EXACT SOLUTIONS IN A 2+1 DIMENSIONAL VARIANT OF THE STANDARD MODEL

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PLAN

- Vortices: what are, what they do represent
  (singular and soliton-like vortices)

- monopoles (SU(2) Y-M Higgs)
  ('t Hooft tensor $\mathcal{F}_{\mu\nu}$, P.S. limit)

- Planar counterpart: NLSM with local symmetry
  (all solutions → Liouville vortices
  Bianchi identities)

- 2+1 dim. model reproducing $\mathcal{F}_{\mu\nu}$ as its field strength
  (solutions, and θ_W)

- Conclusions and extensions to 3+1 D

VORTICES

- 2+1 D configuration with:
  B localized
  $\Phi(B) = n\Phi_0$

- research area: superconductivity (superfluidity)
  solitonic version of strings
  cosmic strings

Simplest example: A.B.

$$\vec{A} = \frac{\Phi_0}{2\pi} n \vec{\nabla}\theta \longrightarrow B = n\Phi_0 \delta(\vec{r})$$

But...

1) singular
2) ill defined energy
3) do not come from a local f.t.

Elementary justification
to vortices, we need

Soliton like solutions

Nielsen - Olesen (1974)

Maxwell - Higgs

$\vec{E} = 0$

$\vec{B} \neq 0$

2D monopoles

Hong - Kim - Pac
Jackiw - Weinberg (1990)

C.S. + Higgs

$B \propto \rho$

$\downarrow$

$\Phi(B) = c\, Q$

2D dyons

Third approach: Follow the (properly modified) lines of H.P. in 2+1 D

$\downarrow$

- Liouville vortices
- All the static - finite energy solutions can be found in term of $\omega(z)$
- They generalize and regularize the A.B. vortices

We need to see what happened in 1974 ....

't HOOFT POLYAKOV MONOPOLES

SU(2) Yang-Mills Higgs

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \frac{1}{2} D_\mu \varphi^a D^\mu \varphi^a - \frac{\lambda}{4}(\varphi^a\varphi^a - v^2)^2$$

Ansatz + Asymptotics:

$$A^a_0 = 0 \qquad\qquad F(r) \to v$$

$$\varphi^a = \delta^a_i \frac{x^i}{r} F(r) \qquad\qquad r \sim \infty$$

$$A^a_i = \epsilon^{aij} \frac{x^j}{r} W(r) \qquad\qquad W(r) \to 1/r$$

problem "conceptually" solved: $\Pi_2(S_2) = \mathbb{Z}$
(in the above case, $Q = 1$)

P.S. limit and Bogomolniy's Bound

$\lambda \to 0$ <u>but</u> retaining the asymptotics!

then, $F(r) = \frac{1}{r}\left[\frac{rv}{tgh(rv)} - 1\right]$, $W(r) = \frac{1}{r}\left[1 - \frac{rv}{sh(rv)}\right]$

and $$E = \frac{1}{2}\underbrace{\int d^3x \left[D_i\varphi^a \pm \tilde{F}^a_i\right]^2}_{\text{Bogomolni'y Bound}} \mp \underbrace{\int d^3x\, D_i\varphi^a \tilde{F}^a_i}_{|Q|} \geq |Q|$$

$$D_i\varphi^a = \mp \tilde{F}^a_i$$

$\Rightarrow$ to find a Bogomolu'yi Bound one has to get rid of the scalar potential

The winding # of the solutions is $Q=1$

$$Q = \frac{1}{8\pi}\int d^3x\, \epsilon_{ijk}\,\epsilon^{abc}\,\partial_i\hat{\varphi}^a\,\partial_j\hat{\varphi}^b\,\partial_k\hat{\varphi}^c = 1$$

as $\hat{\varphi}^a = \dfrac{\varphi^a}{\sqrt{\varphi^b\varphi^b}} = \delta^a{}_i\,\hat{r}^i$

Where are monopoles?

