Pauli–Villars Regulators in Discretized Light-Cone Quantization

S.J. Brodsky, G. McCartor, J.R.H.

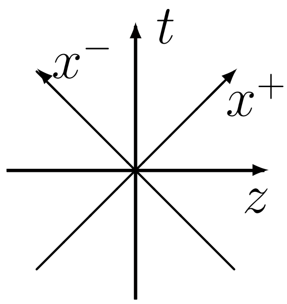
Some computing time granted by
Minnesota Supercomputer Institute

- calculations in QED and QCD require nonperturbative renormalization.
- most attempts have used cutoff-type regularization, which requires counterterms that depend on Fock sector.
- will explore the practicality of Pauli–Villars regularization.
- consider simple heavy fermion model abstracted from Yukawa model.

Outline

1. Light-cone coordinates
   (a) advantages
   (b) discretization
   (c) Tamm–Dancoff truncation
   (d) regularization by cutoff
2. Yukawa theory at one loop
   (a) fermion self energy
   (b) numerical calculations
   (c) boson self energy
3. Heavy fermion model
   (a) effective Hamiltonian
   (b) analytic solution
   (c) coupling renormalization
   (d) form factor slope
   (e) distribution functions
4. Numerical methods and results
   (a) Lanczos algorithm
   (b) efficient form
   (c) references
   (d) basis sizes
   (e) results
5. Summary
6. Future work

1 Light-cone coordinates



We define light-cone coordinates by

$$x^{\pm} = t \pm z \,, \quad \mathbf{x}_{\perp} = (x, y) \,.$$

Momentum variables are similarly constructed as

$$p^{\pm} = E \pm p_z \,, \quad \mathbf{p}_{\perp} = (p_x, p_y).$$

The dot product is written

$$p \cdot x = \frac{1}{2}(p^+ x^- + p^- x^+) - \mathbf{p}_{\perp} \cdot \mathbf{x}_{\perp} \,.$$

The time variable is taken to be x^+, and time evolution of a system is then determined by $\mathcal{P}^-$.

$$\begin{aligned} \text{Energy:} \quad & E \longrightarrow p^- \\ \text{Momentum:} \quad & \mathbf{p} \longrightarrow \underline{p} \equiv (p^+, \mathbf{p}_{\perp}) \end{aligned}$$

The light-cone Hamiltonian is

$$H_{\rm LC} = \mathcal{P}^+ \mathcal{P}^- - \mathcal{P}_{\perp}^2 \,.$$

$\mathcal{P}^+$ and $\mathcal{P}_{\perp}$ are momentum operators conjugate to x^- and $\mathbf{x}_{\perp}$. The eigenvalue problem is

$$H_{\rm LC}\Psi = M^2 \Psi \,, \quad \underline{\mathcal{P}}\Psi = \underline{P}\Psi \,,$$

where M is the mass of the state.

1-a. *advantages of the light-cone*

- largest possible set of nondynamical generators. In particular, boosts are kinematical.
- the perturbative vacuum is the physical vacuum. $(p_i^+ = \sqrt{p^2+m^2} + p_z > 0)$ No need to compute vacuum state.
- existence of well-defined Fock-state expansions. No disconnected vacuum pieces.

Let the states of the Fock basis be written $\{|n : p_i^+, \mathbf{p}_{\perp i}\rangle\}$ where $\mathcal{P}^+$ and $\mathcal{P}_\perp$ are diagonal, with n the number of particles and i ranging between 1 and n. Then

$$\Psi = \sum_n \int [dx]_n \, [d^2k_\perp]_n \, \psi_n(x, \mathbf{k}_\perp)|n : xP^+, x\mathbf{P}_\perp + \mathbf{k}_\perp\rangle \, ,$$

with

$$[dx]_n = 4\pi\delta(1 - \sum_{i=1}^n x_i) \prod_{i=1}^n \frac{dx_i}{4\pi\sqrt{x_i}} \, ,$$

$$[d^2k_\perp]_n = 4\pi^2\delta(\sum_{i=1}^n \mathbf{k}_{\perp i}) \prod_{i=1}^n \frac{d^2k_{\perp i}}{4\pi^2} \, ,$$

$(P^+, \mathbf{P}_\perp)$ is the total light-cone momentum. ψ_n is interpreted as the wave function of the contribution from states with n particles.

1-b. *discretization* (DLCQ)

Periodic BC for bosons and antiperiodic for fermions in a light-cone box $-L < x^- < L$, $-L_\perp < x, y < L_\perp$. Integrals replaced by trapezoidal approximations, with modification at edge of cutoff

$$\text{Grid:} \quad p^+ \to \frac{\pi}{L} n \,, \quad \mathbf{k}_\perp \to (\frac{\pi}{L_\perp} n_x, \frac{\pi}{L_\perp} n_y) \,.$$

The limit $L \to \infty$ can be exchanged for a limit in terms of the integer *resolution*

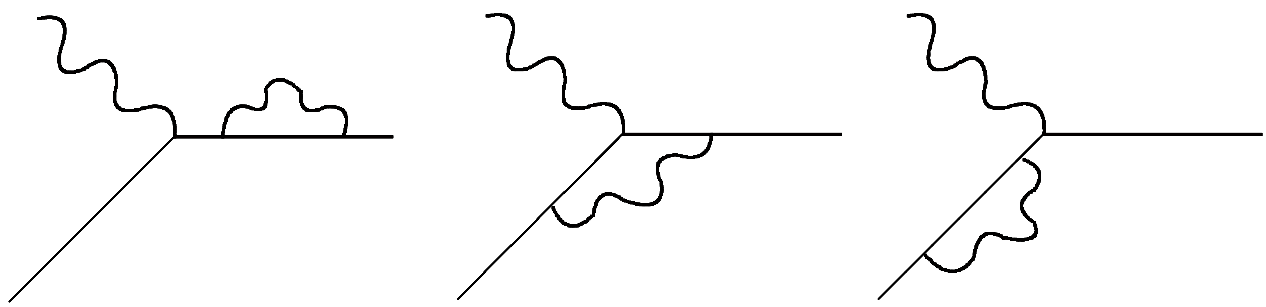
$$K \equiv \frac{L}{\pi} P^+ \,.$$

Also $x = p^+/P^+ \to n/K$, with n odd for fermions and even for bosons. $H_{\rm LC}$ is independent of L.

Because the n_i are all positive, DLCQ automatically limits the number of particles to no more than K. The integers n_x and n_y range between limits associated with some maximum integer $N_\perp$ fixed by the invariant-mass cutoff.

1-c. *Tamm–Dancoff truncation*

- limit the number of particles of each type.
- serious complications for renormalization.
  - severe sector dependence of counterterms.
  - for QED, get violation of Ward identity.

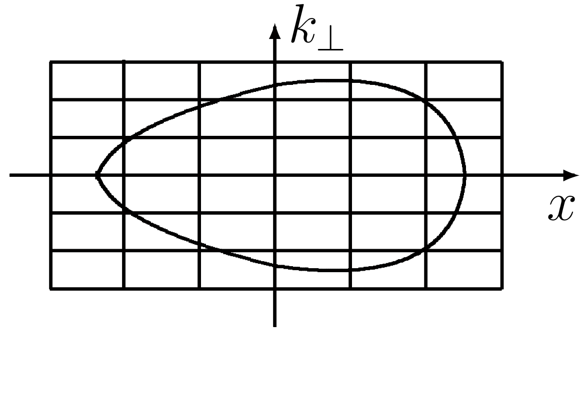


1-d. *regularization*

$$\sum_i \frac{m_i^2 + k_{\perp i}^2}{x_i} \leq \Lambda^2 \,.$$

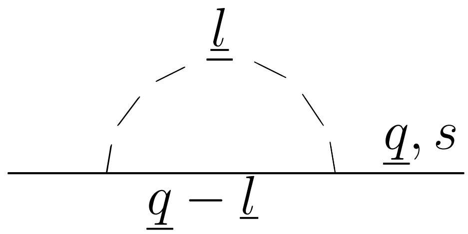
$$\frac{m_i^2 + k_{\perp i}^2}{x_i} \leq \Lambda^2 \;\; \text{for each } i \,.$$

$$\sum_i^n \frac{m_i^2 + k_{\perp i}^2}{x_i} - \sum_j^m \frac{m_j^2 + k_{\perp j}^2}{x_j} \leq \Lambda^2 \,.$$



