Gauss's Law, Gauge Invariance, and Long-Range Forces In QCD

K. Haller
Graduate students:
Mario Belloni
Lusheng Chen

for presentation at the
Eighth International Workshop on
Light-Cone and Nonperturbative QCD;
August 22, 1997.

Phys. Lett. B373, 185 (1996);
Phys. Rev. D55, 2347 (1997);
Phys. Lett. B403, 316 (1997).

The program of this work:

I) Implementing Gauss's Law in temporal gauge ($A_0 = 0$) QCD & Yang-Mills Theory: *i. e.* constructing a set of state vectors that obey the non-Abelian Gauss's Law.

II) Constructing gauge-invariant operator-valued gauge (gluon) and spinor (quark) fields.

III) Transforming the QCD Hamiltonian to a representation in which the interactions between the quarks and the 'pure gauge' components of the gauge field are replaced by nonlocal interactions between quarks — a non-Abelian analog to the Coulomb interaction in QED.

IV) Extracting information — as much as possible — about low-energy, long-range QCD.

In QED, we can unitarily transform the "Gauss's Law Operator",

$$\hat{\mathcal{G}} = \partial_i \Pi_i(\mathbf{r}) + j_0(\mathbf{r}),$$

to $\partial_i \Pi_i(\mathbf{r})$, find perturbative states for which

$$\langle n' \,|\, \partial_i \Pi_i(\mathbf{r}) | n \rangle = 0$$

and then unitarily transform back

$$\partial_i \Pi_i(\mathbf{r}) \rightarrow \partial_i \Pi_i(\mathbf{r}) + j_0(\mathbf{r}) \;\; \text{and} \;\; |n\rangle \rightarrow |\nu\rangle$$

to obtain the states that implement Gauss's Law. Alternatively, we can view $\partial_i \Pi_i(\mathbf{r})$ as the form that the Gauss's Law operator takes in a transformed representation, and transform all operators and states in a similar way. Result:

<u>QED in a representation in which Gauss's Law is implemented and the Hamiltonian is expressed in terms of gauge-invariant operators.</u>

But this trick **does not work** in QCD or Yang-Mills Theory !!

In QED,

$$\left[\partial_i \Pi_i(\mathbf{r}), \partial_j \Pi_j(\mathbf{r}')\right] = 0 \text{ and } \left[\hat{\mathcal{G}}(\mathbf{r}), \hat{\mathcal{G}}(\mathbf{r}')\right] = 0;$$

$\hat{\mathcal{G}}(\mathbf{r})$ and $\partial_i \Pi_i(\mathbf{r})$ are unitarily equivalent.

In QCD, $\left[\partial_i \Pi_i^a(\mathbf{r}), \partial_j \Pi_j^b(\mathbf{r}')\right] = 0$, but

$$\left[\hat{\mathcal{G}}^a(\mathbf{r}), \hat{\mathcal{G}}^b(\mathbf{r}')\right] = igf^{abc}\hat{\mathcal{G}}^c(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'), \text{ so that}$$

$\hat{\mathcal{G}}^a(\mathbf{r})$ and $\partial_i \Pi_i^a(\mathbf{r})$ *can not be* unitarily equivalent ! In QCD,

$$\hat{\mathcal{G}}^a(\mathbf{r}) = \overbrace{\partial_i \Pi_i^a(\mathbf{r}) + gf^{abc}A_i^b(\mathbf{r})\Pi_i^c(\mathbf{r})}^{D_i\Pi_i^a(\mathbf{r})} + j_0^a(\mathbf{r}),$$

with $j_0^a(\mathbf{r}) = g\psi^\dagger(\mathbf{r})\frac{\lambda^a}{2}\psi(\mathbf{r})$.

We will define $J_0^a(\mathbf{r}) = gf^{abc}A_i^b(\mathbf{r})\Pi_i^c(\mathbf{r})$ (the "pure glue" color charge density) and $\mathcal{J}_0^a(\mathbf{r}) = J_0^a(\mathbf{r}) + j_0^a(\mathbf{r})$. THEREFORE:

To find states that implement the non-Abelian Gauss's law, we must solve for them "by hand".

SOLVING THE 'PURE GLUE' GAUSS'S LAW :

We need to construct a state $\Psi\,|\phi\rangle$ for which $\{\, b_Q^a(\mathbf{k}) + J_0^a(\mathbf{k})\,\}\Psi\,|\phi\rangle = 0$, where $b_Q^a(\mathbf{k})$ is the Fourier transform of $\partial_i \Pi_i^a(\mathbf{r})$, and $J_0^a(\mathbf{k})$ is the Fourier transform of the "pure glue" color charge density. We choose a state $|\phi\rangle$ annihilated by $b_Q^a(\mathbf{k})$ (they are easy to construct) and seek a Ψ for which

$$\{\, b_Q^a(\mathbf{k}) + J_0^a(\mathbf{k})\,\}\,\Psi\,|\phi\rangle = \Psi\, b_Q^a(\mathbf{k})\,|\phi\rangle\,,$$

or, equivalently,

$$[\, b_Q^a(\mathbf{k}),\, \Psi\,] = -J_0^a(\mathbf{k})\,\Psi + B_Q^a(\mathbf{k}),$$

where $B_Q^a(\mathbf{k})$ is an operator that has $\partial_i \Pi_i^a(\mathbf{r})$ on its extreme right.

THIS IS A KIND OF AN OPERATOR DIFFERENTIAL EQUATION THAT WE MUST NOW SOLVE — The commutator $[\, b_Q^a(\mathbf{k}),\, \Psi\,]$ is, essentially, an operator derivative of Ψ.