't Hooft $\mathcal{F}_{\mu\nu} = -\dfrac{\varphi^a F^a_{\mu\nu}}{|\varphi|} + \epsilon^{abc}\dfrac{\varphi^a D_\mu\varphi^b D_\nu\varphi^c}{|\varphi|^3}$

$\vec{E}=0$ and $\vec{B}\neq 0$ such that

$$\int d^3x\, \bar{\nabla}\cdot\bar{B} = -4\pi$$

PASS to the 2+1 D case, and look at natural modifications

1.) Kinetic term for A_μ^a : YM or CS?, (CS for convenience)

2) P.S. limit : $\lambda \to \infty$ instead of $\lambda \to 0$ (For topological reasons) $V \sim \lambda(\varphi^a\varphi^a - v^2)^2$

$$\varphi^a \longrightarrow N^a, \quad \text{with } N^a N^a = 1$$

(again, $\pi_2(S_2) = \mathbb{Z}$)

$$\Downarrow$$

NLSM

$$\mathcal{L} = -\frac{1}{2}\kappa\, \epsilon^{\mu\nu\rho}\left[\partial_\mu A_\nu^a A_\rho^a + \frac{1}{3}\epsilon^{abc} A_\mu^a A_\nu^b A_\rho^c\right] + \frac{1}{2} D_\mu N^a D^\mu N^a$$

All the static solutions are

$$A_0^a = \mp \frac{1}{\kappa} N^a, \quad A_i^a = i\,\mathrm{Tr}(\sigma^a U^{-1}\partial_i U), \quad N^3 = \frac{|w|^2 - 1}{|w|^2+1},$$

$$N^1 = \frac{2\,\mathrm{Re}\, w}{1+|w|^2}, \quad N^2 = \frac{2\,\mathrm{Im}\, w}{1+|w|^2} \qquad w = w(z) \text{ arbitrary}$$

and the solutions are O(3) self-dual

$$D_i N^a \mp \epsilon^{abc}\epsilon_{ij} N^b D_j N^c = 0 \quad (\text{Bog. Bound})$$

$$E_\sigma = \pm\frac{1}{2}\int d^2x\, \epsilon_{ij}\epsilon^{abc} N^a \partial_i N^b \partial_j N^c = 4\int d^2x \frac{|w'|^2}{(1+|w|^2)^2} = 4\pi|Q|$$

if $w = (z/z_0)^m$, $m \in \mathbb{Z} \Rightarrow Q = m$

What does the 't Hooft tensor reproduce in this case? Liouville vortices

$$\mathcal{F}_{\mu\nu} = - N^a F^a_{\mu\nu} + \epsilon^{abc} N^a D_\mu N^b D_\nu N^c$$

$$\downarrow$$

$$\vec{E} = 0 \quad \text{and} \quad B = \mathcal{H}_r = \frac{4|w'|^2}{(1+|w|^2)^2}$$

vortex: B localized and

$$\Phi(B) = 4\pi |Q|$$

Notes:

B gen. sol. of $\Delta \log B = -2B$

$W(z) = (z/z_0)^m$ gives the most. gen. rad. symm. sol.

$$B = \frac{4m^2}{r^2}\left(\left(\frac{r}{r_0}\right)^m + \left(\frac{r_0}{r}\right)^m\right)^{-2}$$

if $z_0 \to 0 \Rightarrow B = 2|m|\,\delta(\vec{r}) \to$ A.B. vortices 't Hooft tensor and Bianchi identities

$$F_{\mu\nu} = -N^a F^a_{\mu\nu} + \epsilon^{abc} N^a D_\mu N^b D_\nu N^c$$

is antisymm., gauge invariant tensor in any D

BUT

in 3+1 D

Bianchi id. $\boxed{\text{not}}$ satisfied

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = k_\mu \neq 0$$

$$\partial_\mu k^\mu = 0$$

$$k = \int d^3x \, k^0 = \text{monop. charge}$$

$F_{\mu\nu}$ $\boxed{\text{cannot}}$ be interpreted as an Abelian Field Strength

↓

in 2+1 D

Bianchi id $\boxed{\text{are}}$ satisfied

$$\epsilon_{\mu\nu\rho}\partial^\mu F^{\nu\rho} = \partial^\mu k_\mu = 0$$

$$k = \int d^2x \, k^0 = \text{vorticity}$$

$F_{\mu\nu}$ $\boxed{\text{can}}$ be interpreted as an Abelian Field Strength

- Does it exist a gauge theory in 2+1 D reproducing 't Hooft tensor through its equation of motion?
- Does it exist an analogous question in 3+1 dimension?