2 Yukawa theory at one loop

2-a. *fermion self energy*



$q_\perp = 0$

$$I(\mu^2, M^2) \equiv -\frac{1}{\mu^2}\int \frac{dl^+ d^2 l_\perp}{l^+(q^+ - l^+)^2} \times \frac{(q^+)^2 \mathbf{l}_\perp^2 + (2q^+ - l^+)^2 M^2}{M^2 - D_1}\theta(\Lambda^2 - D_1)\,,$$

where μ is the boson mass, M is the fermion mass, and

$$D_1 = \frac{\mu^2 + \mathbf{l}_\perp^2}{l^+/q^+} + \frac{M^2 + \mathbf{l}_\perp^2}{(q^+ - l^+)/q^+}\,.$$

The boson mass μ sets the energy scale.

$$I(\mu^2, 0) = \frac{\pi}{\mu^2}\left[\frac{\Lambda^2}{2} - \frac{\mu^4}{2\Lambda^2} - \mu^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right)\right]\,.$$

In order to maintain $I(\mu^2, M^2) \propto M^2$, three Pauli-Villars bosons are needed:

$$I_{\rm sub}(\mu^2, M^2, \mu_i^2) = I(\mu^2, M^2) + \sum_{i=1}^{3} C_i I(\mu_i^2, M^2)\,.$$

The C_i are chosen to satisfy

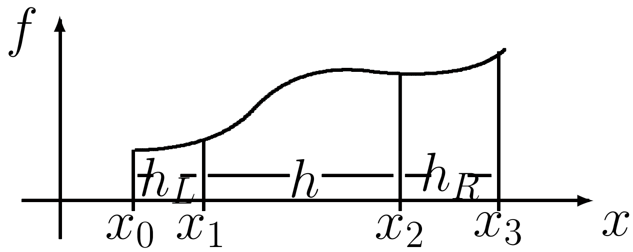
$$1 + \sum_{i=1}^{3} C_i = 0\,, \quad \mu^2 + \sum_{i=1}^{3} C_i \mu_i^2 = 0\,, \quad \sum_{i=1}^{3} C_i \mu_i^2 \ln(\mu_i^2/\mu^2) = 0\,.$$

2-b. *numerical calculations*

Ordinary DLCQ integration is not sufficiently accurate because the cutoff is incommensurate with the DLCQ grid. Alternative integration schemes are of the general form

$$\int d\vec{r} f(\vec{r}) \simeq \sum_{i,j,\ldots} w_{i,j,\ldots} f(\vec{r}_{i,j,\ldots}) \,,$$

where, unlike the case of DLCQ, the weights $w_{i,j,\ldots}$ will not all be equal. Such a new integration rule is obtained from consideration of an integral from, say, x_0 to x_3.



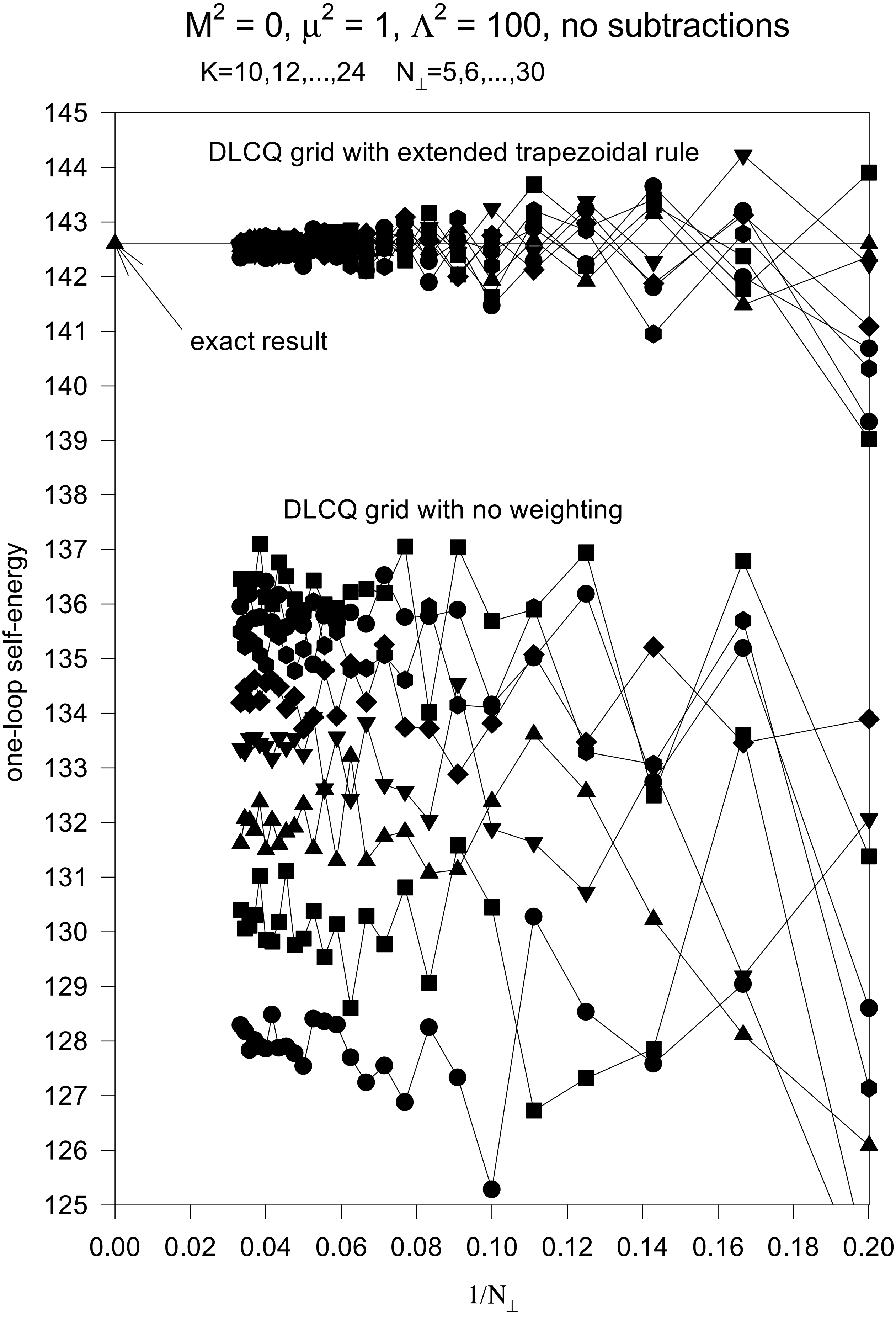
The integral of a function f is then approximated by

$$\int_{x_0}^{x_3} f dx \simeq a_1 f(x_1) + a_2 f(x_2) \,,$$

with

$$\begin{aligned} a_1 &= (h + h_L + h_R)(h + h_L - h_R)/2h \,, \\ a_2 &= (h + h_L + h_R)(h + h_R - h_L)/2h \,. \end{aligned}$$

The coefficients a_i are chosen to provide exact results for linear functions. The standard trapezoidal rule is recovered when $h_L = h_R = 0$. If $h_L = h_R = h$, a standard open Newton–Cotes formula results. When the extended rule is combined with the standard rule for interior intervals, a general composite rule is obtained. The extended rule is then used twice, once at each end, with h_R or h_L set to zero.