CONSIDER $\mathcal{A}_1 = ig\int d\mathbf{r}\,\psi^\gamma_{(1)i}(\mathbf{r})\,\Pi^\gamma_i(\mathbf{r})$,

with $\psi^\gamma_{(1)i}(\mathbf{r}) = f^{\alpha\beta\gamma}\,\mathcal{X}^\alpha(\mathbf{r})\,[a^\beta_i(\mathbf{r}) + \frac{1}{2}x^\beta_i(\mathbf{r})]$

(where $\mathcal{X}^\alpha(\mathbf{r}) = [\frac{\partial_i}{\partial^2}A^\alpha_i(\mathbf{r})]$ and where $a^\beta_i(\mathbf{r})$ and $x^\beta_i(\mathbf{r})$ are the transverse and longitudinal components of the gauge field $A^\beta_i(\mathbf{r})$ respectively).

WE OBSERVE THAT

$$[\,b^a_Q(\mathbf{k}),\, ig\int d\mathbf{r}\,\psi^\gamma_{(1)i}(\mathbf{r})\,\Pi^\gamma_i(\mathbf{r})\,] =$$

$$-g\, f^{a\beta\gamma}\int d\mathbf{r}\; e^{-i\mathbf{k}\cdot\mathbf{r}}\; A^\beta_i(\mathbf{r})\;\Pi^\gamma_i(\mathbf{r})$$

$$-\frac{g}{2}\, f^{a\beta\gamma}\int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\,\mathcal{X}^\beta\;[\,\partial_i\Pi^\gamma_i(\mathbf{r})\,]\,.$$

NOTICE THAT: the first line of the right-hand side of this equation is exactly $= -J^a_0(\mathbf{k})$

This equation raises the question: Would $\exp[\mathcal{A}_1]$ be a satisfactory Ψ? The answer is NO !

Both, the $J_0^a(\mathbf{k})$ and the $\partial_i\Pi_i^\gamma(\mathbf{r})$ produced as shown above, fail to commute with $[\mathcal{A}_1]^n$, and neither can move as required — $J_0^a(\mathbf{k})$ to the left, $\partial_i\Pi_i^\gamma(\mathbf{r})$ to the right — with producing commutator "debris" as it moves..

We have the following remedies for this problem :

The ordered exponential :

We define the ordered product $\| \exp(\mathcal{A}_1) \|$, in which the n^{th} order term, $\|(\mathcal{A}_1)^n\|$, represents the product in which all functionals of the gauge field A_i^a are to the left of all functionals of the canonical momenta Π_j^b.

All $\partial_i \Pi_i^\gamma(\mathbf{r})$ produced by commutators now will automatically be at the right hand side of the expression, as is required. But the result of commuting $\| \exp(\mathcal{A}_1) \|$, with $b_Q^a(\mathbf{k})$ is **not** the formation of $J_0^a(\mathbf{k})$ to the left of $\| \exp(\mathcal{A}_1) \|$, as required. Only $f^{a\beta\gamma} \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, A_i^\beta(\mathbf{r})$ forms to the *left* of $\| \exp(\mathcal{A}_1) \|$, and $\Pi_i^\gamma(\mathbf{r})$ remains to the extreme *right* of all the gauge fields.

Further unwanted terms will be generated as $\Pi_i^\gamma(\mathbf{r})$ is commuted, term by term, to the extreme left to form the desired $J_0^a(\mathbf{k})$.

What is the remedy to this problem ?

1) The replacement of $\mathcal{A}_1$ with $\mathcal{A}$:

$\mathcal{A}$ is multilinear in the gauge field $A_i^b(\mathbf{r})$, but only linear in the canonical momentum $\Pi_i^c(\mathbf{r})$; we represent it as $\mathcal{A} = i\int d\mathbf{r}\, \overline{\mathcal{A}_i^\gamma}(\mathbf{r})\, \Pi_i^\gamma(\mathbf{r})$, where $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ is the series

$$\overline{\mathcal{A}_i^\gamma}(\mathbf{r}) = \sum_{n=1}^{\infty} g^n \mathcal{A}_{(n)i}^\gamma(\mathbf{r}) \, .$$

LEMMA: $\|[\exp(\mathcal{A}), \Pi_i^\gamma(\mathbf{r})]\| = \|[\mathcal{A}, \Pi_i^\gamma(\mathbf{r})]\exp(\mathcal{A})\|$

We must construct the series representation of $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ so that

$$\| \left([b_Q^a(\mathbf{k}), \sum_{n=2}^{\infty} \mathcal{A}_n] + [b_Q^a(\mathbf{k}), \mathcal{A}_1] \right) \exp(\mathcal{A}) \|$$

$$+J_0^a(\mathbf{k}) \| \exp(\mathcal{A}) \| \approx 0 \, ,$$

which, because $\exp(\mathcal{A}) \not\approx 0$, leads to

$$[b_Q^a(\mathbf{k}),\, \sum_{n=2}^{\infty} \mathcal{A}_n] - g\, f^{a\beta\gamma} \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \times$$
$$[a_i^\beta(\mathbf{r}) + x_i^\beta(\mathbf{r})]\, [\sum_{n=1}^{\infty} \mathcal{A}_n,\, \Pi_i^\gamma(\mathbf{r})] \approx 0\,,$$

where $\approx$ indicates a 'soft' equality, that holds only when the equation acts on a state annihilated by $b_Q^a(\mathbf{k})$.

