Variational Tamm-Dancoff Approach to Hadron Spectroscopy

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Part I. Renormalization within Effective Hamiltonian Approach

Part II. Variational Tamm-Dancoff for QCD

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Renormalization within Effective Hamiltonian Approach

OUTLINE

Definition of the Effective Hamiltonian Problem
Method of Solution
Application to Soluble Problems
Renormalization within this Heff approach
Conclusions and Outlook

RECENT COLLABORATORS

<u>Heff for Many-Body Applications</u>

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C-L Song, University of Washington
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<u>Renormalization within this Heff</u>

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1. Definition of the problem

Given a Hamiltonian, $H = H^{\dagger}$, we want to solve

$$H \, |\Psi_i\rangle = E_i \, |\Psi_i\rangle , \qquad i = 1, \ldots \qquad (1)$$

- Suppose perturbation theory is inadequate and there are too many degrees of freedom to solve exactly by diagonalization.

- Suppose we are satisfied with a <u>subset</u> of all solutions. Define an effective Hamiltonian Heff such that

$$H_{eff} \, |\psi_i\rangle = E_i \, |\psi_i\rangle , \qquad i = 1, \ldots, d \qquad (2)$$

- For observables corresponding to Hermitian operators $\mathcal{O}$ we are satisfied with

$$\langle \Psi_i | \mathcal{O} | \Psi_j \rangle = \langle \psi_i | \mathcal{O}_{eff} | \psi_j \rangle , \quad i,j = 1, \ldots, d \qquad (3)$$

- Let H possess a set of symmetries whereby there are C mutually commuting generators K_j such that

$$[H, K_j] = 0 , \qquad j = 1, \ldots, C \qquad (4)$$

It may be appealing to have

$$[H_{eff}, K_j] = 0 , \qquad j = 1, \ldots, C \qquad (5)$$

But this is <u>not</u> <u>necessary</u> as long as conditions (2) and (3) are satisfied.

A. Method [based on a generalization of Bloch's method for degenerate state perturbation theory]

$$H = H_0 + H_I$$
$$H|\Psi_i\rangle = E_i|\Psi_i\rangle, \quad i = 1, \dots$$
$$H_0|\mu\rangle = \mathcal{E}_\mu|\mu\rangle, \quad \mu = 1, \dots$$

Define a model space $\mathcal{M}$ and projectors P and Q ∋

$$P = \sum_{\mu \in \mathcal{M}}^{d} |\mu\rangle\langle\mu|$$

$$Q = \sum_{\mu \notin \mathcal{M}} |\mu\rangle\langle\mu|$$

Clearly

$$P + Q = 1$$
$$P^2 = P$$
$$Q^2 = Q$$
$$PQ = QP = 0$$
$$PH_0Q = 0 = QH_0P$$

but $PH_IQ \neq 0$ + $QH_IP \neq 0$ in general

Consider a similarity transformation on H and on $|\Psi_i\rangle$

$$\tilde{H} = e^{-S} H e^{S}$$
$$|\tilde{\Psi}_i\rangle = e^{-S}|\Psi_i\rangle$$

Now, $\tilde{H}|\tilde{\Psi}_i\rangle = E_i|\tilde{\Psi}_i\rangle, \quad i = 1, \dots$

Exploit freedom in the similarity transformation to <u>require</u>:

$$\tilde{H} P |\tilde{\Psi}_i\rangle = E_i P |\tilde{\Psi}_i\rangle \quad , \quad i = 1, \ldots, d$$

$(P+Q) \rightarrow$

$$\Rightarrow \quad (P\tilde{H}P + Q\tilde{H}P) P |\tilde{\Psi}_i\rangle = E_i P |\tilde{\Psi}_i\rangle$$

and <u>require</u>:

$$Q\tilde{H}P = 0$$

$$\Rightarrow \quad \underbrace{(P\tilde{H}P)}_{H_{eff}} P |\tilde{\Psi}_i\rangle = E_i P |\tilde{\Psi}_i\rangle \quad , \quad i = 1, \ldots, d$$

Goal: Find S (and hence H_{eff}) that achieves these requirements

Suzuki and Lee obtained 2 solutions to this problem [one is equivalent to method of Krenciglowa and Kuo]