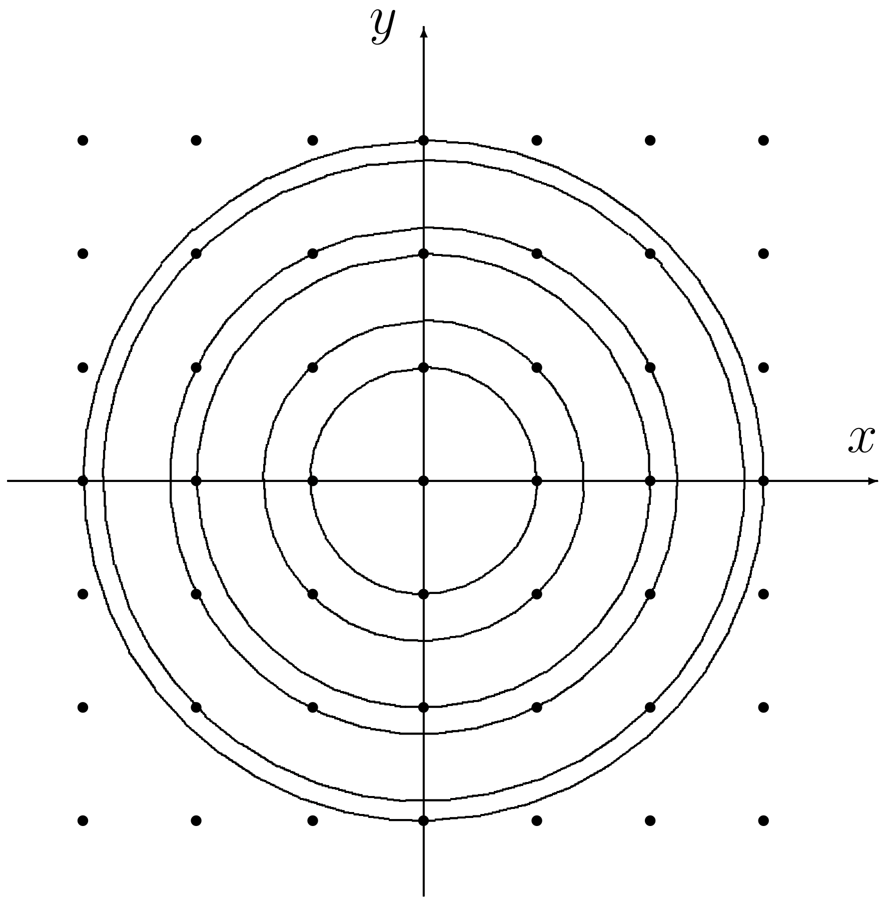


M^2 = 0, μ^2 = 1, Λ^2 = 100, no subtractions
K=10,12,...,24 N⊥=5,6,...,30
DLCQ grid with extended trapezoidal rule
exact result
DLCQ grid with no weighting
one-loop self-energy
1/N⊥
145
144
143
142
141
140
139
138
137
136
135
134
133
132
131
130
129
128
127
126
125
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20

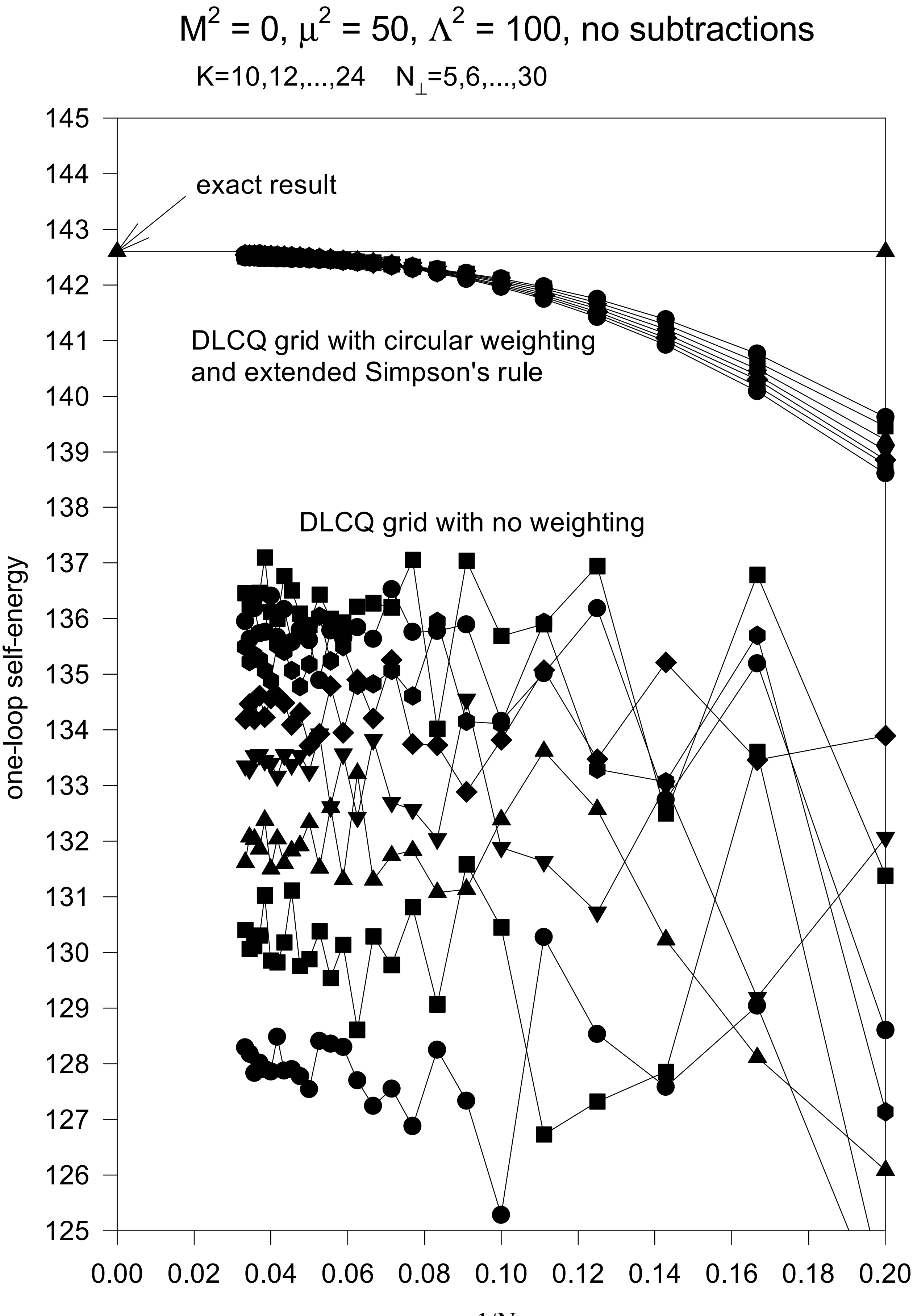
Additional improvement can be obtained by taking into account the cylindrical symmetry of the integration domain. The integral is written in polar coordinates

$$\int dxdy f(x,y) = \frac{1}{2}\int_0^{2\pi} d\phi \int_0^{R^2} d(r^2)\tilde{f}(r^2,\phi)\,.$$

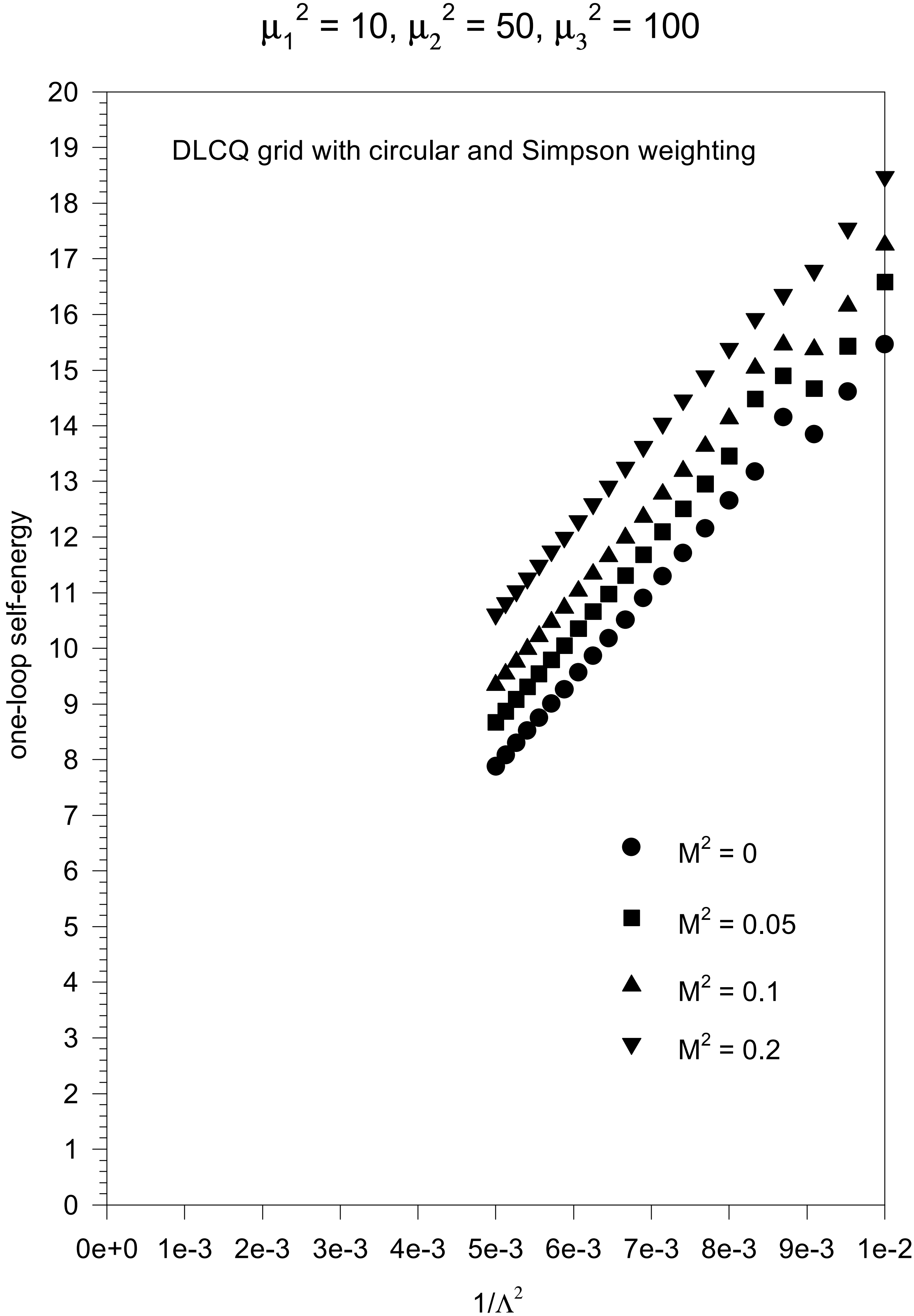
The points of the square grid lie on circles of varying radii r_i



The r_i are easily computed from the coordinates of the square grid. Clearly, the intervals are not of equal length; however, they are on average of order $3h^2$, where h is the spacing in the square grid. For the first 10 circles, the average spacing in r^2 is actually closer to $2h^2$.