SM in 2+1 Dimensions

Adopt standard notations:

$$G = \underset{\substack{\downarrow \\ g}}{SU(2)} \times \underset{\substack{\downarrow \\ g'}}{U(1)}$$

$$A_\mu = A_\mu^a \frac{\sigma^a}{2i} \qquad B_\mu \qquad \rightsquigarrow T^a = \frac{\sigma^a}{2i}\,,\ [T^a, T^b] = \epsilon^{abc} T^c$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu]$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

matter field : $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$

such that $Q\varphi = e \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix}$

then $Q = e\left(iT^3 + \frac{1}{2} Y\right) = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$

$$\varphi \to e^{i\frac{\Lambda}{2}}\, U^{-1}\varphi$$

$$\varphi^\dagger \to e^{-i\frac{\Lambda}{2}}\, \varphi^\dagger U$$

$$g' B_\mu \longrightarrow g' B_\mu + \partial_\mu \Lambda$$

$$g A_\mu \longrightarrow g\, U^{-1} A_\mu U + U^{-1}\partial_\mu U$$

$$\mathcal{D}_\mu \varphi = (\partial_\mu + g A_\mu)\varphi - \frac{i g' B_\mu}{2}\varphi \equiv D_\mu \varphi - i\frac{g' B_\mu}{2}\varphi$$

$$\mathcal{D}_\mu \varphi^\dagger = \partial_\mu \varphi^\dagger - g\varphi^\dagger A_\mu + i\frac{g' B_\mu}{2}\varphi^\dagger \equiv D_\mu \varphi^\dagger + i\frac{g' B_\mu}{2}\varphi^\dagger$$

Matter Potential:

$$V(\varphi^+\varphi) = \lambda\,(\varphi^+\varphi - v^2)^2$$

to have a B.B., use the "natural limit" in 2+1 D, i.e. $\lambda \to \infty$

and $\varphi = v\,n$, $\quad n^+ n = 1$

and $n \longrightarrow e^{i\frac{\Lambda}{2}}\, U^{-1} n$

like a CP_1 field with local $SU(2)$ symmetry [but the $U(1)$ gauge field here will be provided by dynamics]

Matter kinetic term + currents:

$$\mathcal{L}_m = \mathcal{D}_\mu \varphi^+ \mathcal{D}^\mu \varphi = v^2 \mathcal{D}_\mu n^+ \mathcal{D}^\mu n$$

$SU(2)$ current $\quad J^a_\mu = \frac{v^2 g}{2i}\left[\mathcal{D}_\mu n^+ \sigma^a n - n^+ \sigma^a \mathcal{D}_\mu n\right]$

$U(1)$ current $\quad J_\mu = \frac{v^2 g'^2}{2i}\left[\mathcal{D}_\mu n^+ n - n^+ \mathcal{D}_\mu n\right]$

Dynamics for the gauge fields: C.S. dynamics

Topological couplings: $\tilde{F}^{\mu,a} = \frac{1}{2}\epsilon^{\mu\nu\rho} F^a_{\nu\rho}$, $\tilde{G}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho} G_{\nu\rho}$

$$\rightarrow \mathcal{L}_{tc} \sim \tilde{F}^{\mu,a} J^a_\mu + \tilde{G}^\mu J_\mu$$

Complete Lagrangian

$$\mathcal{L} = v^2 \mathcal{D}_\mu n^\dagger \mathcal{D}^\mu n - \frac{k_\alpha}{2} \epsilon^{\mu\nu\rho}\left[\partial_\mu A_\nu^a A_\rho^a + \frac{g}{3}\epsilon^{abc} A_\mu^a A_\nu^b A_\rho^c \right]$$

$$- \frac{k_\beta}{2} \epsilon^{\mu\nu\rho} \partial_\mu B_\nu B_\rho + \sigma_\alpha \epsilon^{\mu\nu\rho} F_{\mu\nu}^a J_\rho^a + \sigma_\beta \epsilon^{\mu\nu\rho} G_{\mu\nu} J_\rho$$

Eqs. of motion:

$$(k_\alpha - g^2 v^2 \sigma_\alpha)\tilde{F}^{a,\mu} - 2\sigma_\alpha \epsilon^{\mu\nu\rho} D_\nu J_\rho^a - g g' v^2 \sigma_\beta n^\dagger \sigma^a n \tilde{G}^\mu - J^{a,\mu} = 0$$

$$(k_\beta - g'^2 v^2 \sigma_\beta)\tilde{G}^\mu - 2\sigma_\beta \epsilon^{\mu\nu\rho}\partial_\nu J_\rho - g g' v^2 \sigma_\alpha n^\dagger \sigma^a n \tilde{F}^{a,\mu} - J^\mu = 0$$

$$\left[(\mathcal{D}_\mu \mathcal{D}^\mu n)^\dagger + 2ig\sigma_\alpha \tilde{F}^{a,\mu} \mathcal{D}_\mu n^\dagger \sigma^a + 2ig'\sigma_\beta \tilde{G}^\mu \mathcal{D}_\mu n^\dagger \right] \cdot$$

$$\cdot (1 - n n^\dagger) \cdot \sigma^b n = 0$$

Look for static configurations providing a Bogomolnyi bound to the energy

"Change variables"

$$N^a = n^\dagger \sigma^a n \quad ; \quad g'' C_\mu = 2i\, \mathcal{D}_\mu n^\dagger n = -2i\, n^\dagger \mathcal{D}_\mu n$$

$$[\, g'' C_0 = -g A_0^a N^a \,]$$

Disentangles the complicated g.t. properties:

$$N^a T^a \xrightarrow[SU(2)\times U(1)]{} U^{-1} N^a T^a U; \quad g'' C_\mu \xrightarrow[SU(2)\times U(1)]{} g'' C_\mu + \partial_\mu \Lambda$$