Take

$$S \equiv QSP$$

$$\Rightarrow \quad S^2 = S^3 = S^4 = \cdots = S^n = 0$$

$$\Rightarrow \quad e^{\pm S} = 1 \pm S$$

$$\Rightarrow \quad H_{eff} = P\tilde{H}P = P(1 - \cancel{QS}P)\,H\,(1 + QSP)\,P$$

$$= PHP + \underbrace{PHQ}_{PH_I}\underbrace{SP}_{S}$$

$$H_{eff} = PHP + PH_I S$$

$$\equiv H_0 + V_{eff}$$

$$\Rightarrow \quad V_{eff} = PH_I P + PH_I S$$

Basic equations to solve:

$$Q\tilde{H}P = 0$$

$$S = QSP$$

$$\Rightarrow \quad H_{eff} = PHP + PH_I S$$

Now, set up an equation for S

$$0 = Q\tilde{H}P = Q(1-S)H(1+S)P$$

$$= QHP - \cancel{Q}SHP - \cancel{Q}SHS\cancel{P} + QHS\cancel{P}$$

$$= QH_IP - SHP - SH_IS + QHS$$

$$= QH_IP - SH_{eff} + QHS$$

$\Rightarrow$ non-linear equation for S $\Rightarrow$ arrange for an iterative method of solution

$$-QHS\ [+\omega S] = QH_IP - SH_{eff}\ [+\omega S]$$

or

$$[\omega - QHQ]S = QH_IP - S(H_{eff} - \omega)$$

$$\Rightarrow \quad S = \frac{1}{\omega - QHQ}\left[QH_IP - S(H_{eff} - \omega)\right]$$

In some cases, e.g. both the P-space and Q-space are finite dimensional, one could solve for Heff now. <u>One</u> choice of iteration scheme:

$$S_n = \frac{1}{\omega - QHQ}\left[QH_IP - S_{n-1}\left(H_{eff}^{(n)} - \omega\right)\right]$$

$$H_{eff}^{(n+1)} = PHP + PH_I S_n$$

$$S_0 = 0 \rightarrow H_{eff}^{(1)} \rightarrow S_1 \rightarrow H_{eff}^{(2)} \rightarrow \cdots$$

<u>Consistency check:</u> Results converge and are indep. of ω

<u>Note:</u> Generates Heff as non-perturbative series in H_I

<u>More general case:</u> Q-space is infinite and/or need to perform an infinite partial summation of H_I interactions to obtain a reasonable approximation to Heff (e.g. Brueckner ladder series)

Begin with our equation for S

$$S = \frac{1}{\omega - QHQ}\left[QH_IP - S\underbrace{(H_{eff} - \omega)}_{Z}\right]$$

Rearrange:

$$\frac{1}{\omega - QHQ} SZ = -S + \frac{1}{\omega - QHQ} QH_IP$$

$PH_IQ \rightarrow$

$$\Rightarrow \quad PH_IQ \frac{1}{\omega - QHQ} SZ = -PH_I\cancel{Q}S + PH_IQ\frac{1}{\omega - QHQ} QH_IP$$

Now, add $Z = H_{eff} - \omega = PHP + PH_IS - \omega P$ to each side

$$\left(1 + PH_IQ\frac{1}{\omega - QHQ} S\right) Z = PHP + \cancel{PH_IS} - \omega P - \cancel{PH_IS} + PH_IQ\frac{1}{\omega - QHQ}QH_IP$$

$$= \overbrace{PH_0P + PH_IP} + PH_IQ\frac{1}{\omega - QHQ}QH_IP - \omega P$$

(the braced terms $PH_IP + PH_IQ\frac{1}{\omega - QHQ}QH_IP$) $\equiv G(\omega)$ "Generalized G-matrix"

$$\Rightarrow Z = \frac{1}{1 + PH_IQ\frac{1}{\omega - QHQ} S}\left[PH_0P + G(\omega) - \omega P\right]$$

Now, substitute for S from above (i.e. $S(Z)$) and generate iterative series for

$$Z_n = H_{eff}^{(n)} - \omega P$$

Steps:

Calculate $$G(\omega) = PH_I P + PH_I Q \frac{1}{\omega - QHQ} QH_I P$$

$$S_0 = 0$$

$$Z_1 = PH_0 P + G(\omega) - \omega P$$

$$S_1 = \frac{1}{\omega - QHQ} QH_I P$$

$$Z_2 = \frac{1}{1 + \underbrace{PH_I Q \left(\frac{1}{\omega - QHQ}\right)^2 QH_I P}_{-\frac{dG(\omega)}{d\omega} \equiv -G_1}} Z_1 = \frac{1}{1 - G_1} Z_1$$