M2 = 0, μ2 = 50, Λ2 = 100, no subtractions
K=10,12,...,24 N⊥=5,6,...,30
exact result
DLCQ grid with circular weighting
and extended Simpson's rule
DLCQ grid with no weighting
one-loop self-energy
1/N⊥
145
144
143
142
141
140
139
138
137
136
135
134
133
132
131
130
129
128
127
126
125
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20



$\mu_1{}^2 = 10, \mu_2{}^2 = 50, \mu_3{}^2 = 100$
DLCQ grid with circular and Simpson weighting
one-loop self-energy
$1/\Lambda^2$
$M^2 = 0$
$M^2 = 0.05$
$M^2 = 0.1$
$M^2 = 0.2$
0e+0
1e-3
2e-3
3e-3
4e-3
5e-3
6e-3
7e-3
8e-3
9e-3
1e-2

Values of the subtracted integral $I_{\rm sub}(\mu^2, M^2, \mu_i^2)$ in the limit of infinite cutoff. The Pauli-Villars masses are $\mu_1^2 = 10\mu^2$, $\mu_2^2 = 50\mu^2$ and $\mu_3^2 = 100\mu^2$.

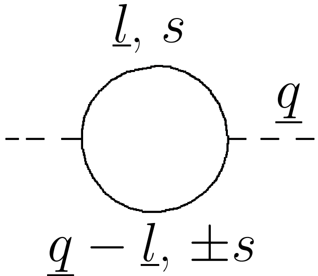
M^2:	0	0.05	0.1	0.2
$I_{\rm sub}$:	-0.064	0.70	1.37	2.70

Number of Fock states used in two typical cases.

				Pauli–Villars bosons			
Λ^2/μ^2	K	$N_\perp$	physical bosons	1	2	3	total
200	20	25	25975	22602	11142	3305	37049
200	24	30	44943	39162	19293	5695	64150

2-c. *boson self energy*

Fermion contribution:



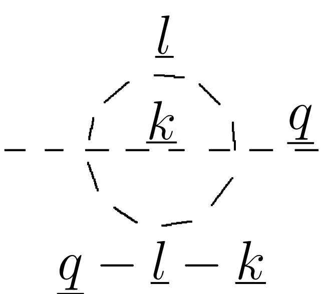
$$\int \frac{dl^+ d^2 l_\perp}{4LL_\perp^2} \frac{q^+(l_\perp^2 + M^2)}{l^{+2}(q^+ - l^+)^2} \theta\left(\Lambda^2 - D_2\right) \left[\mu^2 - D_2\right]^{-1} ,$$

where

$$D_2 \equiv q^{+2}(M^2 + l_\perp^2)/[l^+(q^+ - l^+)] ,$$

μ is the boson mass, and M the fermion mass.

ϕ^4 contribution:



$$\int \frac{dl^+ d^2 l_\perp dk^+ d^2 k_\perp}{q^+ l^+ k^+ (q^+ - l^+ - k^+)} \theta(\Lambda^2 - D_4)/(\mu^2 - D_4)$$

where

$$D_4 \equiv \frac{\mu^2 + l_\perp^2}{l^+/q^+} + \frac{\mu^2 + k_\perp^2}{k^+/q^+} + \frac{\mu^2 + (l_\perp + k_\perp)^2}{(q^+ - l^+ - k^+)/q^+} .$$

3 Heavy fermion model

3-a. *effective Hamiltonian*

[Greenberg & Schweber, Głazek & Perry]

$$
\begin{aligned}
H_{\rm LC}^{\rm eff} &= M_0^2 \int \frac{dp^+ d^2 p_\perp}{16\pi^3 p^+} \sum_\sigma b^\dagger_{\underline{p}\sigma} b_{\underline{p}\sigma} \\
&+ P^+ \int \frac{dq^+ d^2 q_\perp}{16\pi^3 q^+} \left[\frac{\mu^2 + q_\perp^2}{q^+} a^\dagger_{\underline{q}} a_{\underline{q}} + \frac{\mu_1^2 + q_\perp^2}{q^+} a^\dagger_{1\underline{q}} a_{1\underline{q}} \right] \\
&+ g \int \frac{dp_1^+ d^2 p_{\perp 1}}{\sqrt{16\pi^3 p_1^+}} \int \frac{dp_2^+ d^2 p_{\perp 2}}{\sqrt{16\pi^3 p_2^+}} \int \frac{dq^+ d^2 q_\perp}{16\pi^3 q^+} \sum_\sigma b^\dagger_{\underline{p}_1\sigma} b_{\underline{p}_2\sigma} \\
&\quad \times \Big[a^\dagger_{\underline{q}} \delta(\underline{p}_1 - \underline{p}_2 + \underline{q}) + a_{\underline{q}} \delta(\underline{p}_1 - \underline{p}_2 - \underline{q}) \\
&\quad\quad + i a^\dagger_{1\underline{q}} \delta(\underline{p}_1 - \underline{p}_2 + \underline{q}) + i a_{1\underline{q}} \delta(\underline{p}_1 - \underline{p}_2 - \underline{q}) \Big] ,
\end{aligned}
$$

where

$$
\begin{aligned}
\left[a_{\underline{q}}, a^\dagger_{\underline{q}'} \right] &= 16\pi^3 q^+ \delta(\underline{q} - \underline{q}') , \\
\left\{ b_{\underline{p}\sigma}, b^\dagger_{\underline{p}'\sigma'} \right\} &= 16\pi^3 p^+ \delta(\underline{p} - \underline{p}') \delta_{\sigma\sigma'} .
\end{aligned}
$$

The state vector is

$$
\begin{aligned}
\Phi_\sigma &= \sqrt{16\pi^3 P^+} \sum_{n,n_1} \int \frac{dp^+ d^2 p_\perp}{\sqrt{16\pi^3 p^+}} \prod_{i=1}^{n} \int \frac{dq_i^+ d^2 q_{\perp i}}{\sqrt{16\pi^3 q_i^+}} \prod_{j=1}^{n_1} \int \frac{dr_j^+ d^2 r_{\perp j}}{\sqrt{16\pi^3 r_j^+}} \\
&\quad \times \delta(\underline{P} - \underline{p} - \sum_i^n \underline{q}_i - \sum_j^{n_1} \underline{r}_j) \phi^{(n,n_1)}(\underline{q}_i, \underline{r}_j; \underline{p}) \\
&\quad\quad \times \frac{1}{\sqrt{n! n_1!}} b^\dagger_{\underline{p}\sigma} \prod_i^n a^\dagger_{\underline{q}_i} \prod_j^{n_1} a^\dagger_{1\underline{r}_j} |0\rangle .
\end{aligned}
$$

Normalization:

$$\Phi'^{\dagger}_{\sigma} \cdot \Phi_{\sigma} = 16\pi^3 P^+ \delta(\underline{P}' - \underline{P})$$

$$\Rightarrow \quad 1 \; = \; \sum_{n,n_1} \prod_{i}^{n} \int \, dq_i^+ d^2 q_{\perp i} \prod_{j}^{n_1} \int \, dr_j^+ d^2 r_{\perp j}$$

$$\times \left| \phi^{(n,n_1)}(\underline{q}_i, \underline{r}_j; \underline{P} - \sum_i \underline{q}_i - \sum_j \underline{r}_j) \right|^2$$