We now replace the missing $[b_Q^a(\mathbf{k}), \mathcal{A}_1]$ on the left-hand-side of this equation to form $[b_Q^a(\mathbf{k}), \mathcal{A}]$, make a similar addition to the right hand side of the equation, and extend the 'soft' equality to an equation, valid for any operator-valued field, $V_j^\gamma(\mathbf{r})$, for which we are free to, but not required to substitute $\Pi_j^\gamma(\mathbf{r})$.

$$i\int d\mathbf{r}'[\,\partial_i\Pi_i^a(\mathbf{r}),\,\overline{\mathcal{A}_j^\gamma}(\mathbf{r}')\,]\,V_j^\gamma(\mathbf{r}')+$$

$$igf^{a\beta d}A_i^\beta(\mathbf{r})\int d\mathbf{r}'[\,\Pi_i^d(\mathbf{r}),\,\overline{\mathcal{A}_j^\gamma}(\mathbf{r}')\,]\,V_j^\gamma(\mathbf{r}')$$

$$=-gf^{a\mu d}\,A_i^\mu(\mathbf{r})\,V_i^d(\mathbf{r})+$$

$$\sum_{\eta=1}\frac{g^{\eta+1}B(\eta)}{\eta!}f^{a\beta c}f_{(\eta)}^{\vec{\alpha}c\gamma}\,A_i^\beta(\mathbf{r})\;\times$$

$$\frac{\partial_i}{\partial^2}\Big(\mathcal{M}_{(\eta)}^{\vec{\alpha}}(\mathbf{r})\,\partial_jV_j^\gamma(\mathbf{r})\Big)$$

$$-\sum_{\eta=0}\sum_{t=1}(-1)^{t-1}g^{t+\eta}\frac{B(\eta)}{\eta!(t-1)!(t+1)}\;\times$$

$$f_{(t)}^{\vec{\mu}a\lambda}f_{(\eta)}^{\vec{\alpha}\lambda\gamma}\,\mathcal{R}_{(t)}^{\vec{\mu}}(\mathbf{r})\,\mathcal{M}_{(\eta)}^{\vec{\alpha}}(\mathbf{r})\,\partial_iV_i^\gamma(\mathbf{r})$$

$$-gf^{a\beta d}A_i^\beta(\mathbf{r})\;\sum_{\eta=0}\sum_{t=1}(-1)^tg^{t+\eta}\frac{B(\eta)}{\eta!(t+1)!}\times$$

$$f_{(t)}^{\vec{\mu}d\lambda}f_{(\eta)}^{\vec{\alpha}\lambda\gamma}\frac{\partial_i}{\partial^2}\left(\mathcal{R}_{(t)}^{\vec{\mu}}(\mathbf{r})\,\mathcal{M}_{(\eta)}^{\vec{\alpha}}(\mathbf{r})\,\partial_jV_j^\gamma(\mathbf{r})\right).$$

For

$$\partial_j V_j^{\gamma}(\mathbf{r}) = \partial_j \Pi_j^{\gamma}(\mathbf{r}) = 0\,,$$

the operator equation reduces to the 'soft' equality on the preceding page.

Here $\mathcal{M}_{(\eta)}^{\vec{\alpha}}(\mathbf{r})$ represents

$$\mathcal{M}_{(\eta)}^{\vec{\alpha}}(\mathbf{r}) = \prod_{m=1}^{\eta} \overline{\mathcal{Y}^{\alpha[m]}}(\mathbf{r}) = \overline{\mathcal{Y}^{\alpha[1]}}(\mathbf{r}) \cdots \overline{\mathcal{Y}^{\alpha[\eta]}}(\mathbf{r})\,,$$

where $\overline{\mathcal{Y}^{\alpha}}(\mathbf{r}) = \frac{\partial_j}{\partial^2}\overline{\mathcal{A}_j^{\alpha}}(\mathbf{r})\,,$

$f_{(\eta)}^{\vec{\alpha}\beta\gamma}$ denotes the chain of structure constants

$$f_{(\eta)}^{\vec{\alpha}\beta\gamma} = \quad f^{\alpha[1]\beta b[1]}\, f^{b[1]\alpha[2]b[2]}\, f^{b[2]\alpha[3]b[3]} \times \cdots$$
$$\times f^{b[\eta-2]\alpha[\eta-1]b[\eta-1]}\, f^{b[\eta-1]\alpha[\eta]\gamma}\,.$$

and $B(\eta)$ denotes the η^{th} Bernoulli number.

We have proven that the solution of this equation is given by an operator-valued functional whose n^{th} order term, $\mathcal{A}^{\gamma}_{(n)i}(\mathbf{r})\,V_i^{\gamma}(\mathbf{r})$, is

$$ig^n \int d\mathbf{r}\mathcal{A}^{\gamma}_{(n)i}(\mathbf{r})\,V_i^{\gamma}(\mathbf{r}) = \frac{ig^n}{n!}\int d\mathbf{r}\,\psi^{\gamma}_{(n)i}(\mathbf{r})\,V_i^{\gamma}(\mathbf{r})$$

$$+\sum_{\eta=1}\frac{ig^\eta}{\eta!}f^{\vec{\alpha}\beta\gamma}_{(\eta)}\sum_{u=0}\sum_{r=\eta}\delta_{r+u+\eta-n}\times$$
$$\int d\mathbf{r}\,\mathcal{M}^{\vec{\alpha}}_{(\eta,r)}(\mathbf{r})\,\mathcal{B}^{\beta}_{(\eta,u)i}(\mathbf{r})\,V_i^{\gamma}(\mathbf{r})\,.$$

Here

$$\overline{\mathcal{B}^{\beta}_{(\eta)i}}(\mathbf{r}) = a_i^{\beta}(\mathbf{r}) + \left(\delta_{ij} - \frac{\eta}{(\eta+1)}\frac{\partial_i\partial_j}{\partial^2}\right)\overline{\mathcal{A}_i^{\beta}}(\mathbf{r})\,;$$

$\psi^{\gamma}_{(\eta)i}(\mathbf{r})$ is the explicitly known inhomogeneous term in the equation:

$$\psi^{\gamma}_{(\eta)i}(\mathbf{r}) = (-1)^{\eta-1}f^{\vec{\alpha}\beta\gamma}_{(\eta)}\,\mathcal{R}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})\;\mathcal{Q}^{\beta}_{(\eta)i}(\mathbf{r})\,,$$

with $\mathcal{R}^{\vec{\alpha}}_{(\eta)}(\mathbf{r}) = \prod_{m=1}^{\eta}\mathcal{X}^{\alpha[m]}(\mathbf{r})\,,$

$$\mathcal{X}^{\alpha}(\mathbf{r}) = [\frac{\partial_i}{\partial^2}A_i^{\alpha}(\mathbf{r})\,]\,,$$

and $\mathcal{Q}^{\beta}_{(\eta)i}(\mathbf{r}) = [\,a_i^{\beta}(\mathbf{r}) + \frac{\eta}{\eta+1}x_i^{\beta}(\mathbf{r})\,]\,.$

