Consistency of the pion form factor and unpolarized TMDs beyond leading twist in the light-front quark model

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   - LF zero mode issue

3. Light-Front Quark Model (LFQM)
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   - Pion Form Factor

4. Unpolarized TMDs of pion

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1. Motivation

- Understanding the **internal structure of hadron** is an important objective in modern nuclear and particle physics.

  - Experimental studies (e.g. JLab, COMPASS, EIC, J-PARC, etc.) are aimed at **probing the 3D structure of hadrons**, particularly focused on Generalized Parton Distributions (GPDs) and Transverse Momentum Dependent Distributions (TMDs).

**For precision 3D imaging of hadrons,** it is essential to measure positions and momenta of the partons transverse to the hadron’s direction of motion.

\[ W(x, k_\perp, b_\perp): \text{Wigner distribution} \]
3D hadron structure from 5D tomography

\[ W(x, k_\perp, b_\perp): \text{Wigner distribution} \]

\[ \int db_\perp \quad \int dk_\perp \]

\[ f(x, k_\perp) \quad f(x, b_\perp) \]

**TMDs**: semi-inclusive processes

**Impact parameter distributions**

\[ \int dk_\perp \quad \int db_\perp \]

\[ f(x) \]

**PDFs**: inclusive and semi-inclusive processes
3D hadron structure from 5D tomography

\[ W(x, k_{\perp}, b_{\perp}) : \text{Wigner distribution} \]

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\[ f(x, k_{\perp}) \]

\[ f(x, b_{\perp}) \]

TMDs: semi-inclusive processes

Impact parameter distributions

\[ \int dk_{\perp} \]

\[ \int db_{\perp} \]

\[ f(x) \]

PDFs: inclusive and semi-inclusive processes

\[ \mathcal{H}(x, 0, t) \]

\[ \mathcal{H}(x, \zeta, t = -\Delta^2) \]

\[ \zeta = 0 \]

\[ \zeta = \frac{(P - P')^+}{P^+} \]

GPDs: exclusive processes

High energy approach

Twist expansion

\[ \zeta = 0 \]

\[ \mathcal{H}(x, \zeta, t = -\Delta^2) \]

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3D hadron structure from 5D tomography

\[ W(x, k_{\perp}, b_{\perp}) : \text{Wigner distribution} \]

\[ \int db_{\perp} \quad \int dk_{\perp} \]

\[ f(x, k_{\perp}) \quad f(x, b_{\perp}) \]

TMDs: semi-inclusive processes

Impact parameter distributions

\[ \int dk_{\perp} \quad \int db_{\perp} \]

\[ f(x) \]

PDFs: inclusive and semi-inclusive processes

\[ \text{FT} \quad b_{\perp} \leftrightarrow \Delta \]

\[ H(x, 0, t) \]

\[ H(x, \zeta, t = -\Delta^2) \]

GPDs: exclusive processes

\[ \zeta = 0 \quad \zeta = \frac{(P - P')^+}{P^+} \]

\[ H(x, 0,0) \]

\[ \int dx \]

\[ F(t) \]

Form factors: elastic scattering

High energy approach

twist expansion

\[ \zeta = (P - P#) \]

\[ \frac{H(x, 0, t)}{P} \]

\[ \frac{\Delta^2}{P^+} \]

\[ \frac{t}{\zeta^2} \]

\[ M^2 \]

\[ M^2_t, M^2_s, m^2, Q_1^2, m^2, \ldots \]
Study the interplay among the pion's Form Factor, TMDs, and PDFs in the LFQM.
Schematic descriptions of form factors, TMDs, and PDFs of the Pion in the LFQM

\[ \langle P' | \bar{q} (0) \gamma^\mu q(0) | P \rangle = \mathcal{D}^\mu F_\pi(q^2) \quad \mathcal{D} \cdot q = 0 \]

\[ FF \quad F^{(\mu)}_{\pi}(Q^2) = \int dx \, d^2 k_\perp \, f^{(\mu)}(x, k_\perp, Q^2), \quad (\mu = +, \perp, -) \]
Schematic descriptions of form factors, TMDs, and PDFs of the Pion in the LFQM

\[ \langle P' | \bar{q}(0) \gamma^\mu q(0) | P \rangle = \xi^\mu F_\pi(q^2) \]
\[ \xi \cdot q = 0 \]