3-b. *analytic solution*

A solution to

$$H_{\rm LC}^{\rm eff}\Phi_\sigma = M^2\Phi_\sigma$$

must satisfy

$$\left[M^2 - M_0^2 - \sum_i \frac{\mu^2+q_{\perp i}^2}{y_i} - \sum_j \frac{\mu_1^2+r_{\perp j}^2}{z_j}\right]\phi^{(n,n_1)}$$

$$= g\left\{\sqrt{n+1}\int \frac{dq^+d^2q_\perp}{\sqrt{16\pi^3q^+}}\phi^{(n+1,n_1)}(\underline{q}_i,\underline{q},\underline{r}_j,\underline{p}) + \frac{1}{\sqrt{n}}\sum_i \frac{1}{\sqrt{16\pi^3q_i^+}}\phi^{(n-1,n_1)}(\underline{q}_1,\ldots,\underline{q}_{i-1},\underline{q}_{i+1},\ldots,\underline{q}_n,\underline{r}_j,\underline{p})\right.$$

$$\left. +i\sqrt{n_1+1}\int\frac{dr^+d^2r_\perp}{\sqrt{16\pi^3r^+}}\phi^{(n,n_1+1)}(\underline{q}_i,\underline{r}_j,\underline{r},\underline{p}) + \frac{i}{\sqrt{n_1}}\sum_j\frac{1}{\sqrt{16\pi^3r_j^+}}\phi^{(n,n_1-1)}(\underline{q}_i,\underline{r}_1,\ldots,\underline{r}_{j-1},\underline{r}_{j+1},\ldots,\underline{r}_{n_1},\underline{p})\right\}$$

The solution is

$$\phi^{(n,n_1)} = \sqrt{Z}\frac{(-g)^n(-ig)^{n_1}}{\sqrt{n!n_1!}}\prod_i\frac{y_i}{\sqrt{16\pi^3q_i^+}(\mu^2+q_{\perp i}^2)} \times\prod_j\frac{z_j}{\sqrt{16\pi^3r_j^+}(\mu_1^2+r_{\perp j}^2)},$$

with normalization

$$\frac{1}{Z} = \sum_{n,n_1}\frac{(g^2/16\pi^3)^{n+n_1}}{(2n+2n_1)!n!n_1!}\left(\int\frac{d^2q_\perp}{(\mu^2+q_\perp^2)^2}\right)^n\left(\int\frac{d^2r_\perp}{(\mu_1^2+r_\perp^2)^2}\right)^{n_1},$$

provided that $M_0^2(p^+)$ is chosen to satisfy

$$M^2 - M_0^2 = -\frac{g^2}{16\pi^3}\frac{p^+}{P^+}\left\{\int\frac{d^2q_\perp}{\mu^2+q_\perp^2} - \int\frac{d^2r_\perp}{\mu_1^2+r_\perp^2}\right\}.$$

3-c. *coupling renormalization*

To fix the coupling we use $\langle{:}\phi^2(0){:}\rangle \equiv \Phi_\sigma^\dagger \,{:}\phi^2(0){:}\, \Phi_\sigma$. For the analytic solution it reduces to

$$\langle{:}\phi^2(0){:}\rangle \;=\; \sum_{n,n_1} \frac{2Zn(g^2/16\pi^3)^{n+n_1}}{(2n+2n_1-1)!n!n_1!} \times \left(\int \frac{d^2q_\perp}{(\mu^2+q_\perp^2)^2}\right)^n \left(\int \frac{d^2r_\perp}{(\mu_1^2+r_\perp^2)^2}\right)^{n_1} .$$

From a numerical solution it can be computed fairly efficiently in a sum similar to the normalization sum

$$\langle{:}\phi^2(0){:}\rangle \;=\; \sum_{n=1,n_1=0} \prod_i^n \int dq_i^+ d^2q_{\perp i} \times \prod_j^{n_1} \int dr_j^+ d^2r_{\perp j} \left(\sum_{k=1}^n \frac{2}{q_k^+/P^+}\right) \times \left|\phi^{(n,n_1)}(\underline{q}_i, \underline{r}_j; \underline{P} - \sum_i \underline{q}_i - \sum_j \underline{r}_j)\right|^2 .$$

3-d. *form factor slope*

From Brodsky and Drell [PRD **22**, 2236 (1980). See p. 2239.]

$$F(Q^2) \;=\; \frac{1}{2P^+}\langle P+\gamma\uparrow|J^+(0)|P\uparrow\rangle$$
$$\;=\; \sum_j e_j \int 16\pi^3\delta(1-\sum_i x_i)\delta(\sum_i \mathbf{k}_{\perp i})\prod_i \frac{dx_i d^2p_{\perp i}}{16\pi^3}$$
$$\times\psi^*_{P+\gamma\uparrow}(x_i,\mathbf{p}'_{\perp i})\psi_{P\uparrow}(x_i,\mathbf{p}_{\perp i})\,,$$

where the matrix element has been evaluated in the frame with

$$P=(P^+,P^-=\frac{M^2}{P^+},\mathbf{0}_\perp)\,,\quad \gamma=(0,\gamma^-=2\gamma\cdot P/P^+,\gamma_\perp)\,,\quad Q^2\equiv\gamma_\perp^2$$

e_j is the charge of the jth constituent, and

$$\mathbf{p}'_{\perp i}=\begin{cases}\mathbf{p}_{\perp i}-x_i\gamma_\perp & i\neq j\\ \mathbf{p}_{\perp i}+(1-x_i)\gamma_\perp & i=j\,.\end{cases}$$

A sum over Fock states is understood.

When the fermion is assigned a charge of 1, the form factor is

$$F(Q^2) \;=\; Z\sum_{n,n_1}\frac{(g^2/16\pi^3)^{n+n_1}}{n!n_1!}\int_0^1\theta(1-\sum_i^n y_i-\sum_j^{n_1} z_j)$$
$$\times\prod_i^n\frac{y_i dy_i d^2q_{\perp i}}{(\mu^2+q'^2_{\perp i})(\mu^2+q^2_{\perp i})}\prod_j^{n_1}\frac{z_j dz_j d^2r_{\perp j}}{(\mu^2+r'^2_{\perp j})(\mu^2+r^2_{\perp j})}\,,$$

with

$$\mathbf{q}'_\perp=\mathbf{q}_\perp-y\gamma_\perp\,,\quad \mathbf{r}'_\perp=\mathbf{r}_\perp-z\gamma_\perp\,.$$

The slope is extracted as

$$F'(0) \;=\; \sum_{n,n_1} \frac{6Zn(g^2/16\pi^3)^{n+n_1}}{(2n+2n_1+2)!n!n_1!} \int \frac{d^2q_\perp}{(\mu^2+q_\perp^2)^3}\left[\frac{2\mu^2}{\mu^2+q_\perp^2}-1\right]$$
$$\times\left(\int \frac{d^2q_\perp}{(\mu^2+q_\perp^2)^2}\right)^{n-1}\left(\int \frac{d^2r_\perp}{(\mu_1^2+r_\perp^2)^2}\right)^{n_1}$$
$$+\text{P-V term}\,.$$

Numerically, one could compute $F'(0)$ from

$$F'(0) \;=\; \sum_{n,n_1}\prod_i^n \int dq_i^+ d^2q_{\perp i}\prod_j^{n_1}\int dr_j^+ d^2r_{\perp j}$$
$$\times\left[\left(\sum_i \frac{y_i^2}{4}\nabla_{\perp i}^2+\sum_j\frac{z_j^2}{4}\nabla_{\perp j}^2\right)\phi^{(n,n_1)}(\underline{q}_i,\underline{r}_j;\underline{P}-\sum_i\underline{q}_i-\sum_j\underline{r}_j)\right]^*$$
$$\times\,\phi^{(n,n_1)}(\underline{q}_i,\underline{r}_j;\underline{P}-\sum_i\underline{q}_i-\sum_j\underline{r}_j)\,,$$

with ∇^2 represented by finite differences. Integration by parts leads to a computationally better quantity

$$\tilde{F}'(0) \;=\; \sum_{n,n_1}\prod_i^n \int dq_i^+ d^2q_{\perp i}\prod_j^{n_1}\int dr_j^+ d^2r_{\perp j}$$
$$\times\left[\sum_i\left|\frac{y_i}{2}\nabla_{\perp i}\phi^{(n,n_1)}(\underline{q}_i,\underline{r}_j;\underline{P}-\sum_i\underline{q}_i-\sum_j\underline{r}_j)\right|^2+\right.$$
$$\left.+\sum_j\left|\frac{z_j}{2}\nabla_{\perp j}\phi^{(n,n_1)}(\underline{q}_i,\underline{r}_j;\underline{P}-\sum_i\underline{q}_i-\sum_j\underline{r}_j)\right|^2\right]\,,$$

which differs from $F'(0)$ by surface terms that vanish as $\Lambda\to\infty$.

