The Running Coupling in a Coupling-Coherent Approach to Light-Front Field Theory

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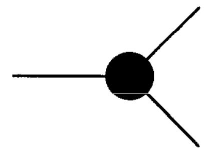
GOAL

- Coupling coherence with a smooth transformation
- Example: $\langle\phi_2\phi_3|\,\mathcal{M}_n^2\,|\phi_1\rangle$ in ϕ^3_{5+1}



OUTLINE

1 - Overview of the Approach

2 - A General Coupling-Coherent Solution

3 - A Second-Order Matrix Element

4 - The Third-Order Matrix Element

  a - The Running Coupling

  b - Noncanonical Operators

5 - Summary

OVERVIEW OF THE APPROACH

- $\mathcal{M}_f^2$ is the free mass-squared operator

$$\mathcal{M}_f^2 \left| K \right\rangle = M_K^2 \left| K \right\rangle$$

$$\Box_{KJ} \equiv M_K^2 - M_J^2$$

- $\mathcal{M}_n^2$ is the mass-squared operator at scale Λ_n.

$$\langle F | \mathcal{M}_n^2 | I \rangle = M_I^2 \langle F | I \rangle + e^{-\Lambda_n^{-4} \Box_{IF}^2} \langle F | \mathcal{M}_{In}^2 | I \rangle$$

- Infinite number of scales

$$\Lambda_0 > \Lambda_1 > \Lambda_2 > \cdots > \Lambda_{n-2} > \Lambda_{n-1} > \Lambda_n > \Lambda_{n+1} > \cdots$$

- Unitary Transformation that lowers the cutoff

$$\mathcal{M}_{n+1}^2 = U(\Lambda_{n+1}, \Lambda_n) \mathcal{M}_n^2 U^\dagger(\Lambda_{n+1}, \Lambda_n)$$

$$\frac{dU(\Lambda_{n+1}, \Lambda_n)}{d\Lambda_{n+1}^{-4}} = \left[\mathcal{M}_f^2, \mathcal{M}_{n+1}^2 \right] U(\Lambda_{n+1}, \Lambda_n)$$

$U(\Lambda_n, \Lambda_n) \equiv 1$ F. Wegner, Ann. Physik **3**, 77 (1994)

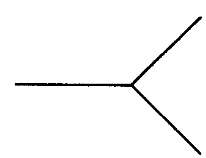
- Perturbative Expansion

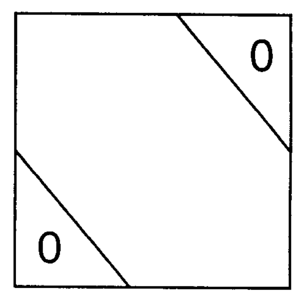
$$\langle F| \mathcal{M}^2_{I(n+1)} |I\rangle = \langle F| \mathcal{M}^2_{In} |I\rangle$$
$$+\frac{1}{2}\sum_K \langle F| \mathcal{M}^2_{In} |K\rangle \langle K| \mathcal{M}^2_{In} |I\rangle D_1(F,K,I) C_1(n,F,K,I)$$
$$+\frac{1}{4}\sum_{K,L} \langle F| \mathcal{M}^2_{In} |K\rangle \langle K| \mathcal{M}^2_{In} |L\rangle \langle L| \mathcal{M}^2_{In} |I\rangle T(n,F,K,L,I)$$
$$+\cdots$$

- Postulate an interaction

$$\mathcal{M}^2_{In} = \hat{\mathcal{P}}^+ (2\pi)^5 g_n \int D_1 D_2 D_3 \Big[a_3^\dagger a_1 a_2 \delta^{(5)}(p_3 - p_1 - p_2) + a_2^\dagger a_3^\dagger a_1 \delta^{(5)}(p_2 + p_3 - p_1) \Big] + \mathcal{O}(g_n^2)$$

$$D_i \equiv \frac{d^4 p_{i\perp} dp_i^+}{64\pi^5 p_i^+}$$





$$g_n \equiv \left. \frac{\langle \phi_2 \phi_3| \mathcal{M}^2_n |\phi_1\rangle \,|_{\vec{p}_{2\perp}=\vec{p}_{3\perp}=0}}{64\pi^5 p_1^+ \delta^{(5)}(p_1 - p_2 - p_3)} \right|_{constant\ part}$$