The equation that defines $\overline{\mathcal{A}^{c}_{j}}(\mathbf{r})$ does so implicitly, in the form of a highly non-linear integral equation, in which $\overline{\mathcal{A}^{c}_{j}}(\mathbf{r})$ recurs in to arbitrarily high powers in $\overline{\mathcal{B}^{\beta}_{(\eta)i}}(\mathbf{r})$ and, to arbitrarily high powers, in $\mathcal{M}^{\vec{\alpha}}_{(\eta)}(\mathbf{r})$.

WE REFER TO THE PROOF THAT THE $\overline{\mathcal{A}^{\gamma}_{j}}(\mathbf{r})$ GIVEN BY THIS NONLINEAR INTEGRAL EQUATION SATISFIES THE RECURSIVE EQUATION THAT GUARANTEES THE IMPLEMENTATION OF THE NON-ABELIAN GAUSS'S LAW, AS THE "FUNDAMENTAL THEOREM".

What are the first few orders of $\overline{\mathcal{A}^{\gamma}_{j}}(\mathbf{r})$?

$$\mathcal{A}_1 = ig \int d\mathbf{r}\; \psi^{\gamma}_{(1)i}(\mathbf{r})\; \Pi^{\gamma}_i(\mathbf{r})\;,$$

$$\mathcal{A}_2 = \frac{ig^2}{2} \int d\mathbf{r}\; \psi^{\gamma}_{(2)i}(\mathbf{r})\; \Pi^{\gamma}_i(\mathbf{r})$$

$$+ \;\; ig^2\, f^{\alpha\beta\gamma} \int d\mathbf{r}\; \frac{\partial_j}{\partial^2}[\mathcal{A}^{\alpha}_{(1)j}(\mathbf{r})]\; a^{\beta}_i(\mathbf{r})\; \Pi^{\gamma}_i(\mathbf{r})\;,$$

$$\mathcal{A}_3 = \frac{ig^3}{3!} \int d\mathbf{r}\; \psi^{\gamma}_{(3)i}(\mathbf{r})\; \Pi^{\gamma}_i(\mathbf{r})$$

$$+ \;\; ig^3\, f^{\alpha\beta\gamma} \int d\mathbf{r}\; \frac{\partial_j}{\partial^2}[\mathcal{A}^{\alpha}_{(2)j}(\mathbf{r})]\; a^{\beta}_i(\mathbf{r})\; \Pi^{\gamma}_i(\mathbf{r})$$

$$+ \;\; ig^3\, f^{\alpha\beta\gamma} \int d\mathbf{r}\; \frac{\partial_j}{\partial^2}[\mathcal{A}^{\alpha}_{(1)j}(\mathbf{r})]\; [\delta_{ik} - \frac{1}{2}\frac{\partial_i\partial_k}{\partial^2}] \times$$

$$\mathcal{A}^{\beta}_{(1)k}(\mathbf{r})\; \Pi^{\gamma}_i(\mathbf{r})\;,$$

$\mathcal{M}$ terms $\quad$ $\mathcal{B}$ terms

$n=4$

$\eta \quad \pi \quad u$

$$\mathcal{A}_4 = \frac{ig^4}{4!}\int d\mathbf{r}\, \psi^{\gamma}_{(4)i}(\mathbf{r})\, \Pi^{\gamma}_i(\mathbf{r})$$

1 3 0
$$+ \; ig^4 f^{\alpha\beta\gamma}\int d\mathbf{r}\, \frac{\partial_j}{\partial^2}[\mathcal{A}^{\alpha}_{(3)j}(\mathbf{r})]\, a^{\beta}_i(\mathbf{r})\Pi^{\gamma}_i(\mathbf{r})$$

1 2 1
$$+ \; ig^4 f^{\alpha\beta\gamma}\int d\mathbf{r}\, \frac{\partial_j}{\partial^2}[\mathcal{A}^{\alpha}_{(2)j}(\mathbf{r})] \times [\delta_{ik} - \frac{1}{2}\frac{\partial_i\partial_k}{\partial^2}]\, \mathcal{A}^{\beta}_{(1)k}(\mathbf{r})\, \Pi^{\gamma}_i(\mathbf{r})$$

1 1 2
$$+ \; ig^4 f^{\alpha\beta\gamma}\int d\mathbf{r}\, \frac{\partial_j}{\partial^2}[\mathcal{A}^{\alpha}_{(1)j}(\mathbf{r})] \times [\delta_{ik} - \frac{1}{2}\frac{\partial_i\partial_k}{\partial^2}]\, \mathcal{A}^{\beta}_{(2)k}(\mathbf{r})\, \Pi^{\gamma}_i(\mathbf{r})$$

2 (1+1) 0
$$+ \; \frac{ig^4}{2} f^{\alpha\beta b} f^{b\delta\gamma}\int d\mathbf{r}\, \frac{\partial_j}{\partial^2}[\mathcal{A}^{\alpha}_{(1)j}(\mathbf{r})] \times \frac{\partial_k}{\partial^2}[\mathcal{A}^{\delta}_{(1)k}(\mathbf{r})]\, a^{\beta}_i(\mathbf{r})\, \Pi^{\gamma}_i(\mathbf{r})\,.$$

Gauge-invariant operator-valued fields:

Attaching quarks to gluons — the basic idea:

$$\hat{\mathcal{G}}^a(\mathbf{r}) = \partial_i \Pi_i^a(\mathbf{r}) + g f^{abc} A_i^b(\mathbf{r}) \Pi_i^c(\mathbf{r}) + j_0^a(\mathbf{r})$$

$$\text{and } \mathcal{G}^a(\mathbf{r}) = \partial_i \Pi_i^a(\mathbf{r}) + g f^{abc} A_i^b(\mathbf{r}) \Pi_i^c(\mathbf{r})$$

do obey the same commutator algebras and *are* unitarily equivalent.

$$\hat{\mathcal{G}}^a(\mathbf{r}) = \mathcal{U}_{\mathcal{C}}\, \mathcal{G}^a(\mathbf{r})\, \mathcal{U}_{\mathcal{C}}^{-1} ,$$

where $\mathcal{U}_{\mathcal{C}} = e^{\mathcal{C}_0} e^{\bar{\mathcal{C}}}$ and $\mathcal{C}_0$ and $\bar{\mathcal{C}}$ are given by $\mathcal{C}_0 = i \int d\mathbf{r}\, \mathcal{X}^\alpha(\mathbf{r})\, j_0^\alpha(\mathbf{r})$ and $\bar{\mathcal{C}} = i \int d\mathbf{r}\, \overline{\mathcal{Y}^\alpha}(\mathbf{r})\, j_0^\alpha(\mathbf{r})$ We can apply the lesson learned in QED in the following way. Here too, we have two choices:

$$\mathcal{O}(\mathbf{r}) \rightarrow \mathcal{O}'(\mathbf{r}) = \exp\left(-\frac{i}{g}\int \hat{\mathcal{G}}^a(\mathbf{r}')\, \omega^a(\mathbf{r}')\, d\mathbf{r}'\right) \times \mathcal{O}(\mathbf{r}) \exp\left(\frac{i}{g}\int \hat{\mathcal{G}}^a(\mathbf{r}')\, \omega^a(\mathbf{r}')\, d\mathbf{r}'\right) ,$$

where $\mathcal{O}(\mathbf{r})$ and $\mathcal{O}'(\mathbf{r})$ are in the "ordinary" or $\mathcal{C}$-representation.

Alternatively, we can transform to the $\mathcal{N}$ representation, in which

$$\mathcal{O}_{\mathcal{N}}(\mathbf{r}) = \mathcal{U}_{\mathcal{C}}^{-1}\mathcal{O}(\mathbf{r})\,\mathcal{U}_{\mathcal{C}}\,,$$

and the gauge transformation $\mathcal{O}_{\mathcal{N}}(\mathbf{r}) \to \mathcal{O}'_{\mathcal{N}}(\mathbf{r})$ is expressed as

$$\begin{aligned}\mathcal{O}'_{\mathcal{N}}(\mathbf{r}) \;&= \exp\left(-\frac{i}{g}\int \mathcal{G}^a(\mathbf{r}')\,\omega^a(\mathbf{r}')\,d\mathbf{r}'\right)\times \\ &\quad \mathcal{O}_{\mathcal{N}}(\mathbf{r})\,\exp\left(\frac{i}{g}\int \mathcal{G}^a(\mathbf{r}')\,\omega^a(\mathbf{r}')\,d\mathbf{r}'\right)\,.\end{aligned}$$

It is easy to see that the spinor field $\psi(\mathbf{r})$ is gauge-invariant in the $\mathcal{N}$ representation, because $\psi(\mathbf{r})$ and $\mathcal{G}^a(\mathbf{r}')$ trivially commute. To produce $\psi_{\mathrm{GI}}(\mathbf{r})$, this gauge-invariant spinor transformed to the $\mathcal{C}$ representation:

$$\psi_{\mathrm{GI}}(\mathbf{r}) = \mathcal{U}_{\mathcal{C}}\,\psi(\mathbf{r})\,\mathcal{U}_{\mathcal{C}}^{-1} \;\text{ and }\; \psi_{\mathrm{GI}}^{\dagger}(\mathbf{r}) = \mathcal{U}_{\mathcal{C}}\,\psi^{\dagger}(\mathbf{r})\,\mathcal{U}_{\mathcal{C}}^{-1}\,;$$

with the Baker-Hausdorff-Campbell theorem,

$$\psi_{\mathrm{GI}}(\mathbf{r}) = V_{\mathcal{C}}(\mathbf{r})\,\psi(\mathbf{r}) \quad \text{and} \quad \psi_{\mathrm{GI}}^{\dagger}(\mathbf{r}) = \psi^{\dagger}(\mathbf{r})\,V_{\mathcal{C}}^{-1}(\mathbf{r}),$$

where

$$V_{\mathcal{C}}(\mathbf{r}) = \exp\left(-ig\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\frac{\lambda^{\alpha}}{2}\right) \exp\left(-ig\mathcal{X}^{\alpha}(\mathbf{r})\frac{\lambda^{\alpha}}{2}\right) .$$

To produce the gauge-invariant gauge field, we note that the gauge-invariant gauge field $\psi_{\rm GI}(\mathbf{r}) = V_{\mathcal{C}}(\mathbf{r})\,\psi(\mathbf{r})$ itself is *related to* $\psi(\mathbf{r})$ *by a gauge transformation*, because there is an $\exp\left[-ig\mathcal{Z}^{\alpha}\frac{\lambda^{\alpha}}{2}\right]$ such that

$$\exp\left[-ig\mathcal{Z}^{\alpha}\frac{\lambda^{\alpha}}{2}\right] = \exp\left[-ig\overline{\mathcal{Y}}^{\alpha}\frac{\lambda^{\alpha}}{2}\right] \exp\left[-ig\mathcal{X}^{\alpha}\frac{\lambda^{\alpha}}{2}\right] .$$