3-e. *distribution functions*

$$f_B(y) \equiv \sum_{n,n_1} \prod_i^n \int dq_i^+ d^2q_{\perp i} \prod_j^{n_1} \int dr_j^+ d^2r_{\perp j} \sum_{i=1}^n \delta(y - q_i^+/P^+) \times \left|\phi^{(n,n_1)}(\underline{q}_i, \underline{r}_j; \underline{P} - \sum_i \underline{q}_i - \sum_j \underline{r}_j)\right|^2 ,$$

$$f_{PV}(z) \equiv \sum_{n,n_1} \prod_i^n \int dq_i^+ d^2q_{\perp i} \prod_j^{n_1} \int dr_j^+ d^2r_{\perp j} \sum_{j=1}^{n_1} \delta(z - r_j^+/P^+) \times \left|\phi^{(n,n_1)}(\underline{q}_i, \underline{r}_j; \underline{P} - \sum_i \underline{q}_i - \sum_j \underline{r}_j)\right|^2 .$$

Their integrals yield the average multiplicities

$$\langle n_B \rangle = \int_0^1 f_B(y) dy ,$$
$$\langle n_{PV} \rangle = \int_0^1 f_{PV}(z) dz .$$

For the analytic solution we obtain

$$f_B(y) = \left(\frac{\mu_1}{\mu}\right)^2 f_{PV}(y) = \sum_{n,n_1} \frac{Zny(1-y)^{2(n+n_1-1)}(g^2/16\pi^3)^{n+n_1}}{(2n+2n_1-2)!n!n_1!} \times \left(\int \frac{d^2q_\perp}{(\mu^2+q_\perp^2)^2}\right)^n \left(\int \frac{d^2r_\perp}{(\mu_1^2+r_\perp^2)^2}\right)^{n_1}$$

and

$$\langle n_B \rangle = \left(\frac{\mu_1}{\mu}\right)^2 \langle n_{PV} \rangle = \sum_{n,n_1} \frac{Zn(g^2/16\pi^3)^{n+n_1}}{(2n+2n_1)!n!n_1!} \left(\int \frac{d^2q_\perp}{(\mu^2+q_\perp^2)^2}\right)^n \left(\int \frac{d^2r_\perp}{(\mu_1^2+r_\perp^2)^2}\right)^{n_1} .$$

4 Numerical methods and results

4-a. *Lanczos algorithm*

For A real symmetric and $\vec{u}_1$ a chosen vector, construct tridiagonal matrix T

$$A \rightarrow T \equiv \begin{pmatrix} a_1 & b_2 & 0 & \dots & 0 \\ b_2 & a_2 & b_3 & 0 & \dots \\ 0 & b_3 & a_3 & b_4 & \dots \\ 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . \end{pmatrix}$$

where

$$b_{n+1}\vec{u}_{n+1} = A\vec{u}_n - a_n\vec{u}_n - b_n\vec{u}_{n-1}$$

and

$$a_n = \vec{u}_n \cdot A\vec{u}_n \, .$$

The $\vec{u}_n$ are orthonormal. A good choice for $\vec{u}_1$ is important but not necessarily critical.

pitfalls:

- all $\vec{u}_n$ have the symmetries of $\vec{u}_1$ that are conserved by A.
- round-off errors
  - loss of orthogonality.
  - spurious & duplicate eigenvalues.
  - poor representation of degeneracies.

Why transform to T? Easily diagonalized.
However, the complete transformation to tridiagonal form requires effort comparable to ordinary diagonalization (e.g. Householder) and is less reliable. The real use is to stop after a few iterations (∼20-100) and diagonalize in the truncated basis; the extreme eigenvalues are quite good.

So, after N iterations solve

$$T\vec{c}_i = \xi_i \vec{c}_i$$

and construct

$$\vec{y}_i = \sum_{j=1}^{N} (\vec{c}_i)_j \vec{u}_j$$

to find

$$A\vec{y}_i \simeq \xi_i \vec{y}_i \,.$$

Can show that

$$|A\vec{y}_i - \xi_i \vec{y}_i| = b_{N+1} |(\vec{c}_i)_N|$$

Then have two criteria for selecting N:

- explicit convergence of one or more eigenvalues
- $b_{N+1}|(\vec{c}_i)_N| <$ tolerance $\times\ ||\vec{c}_i||$

4-b. *efficient form*

$$\vec{v}_{n+1} = A\vec{u}_n - b_n\vec{u}_{n-1}$$

$$a_n = \vec{v}_{n+1} \cdot \vec{u}_n$$
$$\vec{v}'_{n+1} = \vec{v}_{n+1} - a_n\vec{u}_n$$
$$b_{n+1} = \sqrt{\vec{v}'_{n+1} \cdot \vec{v}'_{n+1}}$$
$$\vec{u}_{n+1} = \vec{v}'_{n+1}/b_{n+1}$$

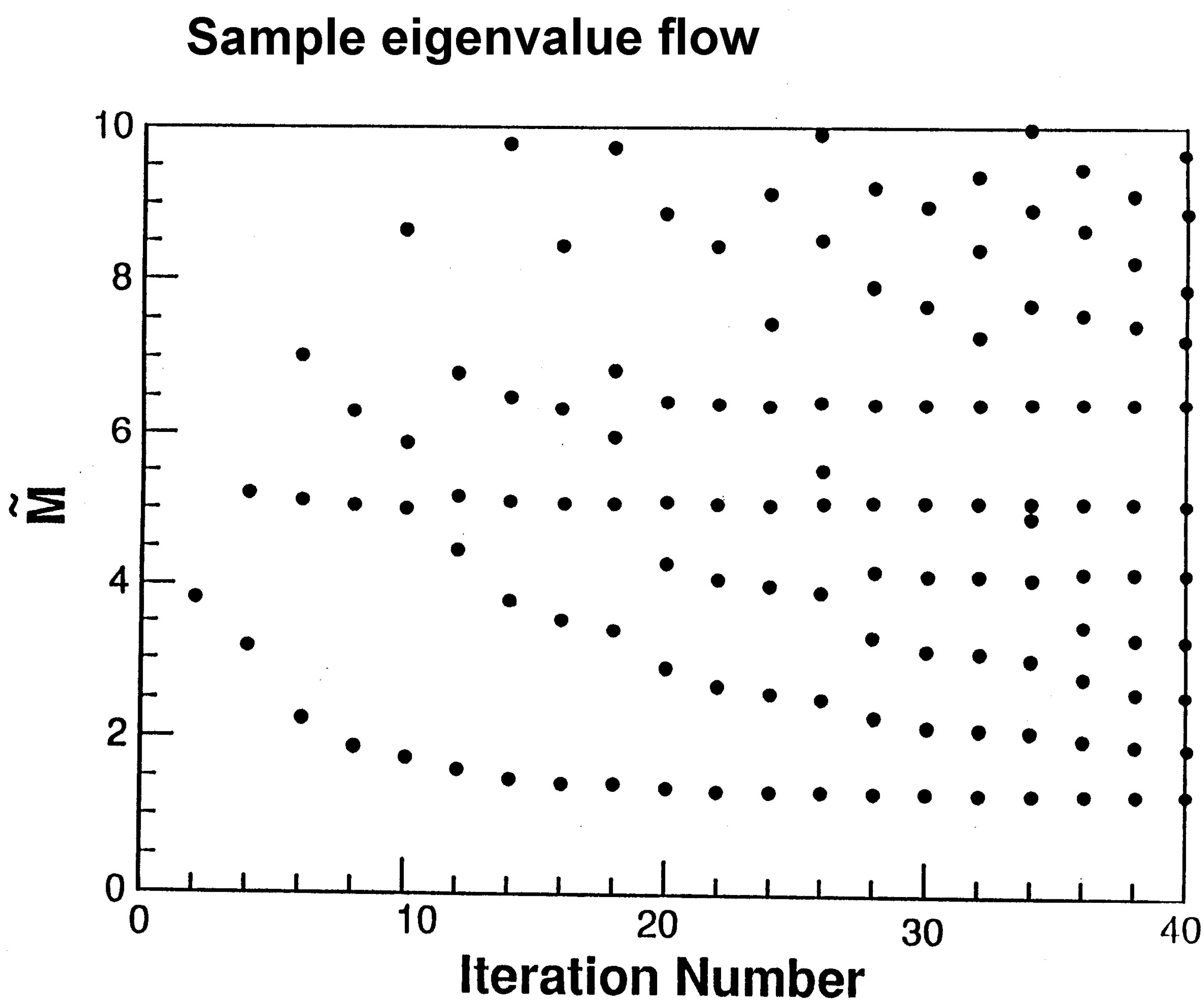
Store $\vec{u}_{n-1}$, $\vec{v}_{n+1}$, $\vec{v}'_{n+1}$, and $\vec{u}_{n+1}$ in same array. Keep only two vectors in RAM. Write others to disk, if want to reconstruct eigenvectors at end.