- Noncanonical operators have strengths governed by coupling coherence

R. J. Perry, Ann. Phys. **232**, 116 (1994)

A GENERAL COUPLING-COHERENT SOLUTION

- Recursion Relation

$$\langle F|\,\mathcal{M}^2_{I(n+1)}\,|I\rangle\,\Big|^{NC}_{\mathcal{O}(g_n^r)} = \langle F|\,\mathcal{M}^2_{In}\,|I\rangle\,\Big|^{NC}_{\mathcal{O}(g_n^r)} + \langle F|\,\Delta_n\,|I\rangle\,\Big|^{NC}_{\mathcal{O}(g_n^r)}$$

- All interactions are approximately transverse-local

$$\langle F|\,\mathcal{M}^2_{In}\,|I\rangle\,\Big|^{NC}_{\mathcal{O}(g_n^r)} = 64\pi^5\mathcal{P}_I^+\delta^{(5)}(\mathcal{P}_I - \mathcal{P}_F)\left[\prod_{i=1}^{N_s} 64\pi^5 s_i^+\delta^{(5)}(s_i - s_i')\right] \times\Lambda_n^{6-2N_{int}} \sum_{\{m_{pc}\}} C^{(n)}_{\{m_{pc}\}} h_{\{m_{pc}\}}\left(\left\{\frac{k_p^+}{\mathcal{P}_I^+}\right\}\right)\prod_{p=1}^{N_{int}}\prod_{c=1}^{4}\left(\frac{k^c_{p\perp}}{\Lambda_n}\right)^{m_{pc}}$$

- $\langle F|\,\Delta_n\,|I\rangle\,\Big|^{NC}_{\mathcal{O}(g_n^r)}$ is restricted to have the form

$$\begin{aligned}\langle F|\,\Delta_n\,|I\rangle\,\Big|^{NC}_{\mathcal{O}(g_n^r)} &= 64\pi^5\mathcal{P}_I^+\delta^{(5)}(\mathcal{P}_I - \mathcal{P}_F)\left[\prod_{i=1}^{N_s} 64\pi^5 s_i^+\delta^{(5)}(s_i - s_i')\right] \\ &\times\; g_n^r \sum_{\{m_{pc}\}} B_{\{m_{pc}\}} h_{\{m_{pc}\}}\left(\left\{\frac{k_p^+}{\mathcal{P}_I^+}\right\}\right) \\ &\times \left[\Lambda_{n+1}^{6-2N_{int}}\prod_{p,c}\left(\frac{k^c_{p\perp}}{\Lambda_{n+1}}\right)^{m_{pc}} - \Lambda_n^{6-2N_{int}}\prod_{p,c}\left(\frac{k^c_{p\perp}}{\Lambda_n}\right)^{m_{pc}}\right]\end{aligned}$$

- Match powers of transverse momentum
- Assume Coupling Coherence

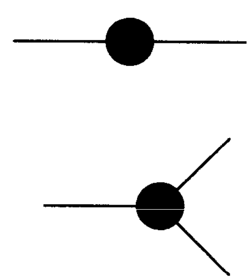
$$C^{(n)}_{\{m_{pc}\}} = \alpha_{\{m_{pc}\}} g_n^r + \mathcal{O}(g_n^{r+1})$$

- This leads to

$$\left(\alpha_{\{m_{pc}\}} - B_{\{m_{pc}\}}\right)\left[\Lambda_{n+1}^{6-2N_{int}} \prod_{p,c} \left(\frac{k^c_{p\perp}}{\Lambda_{n+1}}\right)^{m_{pc}} - \Lambda_n^{6-2N_{int}} \prod_{p,c} \left(\frac{k^c_{p\perp}}{\Lambda_n}\right)^{m_{pc}}\right] = 0$$