Examples:

$$\mathcal{D}_i n^\dagger \mathcal{D}_i n = \frac{1}{4} D_i N^a D_i N^a + \frac{g''^2}{4} C_i C_i$$

$$\mathcal{D}_i n^\dagger \sigma^a n = \frac{1}{2}\left(D_i N^a - i\,\epsilon^{abc} D_i N^b N^c\right) - \frac{i g''}{2} N^a C_i$$

$$\mathcal{D}_i \mathcal{D}_i n^\dagger n = -\frac{i}{2} g'' \partial_i C_i - \frac{1}{4} D_i N^a D_i N^a - \frac{g''^2}{4} C_i C_i$$

$$\vdots$$

etc...

Hamiltonian and B.B.:

$$\mathcal{H} = \frac{v^2}{2}\Big\{ g^2\left(A_0^a A_0^a - (N_a A_0^a)^2\right) + \left(g' B_0 + g A_0^a N^a\right)^2 +$$

$$+ \frac{1}{2}\left| \mathcal{D}_i n \pm i\,\epsilon_{ij} \mathcal{D}_j n\right|^2 \pm \frac{1}{2}\left(g \tilde{F}_0^a N^a + g' \tilde{G}_0\right)\Big\}$$

$$\Downarrow$$

B.B. = (first 3 terms = 0)

but $\mathcal{D}_i n \pm i\,\epsilon_{ij}\mathcal{D}_j n = 0 \Longrightarrow \begin{cases} D_i N^a \mp \epsilon^{abc}\epsilon_{ij} D_j N^b N^c = 0 \\ g'' C_i = g' B_i \end{cases}$

$$\Downarrow$$

$$D_i N^a = \pm\, \epsilon_{ij}\,\epsilon^{abc} D_j N^b N^c$$

$$g'' C_\mu = g' B_\mu \;, \quad A_0^a = A_0 N^a$$

$$\underbrace{}_{2i D_\mu n^\dagger n}$$

the Hypercharge field-strength is precisely 't Hooft tensor!

$$G_{\mu\nu} = -\frac{g}{g'} F^a_{\mu\nu} N^a + \frac{1}{g'} \epsilon^{abc} N^a D_\mu N^b D_\nu N^c$$

1) full agreement $g = g'$ (or $\vartheta_W = \pi/4$)

2) $E = \frac{v^2}{4} \left| \int d^2x \, \epsilon_{ij} \epsilon^{abc} N^a D_i N^b D_j N^c \right|$

3) $G_{\mu\nu} = -\frac{g}{g'} \left(\partial_\mu (A_\nu \cdot N) - \partial_\nu (A_\mu \cdot N) \right) + \frac{1}{g'} \epsilon^{abc} N^a \partial_\mu N^b \partial_\nu N^c$

Explicit solutions (for convenience choose one sign in self-duality conditions)

- Consistency between B.B. and eq. of motion provides further conditions, e.g. $\tilde{F}^a_0 = N^a \tilde{F}_0$

$$A^a_i = \frac{\ell}{g} \epsilon^{abc} \partial_i N^b N^c \quad , \quad \ell \in \mathbb{R}$$

$$\Downarrow$$

1.) s.d.c. simplifies to $\partial_i N^a + \epsilon_{ij} \epsilon^{abc} \partial_j N^b N^c = 0$

and $N^i = \frac{2\omega^i}{|\omega|^2+1}$, $N^3 = \frac{|\omega|^2-1}{|\omega|^2+1}$, $\omega = \omega(z)$

$\omega^1 = \mathrm{Re}\,\omega(z)$, $\omega^2 = \mathrm{Im}\,\omega(z)$

$$Q = \frac{1}{8\pi} \int d^2x \, \epsilon_{ij} \epsilon^{abc} N^a \partial_i N^b \partial_j N^c = -\frac{1}{\pi} \int d^2x \frac{|\omega'|^2}{(1+|\omega|^2)^2} \in \mathbb{Z}$$