Therefore, the gauge-invariant gauge field is

$$\begin{aligned} A^{b}_{{\rm GI}\,i}(\mathbf{r})\,\frac{\lambda^{b}}{2} = \quad & V_{\mathcal{C}}(\mathbf{r})\,[\,A^{b}_{i}(\mathbf{r})\,\frac{\lambda^{b}}{2}]\,V_{\mathcal{C}}^{-1}(\mathbf{r}) \, + \\ & \tfrac{i}{g}\,V_{\mathcal{C}}(\mathbf{r})\,\partial_{i}V_{\mathcal{C}}^{-1}(\mathbf{r}) \, , \end{aligned}$$

or, equivalently,

$$A^{b}_{{\rm GI}\,i}(\mathbf{r}) = A^{\,b}_{T\,i}(\mathbf{r}) + [\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}]\,\overline{\mathcal{A}^{b}_{i}}(\mathbf{r}) \, .$$

Specializing to the $SU(2)$ case, we find

$$\overline{\mathcal{A}^{\gamma}_{i}}(\mathbf{r}) = \overline{\mathcal{A}^{\gamma}_{i}}(\mathbf{r})_{\mathcal{X}} + \overline{\mathcal{A}^{\gamma}_{i}}(\mathbf{r})_{\overline{\mathcal{Y}}} \, ,$$

with

$$\overline{\mathcal{A}_i^{\gamma}}(\mathbf{r})_{\mathcal{X}} = g\epsilon^{\alpha\beta\gamma}\,\mathcal{X}^{\alpha}(\mathbf{r})\,A_i^{\beta}(\mathbf{r})\,[\sin(\mathcal{N})/\mathcal{N}]$$
$$-g\epsilon^{\alpha\beta\gamma}\,\mathcal{X}^{\alpha}(\mathbf{r})\,\partial_i\mathcal{X}^{\beta}(\mathbf{r})\left[(1-\cos(\mathcal{N}))/\mathcal{N}^2\right]$$
$$-g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\,\mathcal{X}^{\mu}(\mathbf{r})\,\mathcal{X}^{\alpha}(\mathbf{r})\,A_i^{\beta}(\mathbf{r})\;\times$$
$$\left[(1-\cos(\mathcal{N}))/\mathcal{N}^2\right]$$
$$+g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\,\mathcal{X}^{\mu}(\mathbf{r})\,\mathcal{X}^{\alpha}(\mathbf{r})\,\partial_i\mathcal{X}^{\beta}(\mathbf{r})\;\left[\frac{1}{\mathcal{N}^2}-\frac{\sin(\mathcal{N})}{\mathcal{N}^3}\right]$$

and

$$\overline{\mathcal{A}_i^{\gamma}}(\mathbf{r})_{\overline{\mathcal{Y}}} = g\epsilon^{\alpha\beta\gamma}\,\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\,A_{\mathrm{GI}\,i}^{\beta}(\mathbf{r})\left[\sin(\overline{\mathcal{N}})/\overline{\mathcal{N}}\right]$$
$$+g\epsilon^{\alpha\beta\gamma}\,\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\,\partial_i\overline{\mathcal{Y}^{\beta}}(\mathbf{r})\left[(1-\cos(\overline{\mathcal{N}}))/\overline{\mathcal{N}}^2\right]$$
$$+g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\,\overline{\mathcal{Y}^{\mu}}(\mathbf{r})\,\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\,A_{\mathrm{GI}\,i}^{\beta}(\mathbf{r})\;\times$$
$$\left[(1-\cos(\overline{\mathcal{N}}))/\overline{\mathcal{N}}^2\right]$$
$$+g^2\epsilon^{\alpha\beta b}\epsilon^{b\mu\gamma}\,\overline{\mathcal{Y}^{\mu}}(\mathbf{r})\,\overline{\mathcal{Y}^{\alpha}}(\mathbf{r})\,\partial_i\overline{\mathcal{Y}^{\beta}}(\mathbf{r})\;\left[\frac{1}{\overline{\mathcal{N}}^2}-\frac{\sin(\overline{\mathcal{N}})}{\overline{\mathcal{N}}^3}\right]$$

$$\text{where } \mathcal{N} = \left[g^2\,\mathcal{X}^{\delta}(\mathbf{r})\,\mathcal{X}^{\delta}(\mathbf{r})\right]^{\frac{1}{2}}$$

and $\overline{\mathcal{N}}(\mathbf{r}) = \left[g^2\,\overline{\mathcal{Y}^{\delta}}(\mathbf{r})\,\overline{\mathcal{Y}^{\delta}}(\mathbf{r})\right]^{\frac{1}{2}}$.

LESSONS FROM QED:

When the Hamiltonian for QED in the temporal gauge is transformed from the $\mathcal{C}$ to the $\mathcal{N}$ representation, the only remaining dynamical interactions between the electron field and the electromagnetic field are the ones between the electrons and the gauge-invariant excitations of the gauge field — which, in QED, correspond to transversely polarized photons.

The interactions which, in the original $\mathcal{C}$ representation, were mediated by the longitudinal, gauge-dependent parts of the gauge field — the interactions mediated by the exchange of longitudinal photon "ghosts" — appear in the $\mathcal{N}$ representation as the non-local Coulomb interaction given by

$$H_c = \frac{1}{2}\int \frac{j_0(\mathbf{r})\, j_0(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\, d\mathbf{r}\, d\mathbf{r}'\,.$$

Here, we extend this approach to the non-Abelian QCD. In the $\mathcal{N}$ representation, the excitations of the "bare" quark field $\psi(\mathbf{r})$ are gauge-invariant, and therefore *already include the gauge-dependent gauge field components required to obey Gauss's law !* The interactions between these components of the gauge fields and quarks therefore will no longer appear explicitly in the Hamiltonian in the $\mathcal{N}$ representation, but will be replaced by non-local interactions among quarks, which form a kind of non-Abelian analog to the Coulomb interaction in QED.