- Marginal contributions to $\langle F| \Delta_n |I\rangle \Big|^{NC}_{\mathcal{O}(g_n^r)}$ have

$$6 - 2N_{int} - \sum_{p,c} m_{pc} = 0$$



$\sim k_\perp^2$

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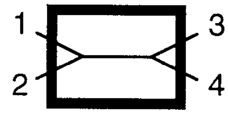
- Nonmarginal contributions to $\langle F| \Delta_n |I\rangle \Big|^{NC}_{\mathcal{O}(g_n^r)}$ have

$$\alpha_{\{m_{pc}\}} = B_{\{m_{pc}\}}$$

• General Coupling-Coherent solution for nonmarginal contributions:

$$\langle F| \mathcal{M}^2_{In} |I\rangle \,\big|_{\mathcal{O}(g_n^r),\, nonmarginal} =$$
$$- \langle F| \Delta_n |I\rangle \,\big|_{\mathcal{O}(g_n^r),\, nonmarginal,\, \Lambda_{n+1}\, terms \to 0}$$

A SECOND-ORDER MATRIX ELEMENT

- As an example, compute 

- Use jacobi variables ($\mathcal{P} \equiv p_1 + p_2$)

$$\begin{aligned}
p_1 &= (x\mathcal{P}^+, x\vec{\mathcal{P}}_\perp + \vec{r}) \\
p_2 &= ([1-x]\mathcal{P}^+, [1-x]\vec{\mathcal{P}}_\perp - \vec{r}) \\
p_3 &= (y\mathcal{P}^+, y\vec{\mathcal{P}}_\perp + \vec{q}) \\
p_4 &= ([1-y]\mathcal{P}^+, [1-y]\vec{\mathcal{P}}_\perp - \vec{q})
\end{aligned}$$

- Define initial and final mass squares

$$u \equiv \frac{r^2}{x(1-x)} \qquad v \equiv \frac{q^2}{y(1-y)}$$

$$\langle \phi_3\phi_4 | \, \Delta_n \, | \phi_1\phi_2 \rangle \Big|_{\mathcal{O}(g_n^2),\, ann} = 64\pi^5 \delta^{(5)}(p_1 + p_2 - p_3 - p_4)$$
$$\times \; \frac{1}{2} g_n^2 \mathcal{P}^+ \left(\frac{1}{u} + \frac{1}{v} \right) \left(e^{-2\Lambda_n^{-4} uv} - e^{-2\Lambda_{n+1}^{-4} uv} \right)$$

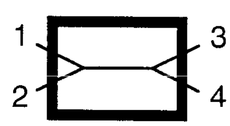
$$\langle \phi_3\phi_4 | \Delta_n | \phi_1\phi_2 \rangle \Big|_{\mathcal{O}(g_n^2),\ ann} = 64\pi^5 \delta^{(5)}(p_1 + p_2 - p_3 - p_4)$$
$$\times \ \frac{1}{2} g_n^2 \mathcal{P}^+ \left(\frac{1}{u} + \frac{1}{v}\right)\left(e^{-2\Lambda_n^{-4} uv} - e^{-2\Lambda_{n+1}^{-4} uv}\right)$$

- Apply the general solution

$$\langle F | \mathcal{M}_{In}^2 | I \rangle \Big|_{\mathcal{O}(g_n^r),\ nonmarginal} =$$
$$- \langle F | \Delta_n | I \rangle \Big|_{\mathcal{O}(g_n^r),\ nonmarginal,\ \Lambda_{n+1}\ terms \to 0}$$

- Leads to the specific solution

$$\langle \phi_3\phi_4 | \mathcal{M}_{In}^2 | \phi_1\phi_2 \rangle \Big|_{\mathcal{O}(g_n^2),\ ann} =$$



$$= \ 64\pi^5 \delta^{(5)}(p_1 + p_2 - p_3 - p_4) \frac{1}{2} g_n^2 \mathcal{P}^+ \left(\frac{1}{u} + \frac{1}{v}\right)\left(1 - e^{-2\Lambda_n^{-4} uv}\right)$$

