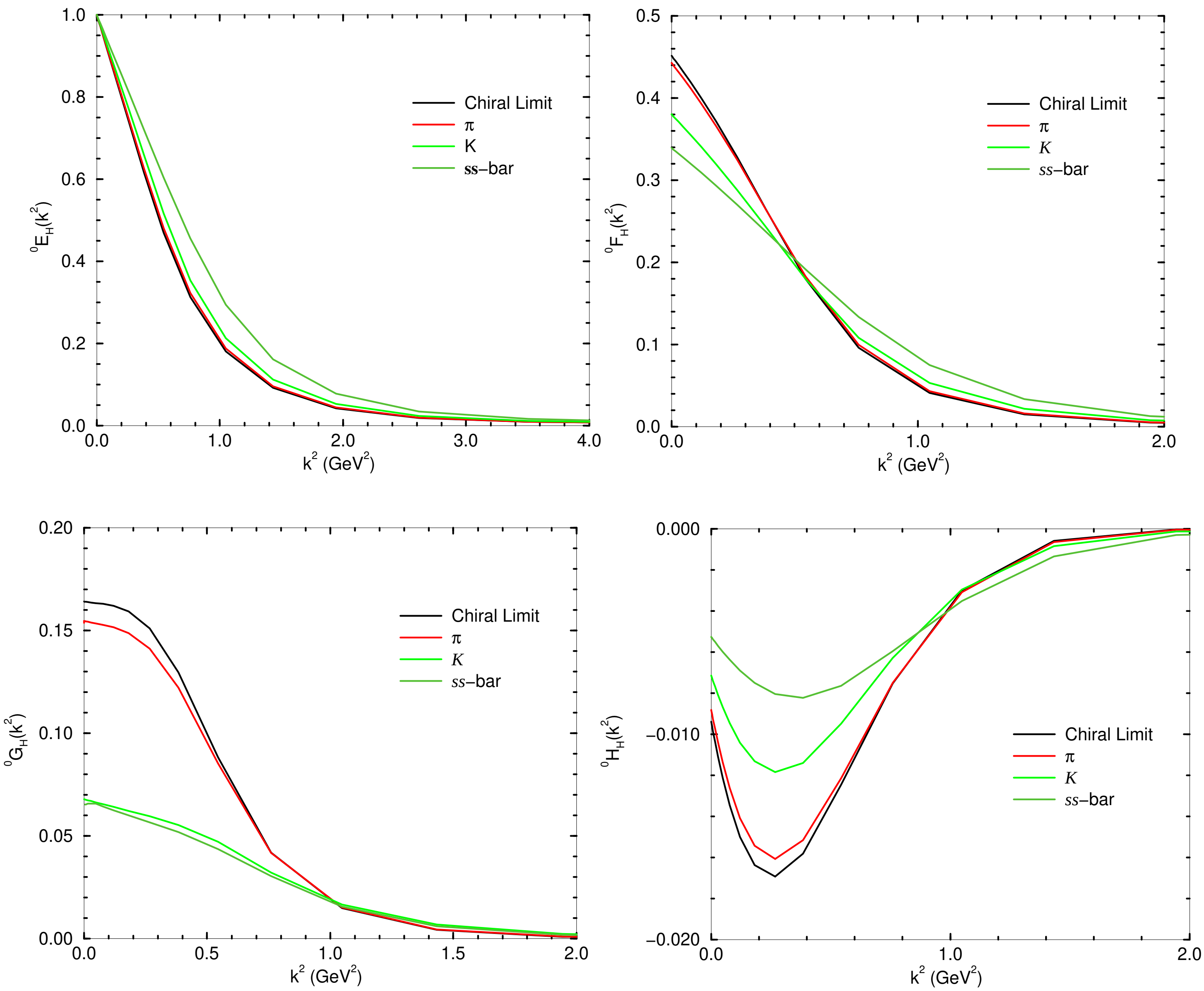
then $(4\, N_{\rm max})$ 1-dimensional integral equations

TEST OF AV-WTI IN CHIRAL LIMIT



Bethe-Salpeter Amplitudes

Zeroth Chebyshev projections of $E(k;P)$, $F(k;P)$, $G(k;P)$ $[k \cdot P\, G(k;P)$ for the kaon$]$ and $H(k;P)$