FF

\[ F_\pi^{(\mu)}(Q^2) = \int dx \, d^2 k_\perp \, f^{(\mu)}(x, k_\perp, Q^2), \quad (\mu = +, \perp, -) \]

as \( Q^2 \to 0 \)

\[ F_\pi^{(\mu)}(0) = 1 = \int dx \, d^2 k_\perp \, f^{(\mu)}(x, k_\perp) \text{ TMDs} \quad (\mu = +, -) \]
Schematic descriptions of form factors, TMDs, and PDFs of the Pion in the LFQM

\[
\langle P' | \bar{q} (0) \gamma^\mu q(0) | P \rangle = \varphi^\mu F_\pi(q^2) \quad \varphi \cdot q = 0
\]

**FF**

\[
F_\pi^{(\mu)}(Q^2) = \int dx \int d^2k_\perp f^{(\mu)}(x, k_\perp, Q^2), \quad (\mu = +, \perp, -)
\]

as \( Q^2 \to 0 \)

\[
F_\pi^{(\mu)}(0) = 1 = \int dx \int d^2k_\perp f^{(\mu)}(x, k_\perp) \quad \text{TMDs} \quad (\mu = +, -)
\]

**PDFs**

\[
f^{(\mu)}(x) = \int d^2k_\perp f^{(\mu)}(x, k_\perp)
\]

\( (\mu = +, -) \)

\[c.f.) \mu = \perp \text{ case later...}\]
2. Form Factors on the Light-Front

\[ J^\mu = q_{\text{LF valence}} + q_{\text{LF nonvalence}} \]
2. Form Factors on the Light-Front

\[ J^\mu \rightarrow \text{facilitates the partonic interpretation of the amplitude!} \]
2. Form Factors on the Light-Front

\[ J^\mu \rightarrow \text{partonic interpretation of the amplitude!} \]

\[ q^+ \rightarrow 0 \quad \text{for } J^+ \& J^\perp \]

\[ q^+ \rightarrow 0 \quad \text{for } J^- \]

Nonvanishing : LF Zero-Mode!
We developed a "new method" to obtain the form factor

\[ F_{\pi}^{(+)}(Q^2) = F_{\pi}^{(\perp)}(Q^2) = F_{\pi}^{(-)}(Q^2) \]

within the valence picture of the LFQM.

\[
\langle P'\bar{q}(0)\gamma^\mu q(0)|P\rangle = \rho^\mu F_\pi(q^2)
\]

\[
F(Q^2) = \int [dx][d^2k_\perp] \psi_i^*(x,k'_\perp)\psi_i(x,k_\perp)
\]

for any \( J^\mu = (J^+,J^\perp,J^-) \)

\( q^+ = 0 \)

\( \psi_i(x,k_\perp) \quad \psi_f(x,k'_\perp) \)

\( \uparrow \quad J^\mu \quad \downarrow \)

☞ We resolved the LF zero mode problems from \( J^- \).
3. Light-Front Quark Model (LFQM)

\[ p = (p^+, p^-, p^\perp) \]

\[ p^\pm = p^0 \pm p^3 \]

\[ p^- = p_1^- + p_2^- \]

\[ M \rightarrow M_0 \]

\[ \frac{M_0^2 + P_1^2}{P^+} = \frac{m_1^2 + p_1^2}{p_1^+} + \frac{m_2^2 + p_2^2}{p_2^+} \]

\[ M_0^2 = \frac{m_1^2 + k_\perp^2}{x} + \frac{m_2^2 + k_\perp^2}{1-x} \]

Meson state: Noninteracting "on-mass" shell Q & \( \bar{Q} \) representation.

Invariant mass
3. Light-Front Quark Model (LFQM)

Meson state: Noninteracting "on-mass" shell $Q$ & $\bar{Q}$ representation.

$$\Psi_{\lambda\bar{\lambda}}(x, k_{\perp}) = \phi(x, k_{\perp}) R_{\lambda\bar{\lambda}}(x, k_{\perp})$$

Spin-Orbit for PS meson

$$R_{\lambda q\bar{\lambda}q} = \frac{\bar{u}_{\lambda q}(p_q) \gamma_5 v_{\bar{\lambda}q}(p_{\bar{q}})}{\sqrt{2M_0}}$$


$$\int_0^1 dx \int \frac{d^2k_{\perp}}{16\pi^3} |\phi(x, k_{\perp})|^2 = 1$$

$$\sum_{\lambda'} R^{\dagger} R = 1.$$
3. Light-Front Quark Model (LFQM)

Meson state: Noninteracting "on-mass" shell $Q$ & $\bar{Q}$ representation.