THE THIRD-ORDER MATRIX ELEMENT

- GOAL: compute $\langle\phi_2\phi_3|\,\mathcal{M}_n^2\,|\phi_1\rangle$ through $\mathcal{O}(g_n^3)$
- The recursion relation:

$$\langle\phi_2\phi_3|\,\mathcal{M}^2_{I(n+1)}\,|\phi_1\rangle = \langle\phi_2\phi_3|\,\mathcal{M}^2_{In}\,|\phi_1\rangle + \langle\phi_2\phi_3|\,\Delta_n\,|\phi_1\rangle$$

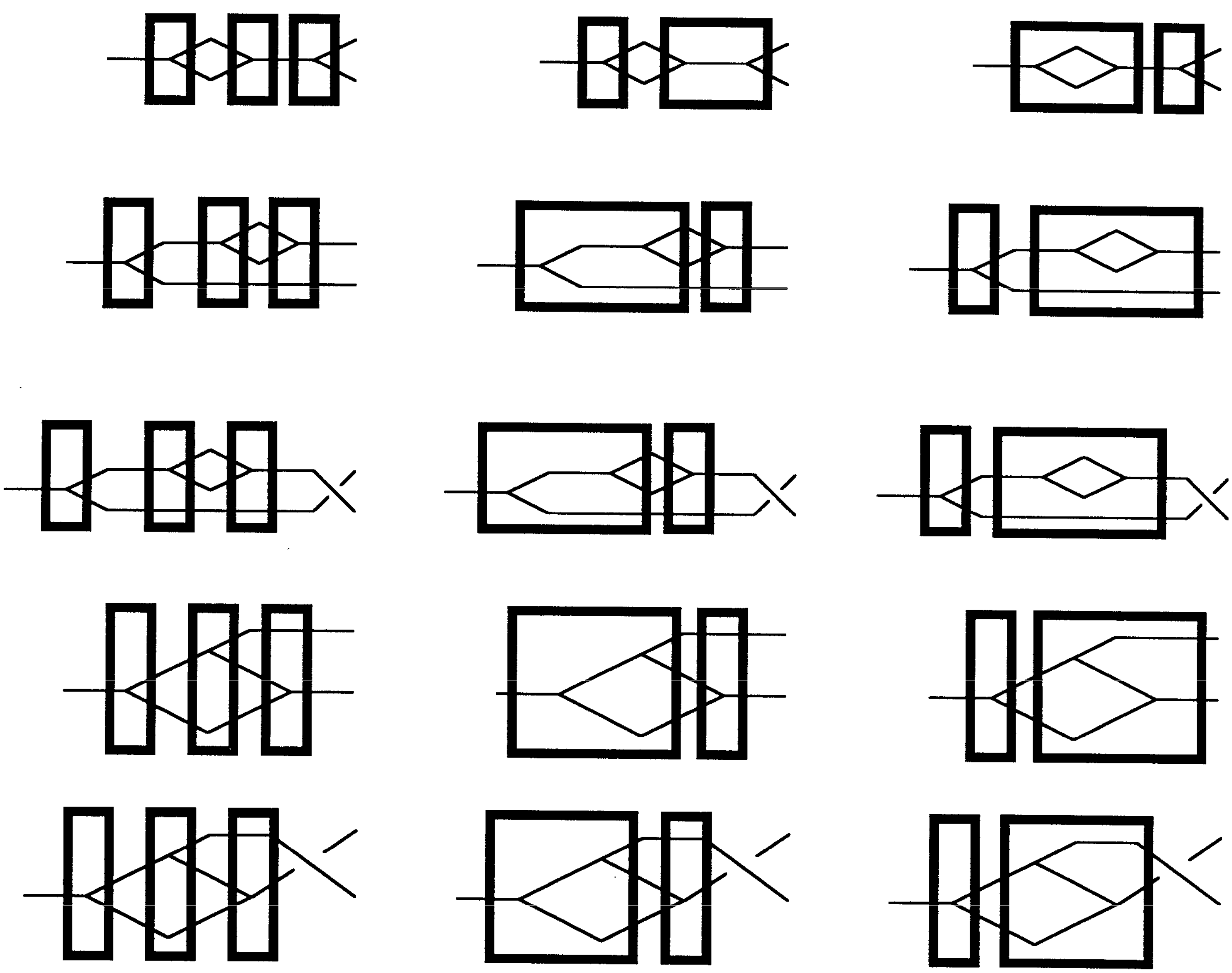
$$\langle\phi_2\phi_3|\,\Delta_n\,|\phi_1\rangle = \frac{1}{2}\sum_K \langle\phi_2\phi_3|\,\mathcal{M}^2_{In}\,|K\rangle\,\langle K|\,\mathcal{M}^2_{In}\,|\phi_1\rangle\,D_1()C_1()$$