Define:
$$G_n \equiv \frac{1}{n!} \frac{d^n G(\omega)}{d\omega^n}$$

$$S_2 = \frac{1}{\omega - QHQ} [QH_I P - S_1 Z_2]$$

$$Z_3 = \frac{1}{1 - G_1 - G_2 Z_2} Z_1$$

$$\vdots$$

$$Z_n = \frac{1}{1 - G_1 - G_2 Z_{n-1} - G_3 Z_{n-2} Z_{n-1} - \cdots - G_{n-1} Z_2 Z_3 \cdots Z_{n-1}} Z_1$$

or

$$Z_n = \frac{1}{1 - G_1 - \sum_{k=2}^{n-1} G_k \prod_{\ell=n-k+1}^{n-1} Z_\ell} Z_1 = H_{eff}^{(n)} - \omega P$$

Consistency check: Results converge and are indep. of ω

Alternative iteration scheme – simpler and relies only on previous iterations' results for Z:

In practical calculations, we expect G_k to decrease to zero in a reasonably rapid manner. Assume $G_k = 0$ for $k > N$. Then, for $n > N$

$$Z_n = \frac{1}{1 - G_1 - G_2 Z_{n-1} - \cdots - G_N Z_{n-N+1} Z_{n-N+2} \cdots Z_{n-1}} Z_1$$

Now, take the limit $n \to \infty$, assume convergence and define $Z_{n\to\infty} \equiv Z$

$$\Rightarrow \quad Z = \frac{1}{1 - G_1 - G_2 Z - G_3 Z^2 - \cdots - G_N Z^{N-1}} Z_1$$

$\Rightarrow$ alternative iteration scheme:

$$Z_n = \frac{1}{1 - G_1 - G_2 Z_{n-1} - G_3 Z_{n-1}^2 - \cdots - G_N Z_{n-1}^{N-1}} Z_1$$

Note on symmetries:

By choosing H_0 to have the same symmetries as H, we can select the P-space to be a set of finite subspaces identified by quantum numbers corresponding to some maximal set of commuting observables.

Thus, if

$$[H, K_i] = 0 \quad , \quad i = 1, \ldots, c$$

Then

$$[H_{eff}, K_i] = 0 \quad , \quad i = 1, \ldots, c$$

However, $\quad H_{eff} \neq H_{eff}^{\dagger} \quad$ in the above

Claim: Can extend the above developments straightforwardly to obtain

$$H'_{eff} = H'^{\dagger}_{eff}$$

3. Extreme case: P projects to a $d=1$ model space – Soluble Models

$\Rightarrow$ matrix equations for H_{eff} (i.e. for Z) become algebraic

$$Z = \frac{1}{1 - G_1 - G_2 Z - \cdots - G_N Z^{N-1}} \left[PH_0P + G - \omega \right]$$

multiplying through and rearranging

$$\omega + Z = \underbrace{PH_0P}_{\varepsilon_0} + G + G_1 Z + G_2 Z^2 + \cdots + G_N Z^N$$

We recognize the Taylor series for $G(\omega + Z)$ as $N \to \infty$ on the rhs.

$$\begin{aligned} \omega + Z &= \varepsilon_0 + G(\omega + Z) \\ &= H_{eff} \\ &= E \qquad \text{in } d=1 \end{aligned}$$

so

$$\boxed{E = \varepsilon_0 + G(E)} \qquad *$$

and the exact energy is a solution of this transcendental equation

<u>Major claim</u>: The equation (*) has as many solutions as there are eigensolutions of the full problem having non-vanishing overlap with the chosen P-space state!

Two simple examples with delta function interaction

$$V(x) = -\mu_0 \, \delta^{(n)}(x)$$

A. Dirac particle in 1 dimension

B. Schroedinger particle in 2 dimensions

Work in momentum space

$$V(k) = -\mu_0$$

select H_0 to be the pure kinetic operator and the P-space to consist of $0 < k < \lambda$ we solve for the generalize G-matrix:

$$G_{k,k'} = \frac{-\mu_0}{1 + \mu_0 I(\omega)} \delta(k - k')$$

[redacted]

$$I(\omega) = \int_{\lambda} d^{(n)}p \, \frac{1}{\omega - E_0(p)}$$

Argue that renormalization requires that the Heff be independent of any cutoff such as an ultraviolet cutoff Λ. Such a cutoff may be necessary for convergence of integrals appearing in the calculation of H_{eff}

That is, we want an H_{eff} which produces cutoff-independent physical results.