2.) $\tilde{F}^a_0 = \frac{\ell(\ell-2)}{2g} N^a \epsilon_{ij} \epsilon^{bcd} N^b \partial_i N^c \partial_j N^d$

$\Downarrow$
Then

solutions are known in terms
of an arbitrary complex analytic function
$\omega(z) = \omega^1(x,y) + i\,\omega^2(x,y)$

+) $N^i = \dfrac{2\omega^i}{|\omega|^2+1}$, $N^3 = \dfrac{|\omega|^2-1}{|\omega|^2+1}$

+) $A_0^a = N^a A_0$, $A_0 = \text{const.}$, $A_i^a = \dfrac{\ell}{g}\epsilon^{abc}\partial_i N^b N^c$

+) n (and $n^\dagger$) from $N^a = n^\dagger \sigma^a n \Longrightarrow n = \dfrac{1}{\sqrt{1+|\omega|^2}}\begin{pmatrix}\bar{\omega}\\ 1\end{pmatrix}$

+) $g' B_\mu = 2i\, D_\mu n^\dagger n \Longrightarrow$ $g' B_0 = -g A_0 = \text{const.}$
$g' B_i = \epsilon_{ij}\partial_j \log(1+|\omega|^2)$

Finally, consistency with gauss' laws:

$$\frac{K_\alpha}{g^2 v^2} = \frac{\sigma_\alpha\left(2(\ell-1)^2 + \ell(\ell-2)\right) + \sigma_\beta}{\ell(\ell-2)}$$

$$\frac{k_\beta}{g^2 v^2} = \sigma_\alpha \ell(\ell-2) + \sigma_\beta$$

$$g A_0 = -g' B_0 = \frac{\ell(\ell-2)}{2\left(\sigma_\alpha((\ell-1)^2+1) + \sigma_\beta\right)}$$

Look at the hypercharge field strength

$$G_{0i} = 0$$

$$G_{ij} = \epsilon_{ij}\,\tilde{G}_0\,, \quad \tilde{G}_0 = \frac{1}{2g'}\,\epsilon_{ij}\,\epsilon^{abc} N^a \partial_i N^b \partial_j N^c$$

$\Downarrow$

It is just the magnetic vortex obtained via 't Hooft contraction from the sol. of the Local NLSM

$$\Phi(\tilde{G}_0) = \frac{4\pi Q}{g'}$$

Energy: $E = 2\pi v^2 (\ell - 1)^2 |Q|$

Weinberg Angle: $\mathcal{A}_\mu^{em} = \cos\vartheta_W B_\mu - \sin\vartheta_W A_\mu^3 \rightarrow$

$\rightarrow$ derive $\mathcal{F}_{\mu\nu}^{em}$, $\tilde{\mathcal{F}}_0^{em} = \epsilon_{ij}\,\partial_i \mathcal{A}_j^{em}$ and

$$\Phi(\tilde{\mathcal{F}}_0^{em}) = \frac{4\pi Q}{g'}\cos\vartheta_W \Rightarrow e_m = \frac{2\cos\vartheta_W}{g'} Q$$

$$e = g'\cos\vartheta_W$$

$\Rightarrow$ Schwinger quant. cond. $e \cdot e_m \in \mathbb{Z}$

$\downarrow$

$\cos^2\vartheta_W = \frac{1}{2}$ and $\vartheta_W = \pi/4$ reobtained!

CONCLUSIONS

– In 3+1 D ?

$\varphi = v n$ not realistic, rather $\varphi = \rho(\vec{r})\, n$, $\rho^2 = \varphi^\dagger \varphi$

and $\rho^2 \to v^2$ for $r \to \infty$

Then, rad. symm. case $\rho(r)$, $n = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\varphi/2} \\ \sin\theta/2\, e^{-i\varphi/2} \end{pmatrix}$

$\Rightarrow N^a = \frac{x^a}{r} = n^\dagger \sigma^a n$ and, in the "radial gauge"

$\frac{x}{r}^a A_i^a = N^a A_i^a = 0$, the composite field $g'' C_i = 2 i D_i n^\dagger n$

gives $\mathcal{F}_{ij} = \frac{1}{g'} \epsilon_{ijk} \frac{x^k}{r^2}$ ⇝ Dirac monopole with charge $1/g$"

Thus, also in 3+1 D it exists a composite field, transforming like $U(1)$ g.p. and such that, almost everywhere, its field strength coincides with 't Hooft one.

– "phenomenology" need to be studied

(possible applications in "planar physics", meaning of $\partial\omega$ in this 2+1 D context, ...)