$$+\ \frac{1}{4}\sum_{K,L} \langle\phi_2\phi_3|\,\mathcal{M}^2_{In}\,|K\rangle\,\langle K|\,\mathcal{M}^2_{In}\,|L\rangle\,\langle L|\,\mathcal{M}^2_{In}\,|\phi_1\rangle\,T()$$

Third-Order Contributions

to

$<\varphi_2 \varphi_3 | \Delta_n | \varphi_1>$



- The recursion relation:

$$\langle \phi_2 \phi_3 | \mathcal{M}^2_{I(n+1)} | \phi_1 \rangle = \langle \phi_2 \phi_3 | \mathcal{M}^2_{In} | \phi_1 \rangle + \langle \phi_2 \phi_3 | \Delta_n | \phi_1 \rangle$$

- Examine the case $\vec{p}_{1\perp} = \vec{p}_{2\perp} = 0$

$$\begin{aligned} \langle \phi_2 \phi_3 | \mathcal{M}^2_{I(n+1)} | \phi_1 \rangle \Big|_{\vec{p}_{1\perp}=\vec{p}_{2\perp}=0} &= \langle \phi_2 \phi_3 | \mathcal{M}^2_{In} | \phi_1 \rangle \Big|_{\vec{p}_{1\perp}=\vec{p}_{2\perp}=0} \\ &+ \frac{3\pi^2}{8} g_n^3 p_1^+ \delta^{(5)}(p_1 - p_2 - p_3) \log \frac{\Lambda_n^2}{\Lambda_{n+1}^2} \end{aligned}$$

- Use the definition of g_n

$$g_n \equiv \left. \frac{\langle \phi_2 \phi_3 | \mathcal{M}^2_n | \phi_1 \rangle \, |_{\vec{p}_{2\perp}=\vec{p}_{3\perp}=0}}{64\pi^5 p_1^+ \delta^{(5)}(p_1 - p_2 - p_3)} \right|_{constant\ part}$$

- The Running Coupling

$$g_{n+1} = g_n + \frac{3}{512\pi^3} g_n^3 \log \frac{\Lambda_n^2}{\Lambda_{n+1}^2}$$