However, the above theory of H_{eff} has manifest cutoff dependence.

Examine what happens if we simply *require* cutoff (Λ) independence as an *additional condition on* H_{eff}.

In its simplest version we allow coupling constants (μ) to acquire cutoff dependence so that:

$$\lim_{\Lambda \to \infty} \frac{\delta H_{eff}(\Lambda, \mu(\Lambda))}{\delta \Lambda} = 0$$

If successful, we can express the result in terms of a "beta function":

$$\beta(\mu) = \Lambda \frac{\delta \mu}{\delta \Lambda}$$

Then obtain the result

$$\beta(\mu) = \mu^2 \Lambda \frac{\delta I}{\delta \Lambda}$$

For case A:

$$E_0(p) = p + m$$

$$I(\omega) = \ln \frac{\omega - (\Lambda + m)}{\omega - (\lambda + m)}$$

$$\beta(\mu) = -\mu^2$$

For case B:

$$E_0(p) = p^2/2m$$

$$I(\omega) = -2\pi \ln \frac{\omega - \Lambda^2}{\omega - \lambda^2}$$

$$\beta(\mu) = -4\pi\mu^2$$

Variational Tamm - Dancoff

Ref. J.R. Spence + J.P. Vary, Phys. Rev. C 52, 1668 (1995)

Assume trial Fock space state

$$\Psi = \sum_{n=1}^{20} \psi(q\bar{q}) \otimes \phi(ng)$$

Adopt H_{QCD} (Coulomb gauge)

Adopt several simplifying approximations

Apply variation on single q and single g degrees of freedom

Obtain Coupled non-linear integral equations for quark and gluon fields

Solve by iteration

Obtain $V_{eff}(q\bar{q})$

Insert into covariant wave equation

Obtain Meson Spectra and Amplitudes

Variational Tamm-Dancoff

Meson Trial State:

$$|\Phi; m\rangle = \sum_{\nu=0}^{N} z_{m,\nu} \, |g^{(\nu)}; m\rangle \, |\Phi_{q\bar{q}(\nu)}; m\rangle$$

Limitations:

- Single $q\bar{q}$ pair
- Finite number of gluons (max = 20)
- cm(gluons) = cm ($q\bar{q}$) "anchoring"

Other initial approximations: ≈ "Mean Field" Treatment

$q\bar{q}$ configurations taken indep. of ν

$$\Rightarrow \quad |\Phi; m\rangle \approx \underbrace{\sum_{\nu=0}^{N} z_{m,\nu} \, |g^{(\nu)}; m\rangle}_{|G; m\rangle} \, |\Phi_{q\bar{q}}; m\rangle$$

Hamiltonian (Coulomb gauge)

$$H_{QCD} = H_Q + H_G + H_{int}$$

Variational Treatment

$$\delta \langle \Phi; m | H_{QCD} | \Phi; m\rangle = 0$$

By separately varying g basis expansion coefficients, q basis expansion coefficients and Tamm-Dancoff amplitudes $\Rightarrow$ 3 coupled equations to solve

e.g.

$$\mathcal{E}_m |\Psi_{q\bar{q}};m\rangle = [H_Q + \underbrace{\langle G;m|H_{int}|G;m\rangle}_{V_{eff}}]|\Psi_{q\bar{q}};m\rangle$$

Treat $V_{eff} = V_{OGEP} + \Delta V_{eff}$

ΔV_{eff} : – arises from 2, 3, 4, ... gluons creating a localized density function coupled to the $q\bar{q}$ system
– quark coupling to gluon density assumed to generate either $\frac{1}{2}(\gamma_0 \otimes \gamma_0 + I \otimes I)$ form or $I \otimes I$ form.

Wave Equation which emerges

Form is equivalent to instantaneous approximation to the Bethe-Salpeter equation in momentum space with singular kernel.