The dot products do not involve conjugation, and the constants a_n and b_n are in general complex. The process will fail if b_{n+1} is zero for nonzero $\mathbf{v}'_{n+1}$, which can happen in principle but does not seem to happen in practice.

For the heavy fermion model the matrix is extremely sparse and all nonzero elements can be stored for use in the matrix-vector multiplication.

Weighting factors are used in all summations that represent integrals, and symmetry is restored to the matrix via

$$\sum_j A_{ij} w_j u_j = \xi u_i \quad \longrightarrow \quad \sum_j \sqrt{w_i w_j} A_{ij} \sqrt{w_j} u_j = \xi \sqrt{w_i} u_i$$


Sample eigenvalue flow
10
8
6
4
2
0
M̃
0
10
20
30
40
Iteration Number

4-c. *references*

1. The original:
   C. Lanczos, J. Res. Natl. Bur. Stand. **45**, 255 (1950).

2. Thorough discussion:
   J. Cullum and R.A. Willoughby, J. Comput. Phys. **44**, 329 (1981); *Lanczos Algorithms for Large Symmetric Eigenvalue Computations* (Birkhauser, Boston, 1985), Vol. I and II.

3. Basic references:
   J.H. Wilkinson, *The Algebraic Eigenvalue Problem* (Clarendon, Oxford, 1965); B.N. Parlett, *The Symmetric Eigenvalue Problem* (Prentice-Hall, Englewood Cliffs, NJ, 1980); G.H. Golub and C.F. van Loan, *Matrix Computations* (Johns Hopkins University Press, Baltimore, 1983).

4. Loss of orthogonality:
   C.C. Paige, BIT **10**, 183 (1970); J. Inst. Math. Appl. **10**, 373 (1972); B.N. Parlett and D.S. Scott, Math. Comput. **33**, 217 (1979); B.N. Parlett and B. Nour-Omid, Comput. Phys. Commun., **53**, 169 (1989).

5. Efficient algorithm:
   D.S. Scott, Comput. Phys. Commun. **53**, 271 (1989).

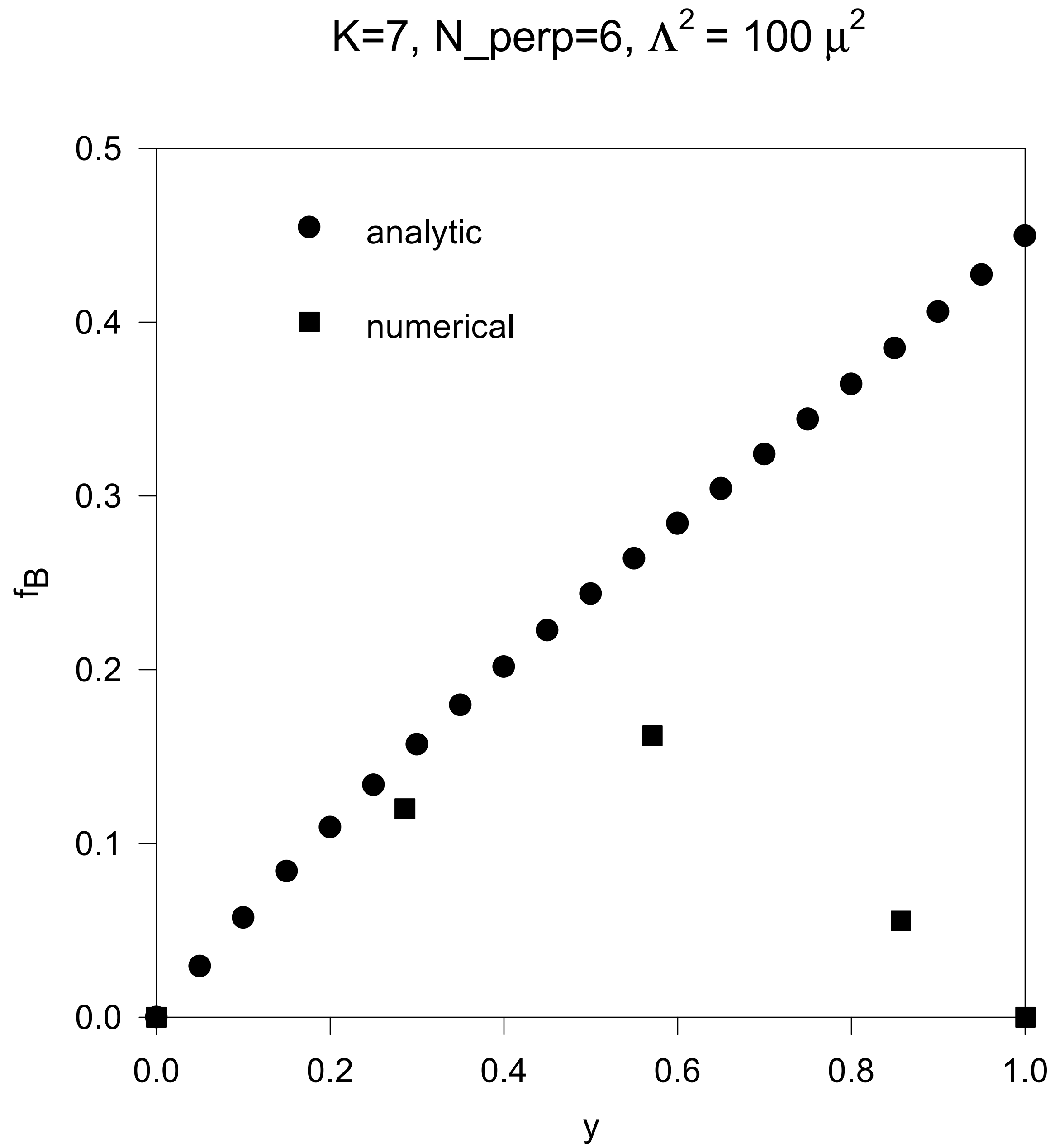
6. Biorthogonal algorithm:
   S.K. Kin and A.T. Chronopoulos, J. Comp. and Appl. Math **42**, 357 (1992); Y. Saad, Comput. Phys. Commun. **53**, 71 (1989); J. Cullum and R.A. Willoughby, in *Large Scale Eigenvalue Problems*, J. Cullum and R.A. Willoughby, eds., (Elsevier, Amsterdam, 1986); J.H. Wilkinson, *The Algebraic Eigenvalue Problem* (Clarendon, Oxford, 1965).

M^2= 1.000
mu_1^2= 10.000
Lambda^2= 20.00

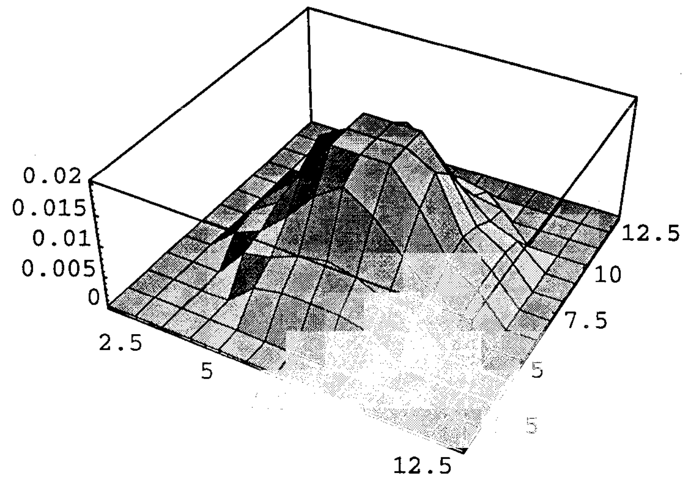
N_perp	3	5	7	9	11	13	15	17	19	21
1	3	5	10	15	26	37	59	81	123	111
	(2)	(4)	(7)	(12)	(19)	(30)	(45)	(67)	(97)	(97)
2	15	37	66	130	296	221	333	660	893	1311
	(10)	(32)	(63)	(119)	(275)	(198)	(299)	(626)	(833)	(1277)
3	31	111	286	783	1578	5285	4686	9598	25675	6609
	(22)	(102)	(251)	(736)	(1539)	(5174)	(4574)	(9358)	(25232)	(6435)
4	59	351	1194	6084	8048	10415	33559	92933	82149	115322
	(38)	(330)	(1095)	(5913)	(7759)	(10224)	(32889)	(92139)	(81512)	(114154)
5	75	675	5918	12854	43408	176093	130820	239849	586423	1544653
	(46)	(650)	(5699)	(12619)	(42339)	(174816)	(129720)	(238503)	(582535)	(1540183)
6	115	1369	10602	49576	224064	511711	1226784	3796491	2924011	3836433
	(70)	(1332)	(10127)	(49061)	(222147)	(508872)	(1219924)	(3789983)	(2912424)	(3827311)
7	159	2397	28186	158720	555176					
	(98)	(2348)	(27375)	(157669)	(551143)					
8	199	3975	51666	261866	2167508					
	(130)	(3906)	(50027)	(260479)	(2155395)					
9	259	6529	99802	897289	3540988					
	(162)	(6440)	(97923)	(894490)	(3524831)					
10	315	9329	202186	1964771	14512958					
	(194)	(9228)	(199031)	(1960548)	(14470631)					