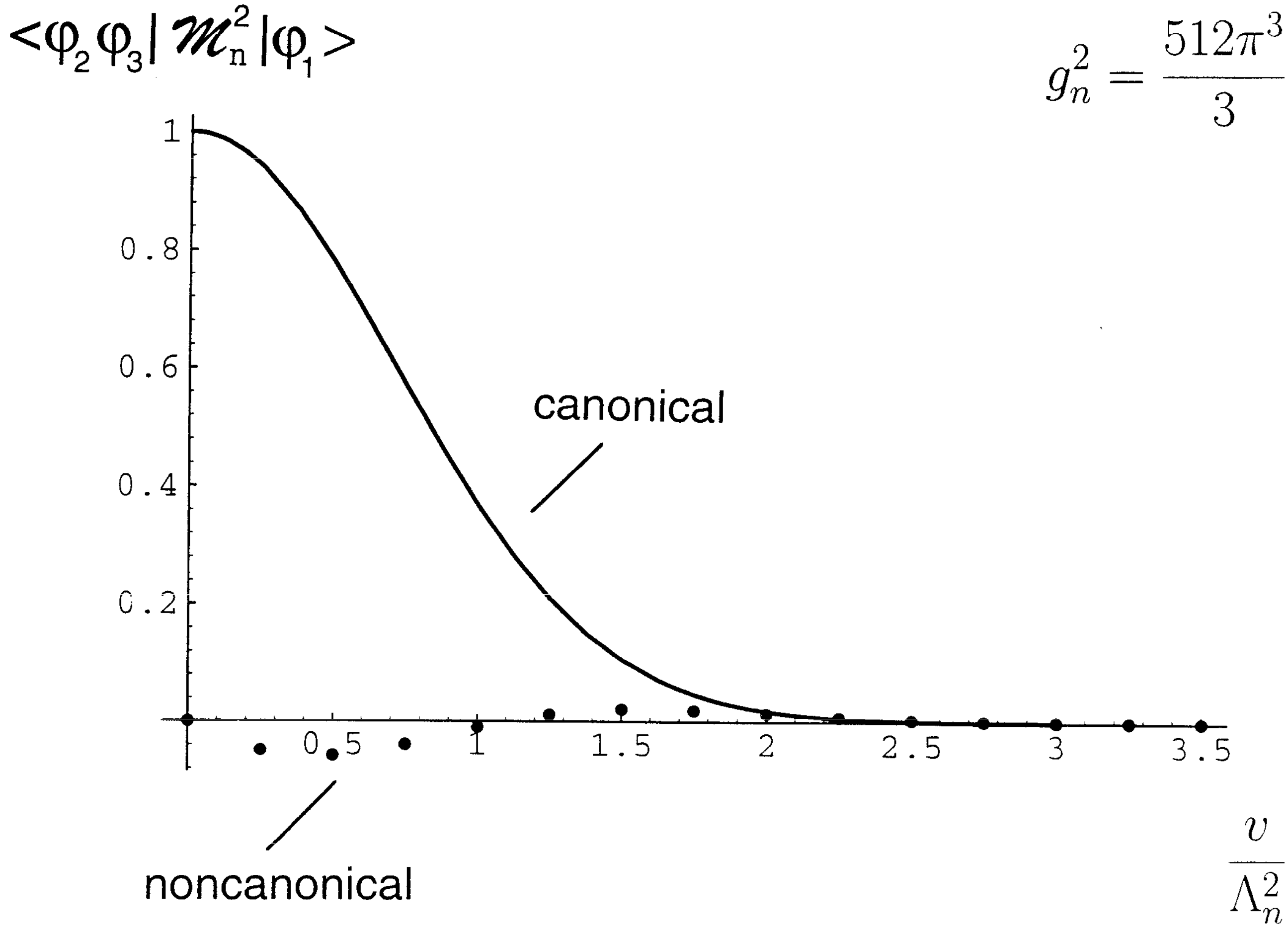
- Now determine the noncanonical operators in $\langle\phi_2\phi_3|\,\mathcal{M}_n^2\,|\phi_1\rangle$

$$\langle\phi_2\phi_3|\,\Delta_n\,|\phi_1\rangle\,\Big|_{1st\ 9\ diagrams} = \frac{\pi^2}{48} g_n^3 p_1^+ \delta^{(5)}(p_1 - p_2 - p_3) \times \int_0^\infty du\,[f(u,v) + 2f(u,-v)]$$

$$f(u,v) = \left[e^{-2\Lambda_n^{-4}u^2} - e^{-2\Lambda_{n+1}^{-4}u^2}\right]\left[\frac{1}{u} + \frac{v}{u^2} + \frac{2}{v}\right] + \left[e^{-2\Lambda_n^{-4}(u^2-uv)} - e^{-2\Lambda_{n+1}^{-4}(u^2-uv)}\right]\left[\frac{1}{v-u} - \frac{1}{u} + \frac{u}{v(v-u)} - \frac{1}{v}\right]$$