- Nonleading Dirac amplitudes F and G are
  - essential in order to satisfy AV-WTI
  - quantitatively important for physical observables
- AV-WTI is satisfied explicitly
  - chiral limit (see figure)
  - explicit current mass

$$f_{\pi,K}\, m_{\pi,K}^2 \quad = \quad r_{\pi,K}\, \left(m_{u/d}(\mu) + m_{u/d,s}(\mu)\right)$$

- "In-Meson condensate" increases with m_H

$-\langle\bar{q}q\rangle^{\pi}_{\mu=1\,\mathrm{GeV}}$	$-\langle\bar{q}q\rangle^{K}_{\mu=1\,\mathrm{GeV}}$	$-\langle\bar{q}q\rangle^{s\bar{s}}_{\mu=1\,\mathrm{GeV}}$
$(0.245\,\mathrm{GeV})^3$	$(0.284\,\mathrm{GeV})^3$	$(0.317\,\mathrm{GeV})^3$

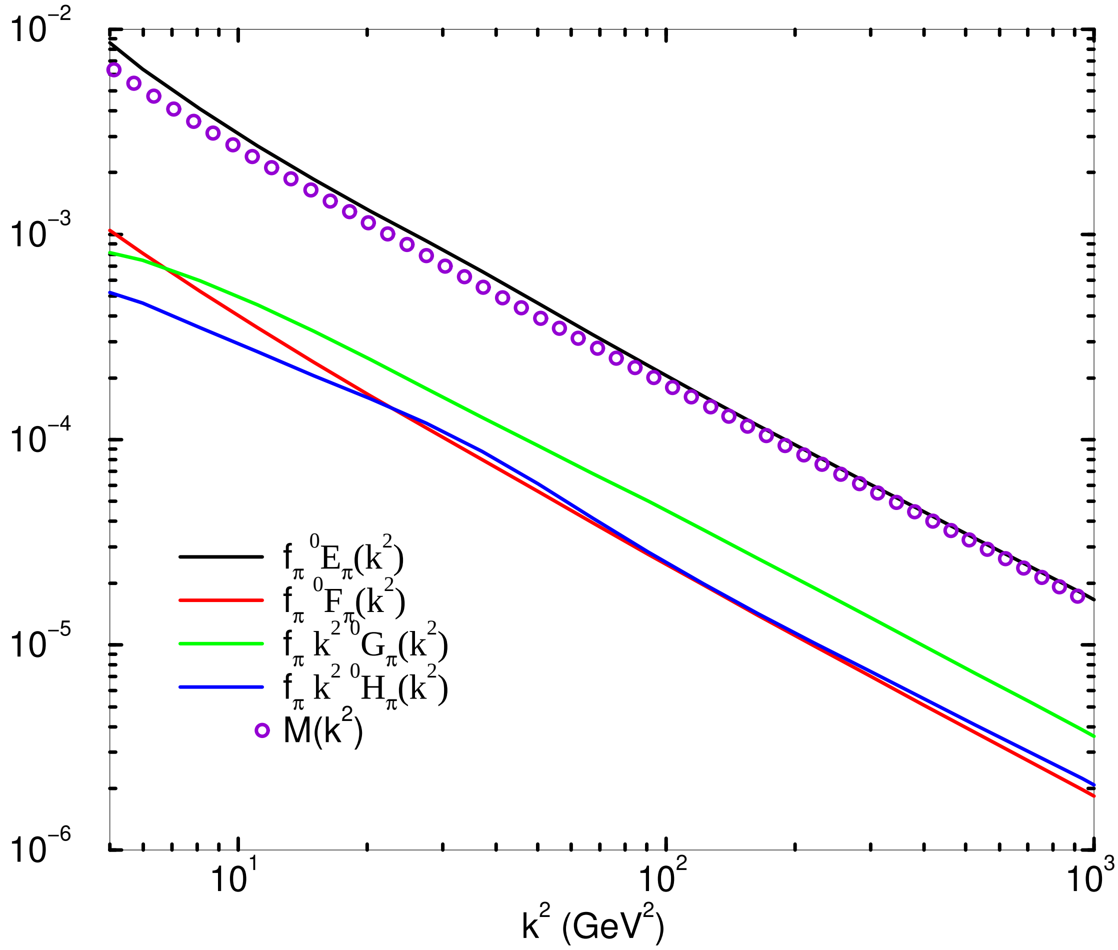
- Same is true for f_H
- Results are independent of the momentum partitioning if both subleading Dirac structures F, G and angular dependence are taken into account

sym.	chiral limit			π			$s\bar{s}$		
	m_0	f_0	$\mathcal{R}_H$	m_π	f_π	$\mathcal{R}_H$	$m_{s\bar{s}}$	$f_{s\bar{s}}$	$\mathcal{R}_H$
All	0.0	0.0898	1.00	0.139	0.0924	1.01	0.685	0.129	1.00
E	0.0	0.0649	1.81	0.105	0.0667	1.82	0.512	0.092	1.68
E, F	0.0	0.0965	0.95	0.136	0.0992	0.95	0.677	0.137	0.95
E, F, G	0.0	0.0891	1.00	0.140	0.0917	1.01	0.688	0.128	1.01