We note the QCD Hamiltonian in the $\mathcal{C}$ representation:

$$H_{\mathcal{C}} = \int d\mathbf{r} \left\{ \tfrac{1}{2}\Pi_i^a(\mathbf{r})\Pi_i^a(\mathbf{r}) + \tfrac{1}{4}F_{ij}^a(\mathbf{r})F_{ij}^a(\mathbf{r}) \right.$$
$$\left. + \ \psi^\dagger(\mathbf{r}) \left[\beta m - i\alpha_i \left(\partial_i - igA_i^a(\mathbf{r})\frac{\lambda^a}{2} \right) \right] \psi(\mathbf{r}) \right\}$$

Transforming to the $\mathcal{N}$ representation:

For $\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{N}}$ we need $\mathcal{O}_{\mathcal{N}} = \mathcal{U}_{\mathcal{C}}^{-1}\mathcal{O}_{\mathcal{C}}\mathcal{U}_{\mathcal{C}}$, so that the part of the Hamiltonian that involves the spinor (quark) field transforms to

$$\psi^{\dagger}(\mathbf{r})V_{\mathcal{C}}\left[\beta m - i\alpha_i\left(\partial_i - igA_i^a(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\right]V_{\mathcal{C}}^{-1}\psi(\mathbf{r})$$

which becomes

$$\psi^{\dagger}(\mathbf{r})\left[\beta m - i\alpha_i\partial_i\right]\psi(\mathbf{r}) - i\psi^{\dagger}(\mathbf{r})\alpha_i\left[V_{\mathcal{C}}(\mathbf{r})\,\partial_i V_{\mathcal{C}}^{-1}(\mathbf{r})\right.$$

$$\left. - igV_{\mathcal{C}}(\mathbf{r})\left(A_i^b(\mathbf{r})\frac{\lambda^b}{2}\right)V_{\mathcal{C}}^{-1}(\mathbf{r})\right]\psi(\mathbf{r})$$

and, finally,

$$\tilde{H}_{quark} = \psi^{\dagger}(\mathbf{r})\left[\beta m - i\alpha_i\left(\partial_i - igA_{\mathrm{GI}\,i}^a(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\right]\psi(\mathbf{r})$$

In the gluon-part of the Hamiltonian, $F_{ij}^a(\mathbf{r})F_{ij}^a(\mathbf{r})$ transforms into itself, and $\Pi_i^a(\mathbf{r})$transforms into

$$\Pi_i^a(\mathbf{r})\rightarrow\Pi_i^a(\mathbf{r})+\sum_{m=0}\sum_{n=0}\sum_{r=0}\frac{g^{m+n+r}}{m!n!}(-1)^{m+n+r}\times$$

$$f_{(m)}^{\vec{\mu}ac}f_{(n)}^{\vec{\nu}cd}f_{(r)}^{\vec{\delta}dh}\mathcal{R}_{(m)}^{\vec{\mu}}(\mathbf{r})\,\mathcal{M}_{(n)}^{\vec{\nu}}(\mathbf{r})\frac{\partial_i}{\partial^2}\left(\mathcal{T}_{(r)}^{\vec{\delta}}(\mathbf{r})j_0^h(\mathbf{r})\right),$$

where $\mathcal{T}_{(r)}^{\vec{\delta}}(\mathbf{r})$ designates

$$\mathcal{T}_{(r)}^{\vec{\delta}}(\mathbf{r})j_0^a(\mathbf{r})=A_{\mathrm{GI}\,j(1)}^{\delta(1)}(\mathbf{r})\,\frac{\partial_{j(1)}}{\partial^2}\left(A_{\mathrm{GI}\,j(2)}^{\delta(2)}(\mathbf{r})\,\frac{\partial_{j(2)}}{\partial^2}\times\right.$$

$$\left.\left(\cdots\left(A_{\mathrm{GI}\,j(r)}^{\delta(r)}(\mathbf{r})\,\frac{\partial_{j(r)}}{\partial^2}\,(j_0^a(\mathbf{r}))\right)\right)\right).$$

The transformed Hamiltonian turns out to be

$$H_{\mathcal{N}}=\int d\mathbf{r}\left\{\tfrac{1}{2}\Pi_i^a(\mathbf{r})\,\Pi_i^a(\mathbf{r})+\tfrac{1}{4}F_{ij}^a(\mathbf{r})F_{ij}^a(\mathbf{r})+\right.$$

$$\left.\psi^\dagger(\mathbf{r})\left[\beta m-i\alpha_i\left(\partial_i-igA_{\mathrm{GI}\,i}^a(\mathbf{r})\frac{\lambda^a}{2}\right)\right]\psi(\mathbf{r})\right\}$$

$$+\tilde{H}_{\mathcal{G}}+\tilde{H}_{LR}$$

where

$$\tilde{H}_{\mathcal{G}} = \int d\mathbf{r} \left\{ -\mathrm{Tr}\left[\sum_{r=0} g^r (-1)^r f_{(r)}^{\vec{\delta} dh} \mathcal{G}^a(\mathbf{r}) \frac{\lambda^a}{2} \times \right.\right.$$

$$\left. V_{\mathcal{C}}^{-1}(\mathbf{r}) \frac{\lambda^d}{2} V_{\mathcal{C}}(\mathbf{r}) \frac{\partial_i}{\partial^2} \mathcal{T}_{(r)}^{\vec{\delta}}(\mathbf{r}) j_0^h(\mathbf{r}) \Big) \right]$$