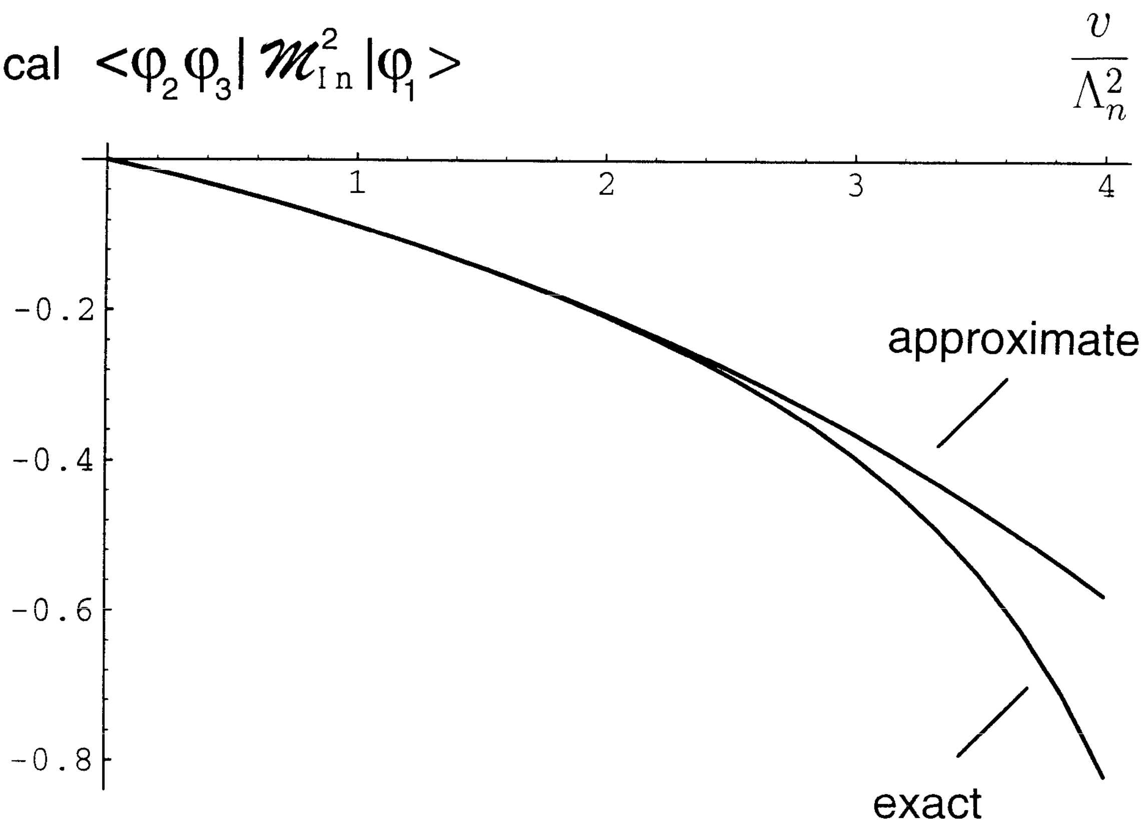
- Apply the general solution

$$\langle\phi_2\phi_3|\,\mathcal{M}_{In}^2\,|\phi_1\rangle\,\Big|_{NM,\ 1st\ 9} = -\,\langle\phi_2\phi_3|\,\Delta_n\,|\phi_1\rangle\,\Big|_{NM,\ 1st\ 9,\ \Lambda_{n+1}\ terms\to 0}$$
$$= -\frac{\pi^2}{48} g_n^3 p_1^+ \delta^{(5)}(p_1 - p_2 - p_3) \times \int_0^\infty du\,[f(u,v) + 2f(u,-v) - 3f(u,0)]\,\Big|_{\Lambda_{n+1}^{-2}\to 0}$$