M^2= 1.000
mu_1^2= 10.000
Lambda^2= 50.00

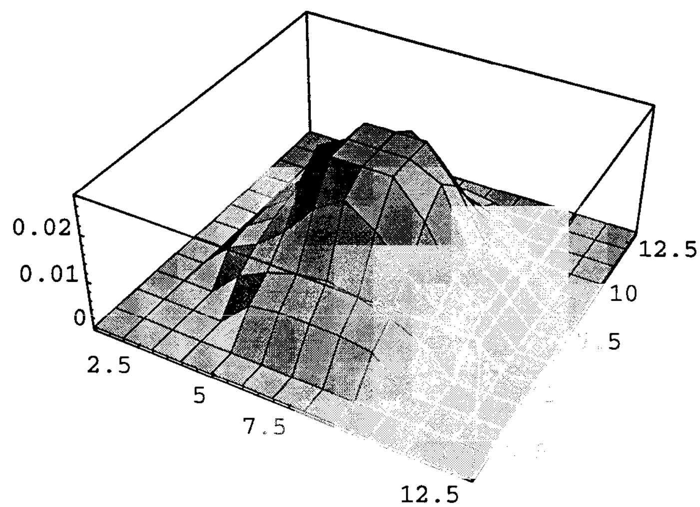
N_perp	3	5	7	9	11	13	15	17	19	21
1	3	8	18	38	36	65	110	185	300	296
	(2)	(4)	(7)	(12)	(19)	(30)	(45)	(67)	(97)	(139)
2	19	70	218	265	590	1120	822	1410	2422	3774
	(10)	(32)	(127)	(119)	(343)	(754)	(453)	(626)	(1285)	(2307)
3	43	222	958	1408	4460	17031	22486	21635	45782	83947
	(22)	(102)	(367)	(736)	(2671)	(9230)	(13213)	(13531)	(25232)	(53411)
4	75	872	3714	9259	49394	50966	110254	328966	671060	1282800
	(38)	(330)	(1399)	(5913)	(32363)	(32124)	(55319)	(172247)	(468491)	(936593)
5	99	2028	13702	54100	95176	386140	1553576			
	(50)	(722)	(5699)	(28065)	(66371)	(232400)	(1038070)			
6	139	3982	35666	126748	536758	2907158				
	(70)	(1548)	(12991)	(69245)	(391511)	(2107688)				
7	195	7734	79794	519325	1317392					
	(98)	(2780)	(32891)	(276299)	(1008539)					
8	275	11736	172118	1165832						
	(138)	(4268)	(61947)	(687394)						


K=7, N_perp=6, Λ² = 100 μ²
0.0
0.1
0.2
0.3
0.4
0.5
f_B
analytic
numerical
0.0
0.2
0.4
0.6
0.8
1.0
y

Analytic two-body wave function,
transverse slice at x=0.286.

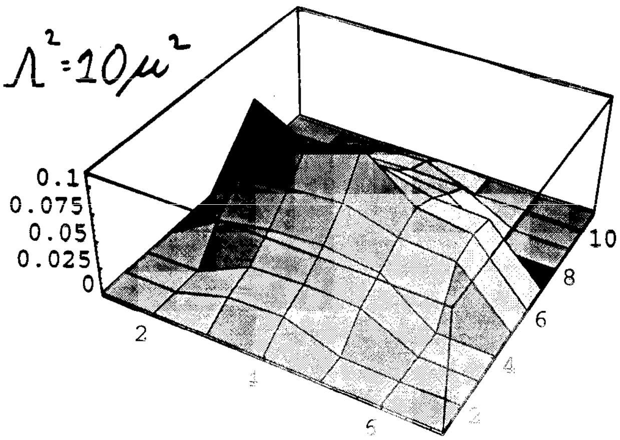


Numerical two-body wave function,
transverse slice at x=0.286.

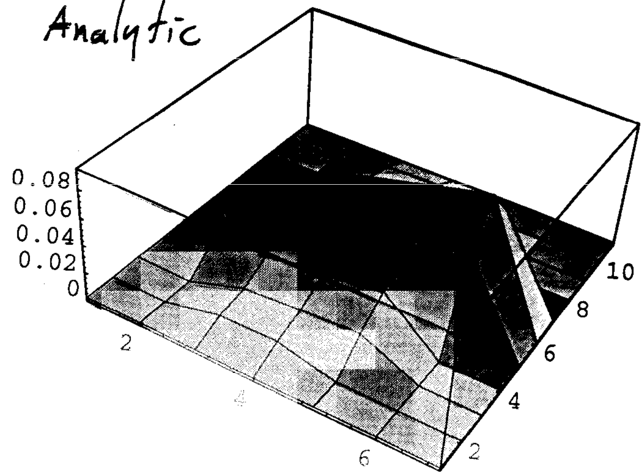


$K = 11,\ N_\perp = 5$

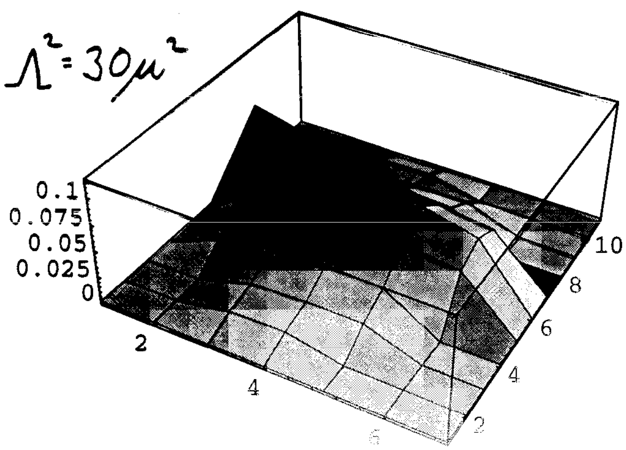
$\Lambda^2 = 10\mu^2$



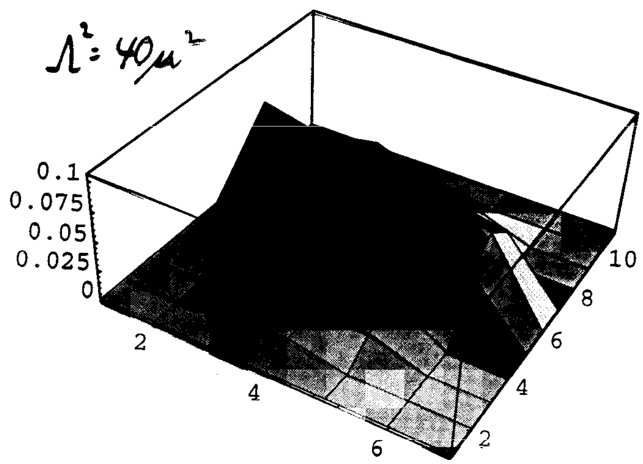
Analytic



$\Lambda^2 = 30\mu^2$



$\Lambda^2 = 40\mu^2$



5 Summary

- the number of PV Fock states may not be prohibitive.
- the numerical accuracy of DLCQ can be significantly improved with weighting factors.
- a simple model exists for the testing of PV regularization.

6 Future work

- extend numerical solution of heavy fermion model.
- reinstate complexity gradually.
- reach Yukawa theory.
- look beyond.