Kaon	$\eta_P = 0.50$			$\eta_P = 0.25$			$\eta_P = 0.00$		
	m_K	f_K	$\mathcal{R}_K$	m_K	f_K	$\mathcal{R}_K$	m_K	f_K	$\mathcal{R}_K$
All, Angles	0.497	0.109	1.01	0.497	0.109	1.01	0.497	0.109	1.01
U_0 only	0.469	0.117	0.96	0.482	0.117	0.95	0.475	0.113	1.02
U_0 and U_1	0.500	0.111	1.00	0.497	0.109	1.01	0.498	0.110	1.00
U_0, U_1, U_2	0.497	0.109	1.01	0.497	0.109	1.01	0.497	0.109	1.01
E only									
Angles	0.430	0.078	1.55	0.430	0.078	1.55	0.429	0.076	1.55
U_0 only	0.380	0.077	1.54	0.401	0.076	1.51	0.415	0.073	1.55
U_0 and U_1	0.439	0.089	1.52	0.430	0.078	1.55	0.431	0.076	1.57
U_0, U_1, U_2	0.430	0.078	1.55	0.430	0.078	1.55	0.427	0.076	1.55
E, F									
Angles	0.587	0.17	0.79	0.557	0.14	0.86	0.533	0.11	0.94
U_0 only	0.505	0.12	0.82	0.518	0.11	0.86	0.512	0.11	0.96
U_0 and U_1	-	-	-	0.556	0.14	0.86	0.537	0.12	0.94
U_0, U_1, U_2	0.583	0.16	0.79	0.557	0.14	0.86	0.532	0.12	0.93
E, F, $\hat{G}$									
Angles	0.500	0.108	1.01	0.500	0.108	1.01	0.500	0.108	1.01
U_0 only	0.471	0.116	0.96	0.484	0.116	0.95	0.477	0.112	1.02
U_0 and U_1	0.504	0.110	1.00	0.500	0.108	1.01	0.502	0.109	1.00
U_0, U_1, U_2	0.500	0.108	1.01	0.500	0.108	1.01	0.499	0.108	1.01

- UV asymptotic behaviour of E, F, $k^2\,G$ and $k^2\,H$ is fully determined by the chiral quark mass function: all behave like

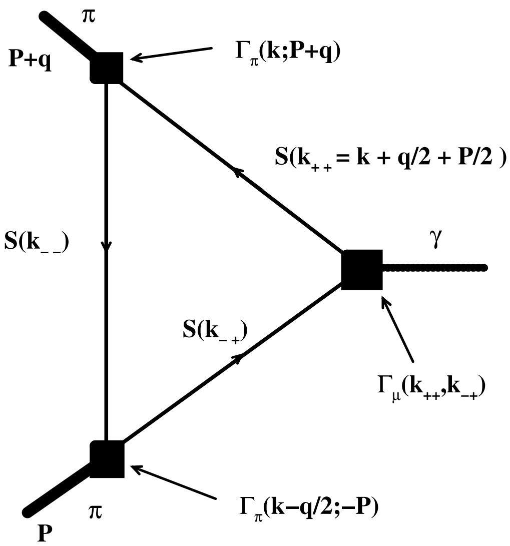
$$\frac{1}{p^2\left(\frac{1}{2}\ln\left[\frac{p^2}{\Lambda^2_{\rm QCD}}\right]\right)^{1-\gamma_m}}$$



$\Longrightarrow$ Constrains phenomenological studies of meson elastic and transition form factors at large Q^2

Pion Electromagnetic Form Factor

- Model dressed quark propagator $S(p)$
  - D χSB: strong momentum dependence in $M(p^2)$
  - Confinement: No Lehmann representation for $S(p)$
- BS amplitude: $E_\pi(p;P) = f_\pi\, B_0(p)$
- Electromagnetic form factor:

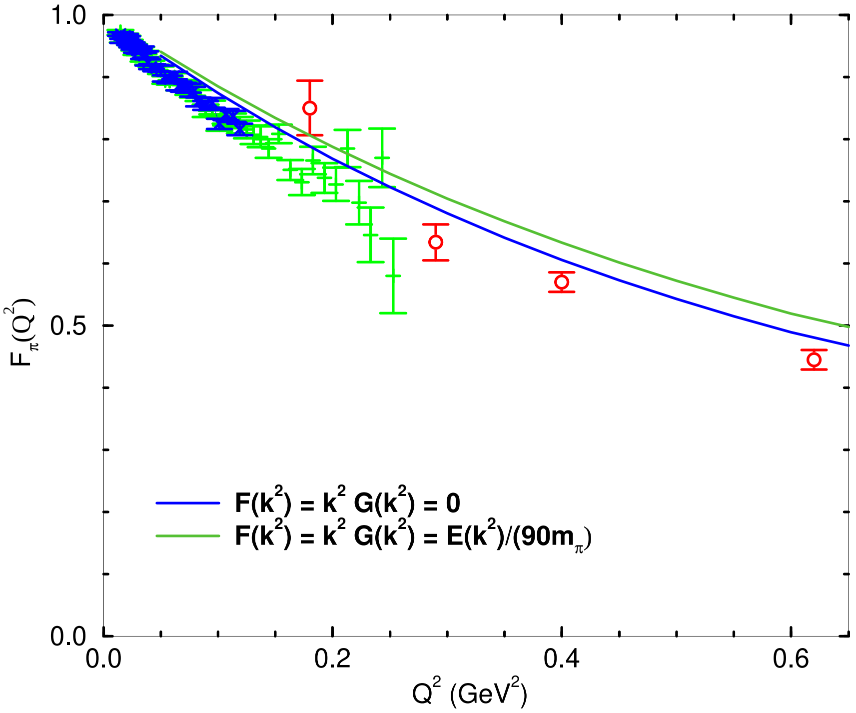


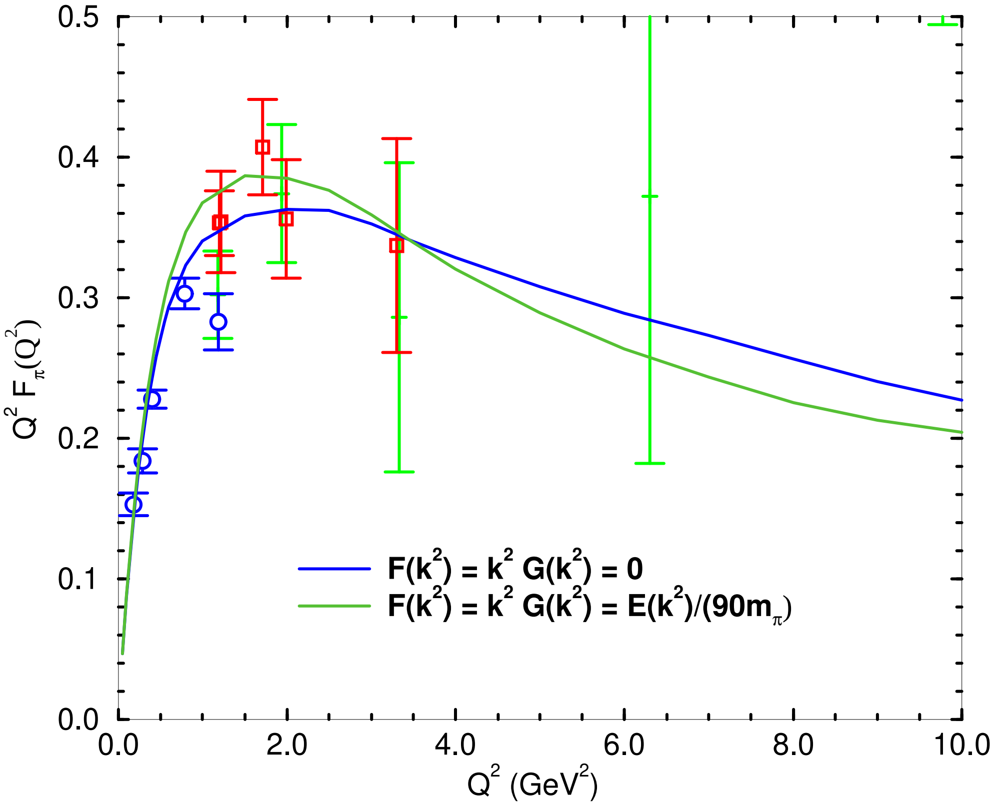
need dressed quark-photon vertex: Ball–Chiu vertex