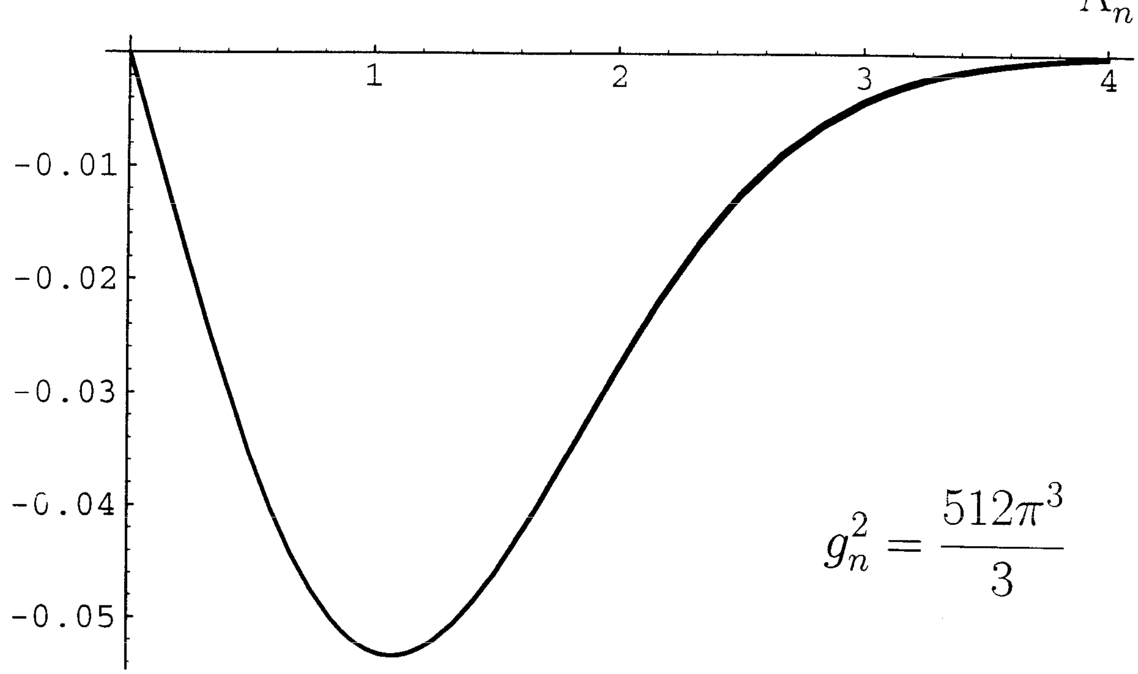
<φ2 φ3| M²n |φ1>
g²n = 512π³/3
1
0.8
0.6
0.4
0.2
canonical
0.5
1
1.5
2
2.5
3
3.5
noncanonical
v/Λ²n

Noncanonical $<\varphi_2 \varphi_3 | \mathcal{M}^2_{\mathrm{In}} | \varphi_1 >$



$$g_n^2 = \frac{512\pi^3}{3}$$

Noncanonical $<\varphi_2 \varphi_3| \mathcal{M}_n^2 |\varphi_1>$



SUMMARY

1 - Developed a general method for coupling coherence with a smooth transformation

2 - Computed second- and third-order matrix elements of $\mathcal{M}_n^2$ in ϕ_{5+1}^3 theory

3 - Showed that the nonmarginal noncanonical contributions to $\langle\phi_2\phi_3|\,\mathcal{M}_n^2\,|\phi_1\rangle$ are small

4 - Verified the assumption of approximate transverse locality

NONCANONICAL MARGINAL OPERATORS

- The general equation for the coupling is

$$h_{n+1} = h_n + a g_n^3 + b g_n^2 h_n + c g_n^5$$

- Use coupling coherence

$$h_n = \alpha g_n^3 + \beta g_n^5 + \mathcal{O}(g_n^7)$$

- Use the running coupling

$$g_{n+1} = g_n + d g_n^3 + \mathcal{O}(g_n^5)$$

- To third order

$$a = 0$$

- To fifth order

$$\alpha = \frac{c}{3d - b}$$