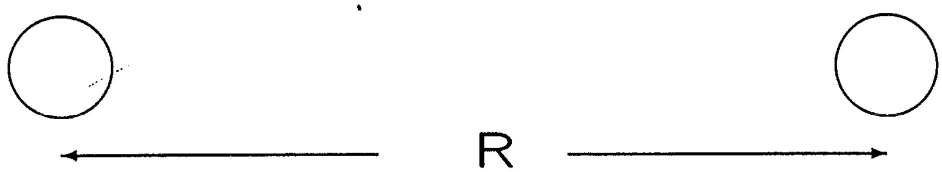
$$-\mathrm{Tr}\left[\sum_{r=0} g^r (-1)^r f_{(r)}^{\vec{\delta} dh} \frac{\partial_i}{\partial^2} \left(\mathcal{T}_{(r)}^{\vec{\delta}}(\mathbf{r}) j_0^h(\mathbf{r}) \right) \times \right.$$

$$\left.\left. V_{\mathcal{C}}^{-1}(\mathbf{r}) \frac{\lambda^d}{2} V_{\mathcal{C}}(\mathbf{r}) \mathcal{G}^b(\mathbf{r}) \frac{\lambda^b}{2} \right] \right\}$$

and

$$
\begin{aligned}
\tilde{H}_{LR} = \int d\mathbf{r} \Biggl\{ &+\mathrm{Tr}\Biggl[\sum_{r=0} g^{r+1}(-1)^r f_{(r)}^{\vec{\delta}dh} f^{d\sigma e} \times \\
&\Pi_i^a(\mathbf{r})\frac{\lambda^a}{2} V_{\mathcal{C}}^{-1}(\mathbf{r})\frac{\lambda^e}{2} V_{\mathcal{C}}(\mathbf{r}) A_{\mathrm{GI}\,i}^{\sigma}(\mathbf{r})\frac{\partial_i}{\partial^2}\left(\mathcal{T}_{(r)}^{\vec{\delta}}(\mathbf{r}) j_0^h(\mathbf{r})\right)\Biggr] \\
&+\mathrm{Tr}\Biggl[\sum_{r=0} g^{r+1}(-1)^r f_{(r)}^{\vec{\delta}dh} f^{d\sigma e}\frac{\partial_i}{\partial^2}\left(\mathcal{T}_{(r)}^{\vec{\delta}}(\mathbf{r}) j_0^h(\mathbf{r})\right) \times \\
&\quad V_{\mathcal{C}}^{-1}(\mathbf{r})\frac{\lambda^e}{2} V_{\mathcal{C}}(\mathbf{r}) A_{\mathrm{GI}\,i}^{\sigma}(\mathbf{r}) \Pi_i^b(\mathbf{r})\frac{\lambda^b}{2}\Biggr] \\
&+\frac{1}{2}\sum_{r=0}\sum_{r'=0} g^{r+r'}(-1)^{r+r'} f_{(r)}^{\vec{\delta}dh} f_{(r')}^{\vec{\delta}'dh'}\frac{\partial_i}{\partial^2} \times \\
&\quad \left(\mathcal{T}_{(r)}^{\vec{\delta}}(\mathbf{r}) j_0^h(\mathbf{r})\right)\frac{\partial_i}{\partial^2}\left(\mathcal{T}_{(r')}^{\vec{\delta}'}(\mathbf{r}) j_0^{h'}(\mathbf{r})\right)\Biggr\} \, .
\end{aligned}
$$

for QED



$$\psi_e = \sum \left(u_n(\mathbf{r}) e_n + v_n(\mathbf{r}) \bar{e}_n^\dagger \right) \text{ and}$$

$$\psi_E = \sum \left(U_n(\mathbf{r}) E_n + V_n(\mathbf{r}) \bar{E}_n^\dagger \right);$$

with negligible overlap between u, v and U, V.

$$j_0 = q \left(\psi_e^\dagger \psi_e + \psi_E^\dagger \psi_E \right),$$

$$\{e_n, \bar{e}_{n'}^\dagger\} = \delta_{n,n'}, \quad \{E_n, \bar{E}_{n'}^\dagger\} = \delta_{n,n'}.$$

$$H_C = \int d\mathbf{r} d\mathbf{r}' \frac{j_0(\mathbf{r})\, j_0(\mathbf{r}')}{8\pi|\mathbf{r} - \mathbf{r}'|}.$$

$$\langle e_n \cdots e_1 E_N \cdots E_1 | H_C | E_1 \cdots E_N e_1 \cdots e_n \rangle = \int \frac{d\mathbf{r} d\mathbf{r}'}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

$$\times \langle e_n \cdots e_1 | j_0(\mathbf{r}) | e_1 \cdots e_n \rangle \langle E_N \cdots E_1 | j_0(\mathbf{r}') | E_1 \cdots E_N \rangle$$

with $e^\dagger \bar{e}^\dagger, E^\dagger \bar{E}^\dagger$ quenched.

To leading order in $1/4\pi|\mathbf{r}-\mathbf{r}'|$, $\to\ Q_e Q_E/4\pi R$; van der Waals corrections are higher order.

for QCD

Instead of $j_0(\mathbf{r})$ in $\int d\mathbf{r}d\mathbf{r}' \frac{j_0(\mathbf{r})\, j_0(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}$, we get

$$j_0^a(\mathbf{r}) + f^{\delta(1)ba} A^{\delta(1)}_{\mathrm{GI}\,i}(\mathbf{r})\partial_i \int \frac{d\mathbf{x}_1}{4\pi|\mathbf{r}-\mathbf{x}_1|} j_0^b(\mathbf{x}_1) +$$

$$f^{\delta(1)bs(1)} f^{s(1)\delta(2)a} A^{\delta(1)}_{\mathrm{GI}\,i}(\mathbf{r})\partial_i \int \frac{d\mathbf{x}_1}{4\pi|\mathbf{r}-\mathbf{x}_1|} A^{\delta(2)}_{\mathrm{GI}\,i}(\mathbf{x}_1)\times$$

$$\partial_j \int \frac{d\mathbf{x}_2}{4\pi|\mathbf{x}_1-\mathbf{x}_2|} j_0^b(\mathbf{x}_2) + \cdots etc. \cdots$$

Suppose the variation in the coefficient of $j_0^b(\mathbf{x}_i)$ is gradual enough so that the series behaves like a multiple of Q_b. Since Q_b is the generator of rotations in SU(3), its matrix elements for singlet states should vanish.