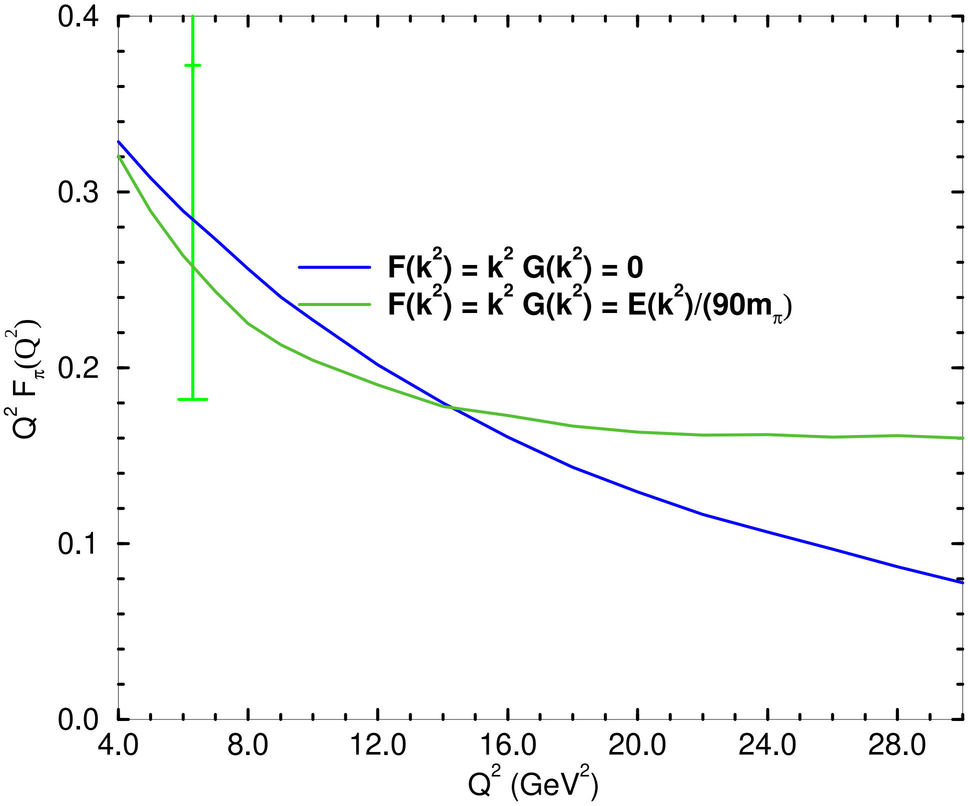
- Reanalysis with Dirac amplitudes F_π and G_π

	Calculated	Experiment
N_π	0.094 GeV	0.0924 ± 0.001
f_π	0.104 GeV	
m_π	0.127	0.138
$m^{\text{ave}}_{1\,\text{GeV}^2}$	0.0062	0.008
$-\langle \bar{q}q \rangle^{\frac{1}{3}}_{1\,\text{GeV}^2}$	0.226	0.236 ± 0.008
$r_{\pi^\pm}$	0.54 fm	0.663 ± 0.006

$F_\pi(Q^2)$ at low Q^2 using $f_\pi E_\pi(p) = B_0(p)$ and F_π and $p^2 G_\pi$ proportional to E_π







- UV behaviour

  $F_\pi(Q^2) \sim 1/Q^4$ with E only

  $F_\pi(Q^2) \sim 1/Q^2$ with E, F, G

- Dirac amplitudes F and G
  - unimportant at low Q^2
  - dominate at large Q^2

Summary and Conclusions

- Dyson–Schwinger Eqns
  - Continuum, Poincaré invariant
  - Perturbative results $\subset$ as weak-coupling expansion
  - Confinement & Dynamical Chiral Symmetry Breaking
  - Bound state properties

- One framework to describe and correlate both infrared (low Q^2) and ultraviolet (high Q^2) physics

- Essential: respect Ward Identities

- Pseudoscalar Mesons
  - Preserve axial-vector Ward–Takahashi identity
  - Importance of subleading Dirac amplitudes
  - Results independent of momentum partitioning
  - UV Asymptotic behaviour of BS amplitudes completely determined by chiral quark mass function
  - Pion form factor $F_\pi(Q^2) \sim 1/Q^2$ at large Q^2 as a consequence of the pion BS amplitudes F and G