Method of solving the wave equation:

J. R. Spence and J. P. Vary, Phys. Rev. C47, 1282 (1993)

Method of Solving the Wave Equation

adapted from

J.R. Spence and J.P. Vary, Phys Rev. C47, 1282 (1993)

Partial Wave Decomposition

$$\langle JML'S|\chi(q)\rangle = \frac{1}{E-2E(q)}\sum_{L'}\int dq'\, q'^2 \langle JML'S|V(\vec{q},\vec{q}')|JMLS\rangle\langle JMLS|\chi(q')\rangle$$

either $L = L' = J$ or $L = J \pm 1$, $L' = J \pm 1$

One gluon exchange component of V_{eff} - semi analytic treatment

$\lim_{\mu\to 0}$ taken in the following

$$\langle JML'S|V_{OGEP}(\vec{q},\vec{q}')|JMLS\rangle = \sum_k A^k_{JMLL'S}\, Q_k(z)$$

$Q_k(z)$ = Legendre functions of 2nd kind

$$z \equiv \frac{q^2+q'^2+\mu^2}{2qq'}$$

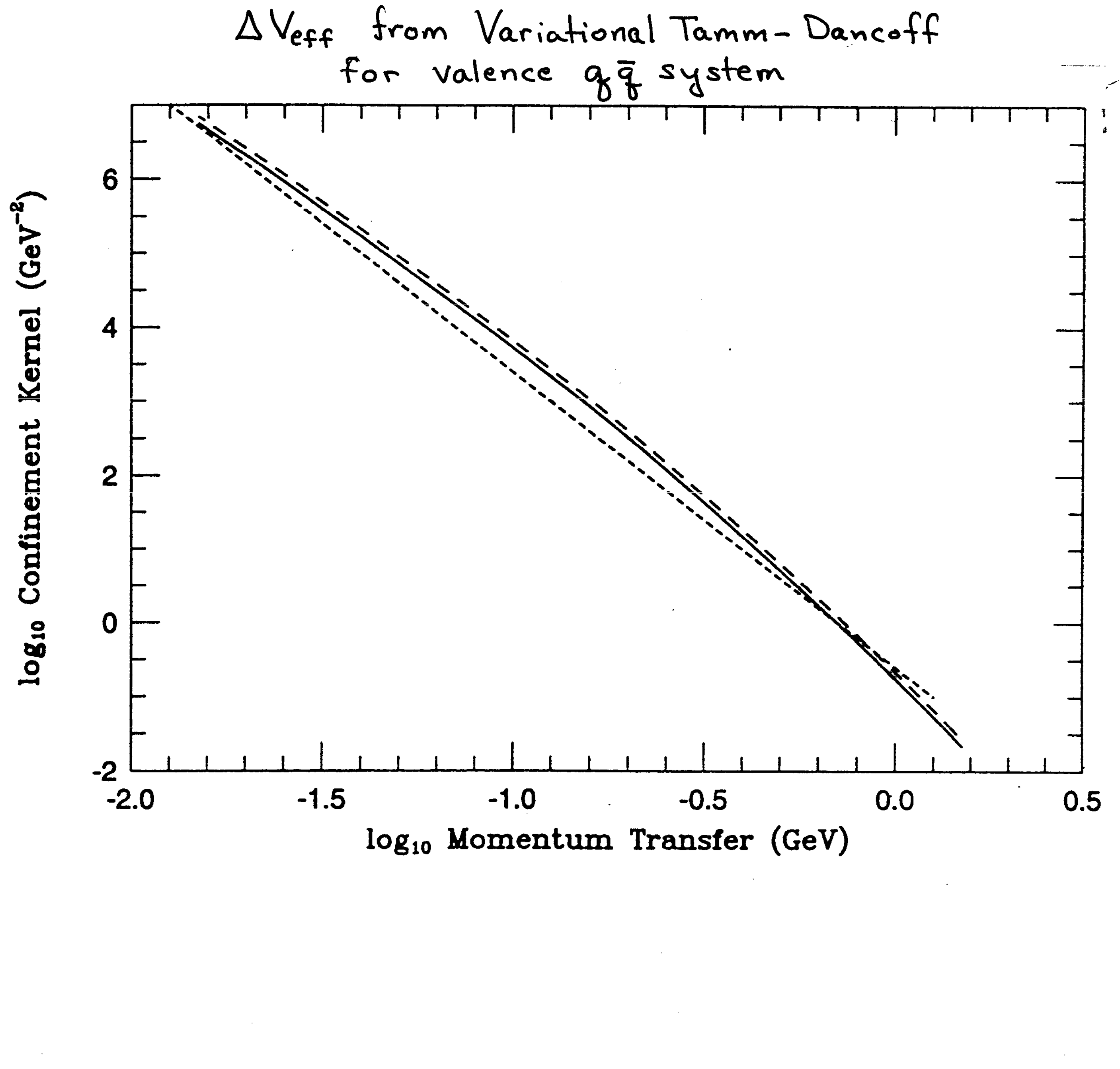
Treatment of ΔV_{eff} - motivated by smooth numerical form

$$\text{Fit}\quad \Delta V_{eff}(q,q') = \frac{\tilde{b}}{(q-q')^4}$$

$$\langle JML's|\Delta V_{eff}(q,q')|JMLS\rangle = \lim_{\mu\to 0}\left(\frac{\partial}{\partial\mu}\right)^2 \sum_k B^k_{JMLL'S}\, Q_k(z)$$

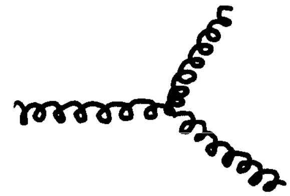
Casting results in form $\Delta V_{eff}(r) = b_i\, r$

i	b_i
$c\bar{c}$	0.2822 GeV^2
$b\bar{b}$	0.2872 GeV^2



ΔV_eff from Variational Tamm-Dancoff
for valence q q̄ system
log10 Confinement Kernel (GeV^-2)
log10 Momentum Transfer (GeV)
6
4
2
0
-2
-2.0
-1.5
-1.0
-0.5
0.0
0.5

How Does a Confining ΔV_{eff} Arise in VTD?

Consider the role of 

taking classical static fields and as functions of scalar momenta (angle averaging). In VTD the mean field treatment implies

$$\varepsilon A_A(p) = p A_A(p) + \frac{4\pi G}{(2\pi)^3}\int_0^\infty dq\, q^2 A_B(p-q) A_C(q)\, U^{ABC}(p,q)$$

now $U^{ABC}(p,q) = f^{ABC}(ap + bq)$

Assume color independence of A and color average to eliminate f^{ABC}

Ansatz for $A(p)$ as $p \to 0 = Cp^{\alpha}$

$$\Rightarrow \quad p^{\alpha} = B' \int dq\, q^2 (ap + bq)(p-q)^{\alpha}(q)^{\alpha}$$

Clearly, $\alpha < 0$ and we take pole dominance at $p \sim q \sim 0$

$$\Rightarrow \quad p^{\alpha} = p^4 (p)^{\alpha}(p)^{\alpha}$$

$$\Rightarrow \quad \boxed{\alpha = -4}$$

Parameters adjusted to fit subset of known states

$M_b = 4.6024$ GeV

$M_c = 1.2339$ GeV

$\alpha_s = 0.1883$

RMS mass deviations over 22 known $c\bar{c}$ and $b\bar{b}$ states

RMS = **52 MeV** VTD with 3 fit parameters

Compare with best fit allowing b to adjust as well

RMS = **43 MeV** BS (IA) with 4 fit parameters

T= 1/2 Channel

Expt	Theory
N(938) P_{11}	937
N(1440) P_{11}	1508
N(1520) D_{13}	1484
N(1535) S_{11}	1409
N(1650) S_{11}	1717
N(1675) P_{15}	1770
N(1680) F_{15}	1634
N(1700) D_{13}	1771
N(1710) P_{11}	1815
N(1720) P_{13}	1814
N(2190) G_{12}	2175
N(2220) H_{19}	2521
N(2250) G_{19}	2859
N(2600) $I_{1,11}$	2861

T = 3/2 Channel

Expt	Theory
$\Delta(1232)\ P_{33}$	1242
$\Delta(1620)\ S_{31}$	1649
$\Delta(1700)\ D_{33}$	1610
$\Delta(1900)\ S_{31}$	1905
$\Delta(1905)\ F_{35}$	1917
$\Delta(1910)\ P_{31}$	1912
$\Delta(1920)\ P_{33}$	1839
$\Delta(1930)\ D_{35}$	1896
$\Delta(1950)\ F_{37}$	1919
$\Delta(2420)\ H_{3,11}$	2346

Conclusions

Variational Tamm-Dancoff has been implemented for $q\bar{q}$ heavy quark systems

Results show confinement behavior arising from triple gluon coupling

Mass spectra are competitive with best fits using phenomenological confinement

Preliminary results show promise for qqq systems

Future Work

Extend to lighter $q\bar{q}$ systems

Evaluate other observables (widths, form factors, ...)