The interaction between $Q\bar{Q}$ is incorporated into the mass operator via $M := M_0 + V_{Q\bar{Q}}$

$H_{Q\bar{Q}} = \sqrt{m_Q^2 + \vec{k}^2} + \sqrt{m_{\bar{Q}}^2 + \vec{\bar{k}}^2} + V_{Q\bar{Q}}$

$V_{Q\bar{Q}} = a + br(br^2) - \frac{4\kappa}{3r} + \frac{2\vec{S}_Q \cdot \vec{\bar{S}}_{\bar{Q}}}{3m_Q m_{\bar{Q}}} \nabla^2 V_{\text{Coul}}$

$H_{Q\bar{Q}} |\Psi\rangle = M_{Q\bar{Q}} |\Psi\rangle$
3. Light-Front Quark Model (LFQM)

Meson state: Noninteracting "on-mass" shell \( Q \) & \( \bar{Q} \) representation.

The interaction between \( Q\bar{Q} \) is incorporated into the mass operator via \( M := M_0 + V_{Q\bar{Q}} \)

\[
H_{Q\bar{Q}} = \sqrt{m_Q^2 + \vec{k}^2} + \sqrt{m_{\bar{Q}}^2 + \vec{k}^2} + V_{Q\bar{Q}} \\
V_{Q\bar{Q}} = a + b r (br^2) - \frac{4\kappa}{3r} + \frac{2\vec{S}_Q \cdot \vec{S}_{\bar{Q}}}{3m_Q m_{\bar{Q}}} \nabla^2 V_{\text{Coul}}
\]

\[ H_{Q\bar{Q}} \left| \Psi \right\rangle = M_{Q\bar{Q}} \left| \Psi \right\rangle \]

Bakamjian-Thomas (BT) construction!
Optimized model parameters (in unit of GeV) and 1S state meson mass spectra

<table>
<thead>
<tr>
<th>Model</th>
<th>$m_q$</th>
<th>$m_s$</th>
<th>$m_c$</th>
<th>$m_b$</th>
<th>$\beta_{qq}$</th>
<th>$\beta_{sq}$</th>
<th>$\beta_{ss}$</th>
<th>$\beta_{qc}$</th>
<th>$\beta_{sc}$</th>
<th>$\beta_{cc}$</th>
<th>$\beta_{qb}$</th>
<th>$\beta_{sb}$</th>
<th>$\beta_{cb}$</th>
<th>$\beta_{bb}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>0.22</td>
<td>0.45</td>
<td>1.8</td>
<td>5.2</td>
<td>0.366</td>
<td>0.389</td>
<td>0.413</td>
<td>0.468</td>
<td>0.502</td>
<td>0.651</td>
<td>0.527</td>
<td>0.571</td>
<td>0.807</td>
<td>1.145</td>
</tr>
<tr>
<td>HO</td>
<td>0.25</td>
<td>0.48</td>
<td>1.8</td>
<td>5.2</td>
<td>0.319</td>
<td>0.342</td>
<td>0.368</td>
<td>0.422</td>
<td>0.469</td>
<td>0.699</td>
<td>0.496</td>
<td>0.574</td>
<td>1.035</td>
<td>1.803</td>
</tr>
</tbody>
</table>

Mass spectroscopy analysis

- PRD59, 074015(99); PLB460, 461(99) by HMC and CRJ
- PRC92, 055203(2015) by HMC, CRJ, Z. Li, and H. Ryu
- PRD 100, 014026(2019) by N. Dhiman, H. Dahiya, HMC, CRJ
- PRD106, 014009(2022) by A. J. Arifi, HMC, CRJ, YO

PRC92, 055203(2015) by HMC, CRJ, Z. Li, and H. Ryu
Analysis of (1S, 2S) state heavy meson spectroscopy

PRD106, 014009(2022) by A. J. Arifi, HMC, CRJ, YO

\[
\begin{align*}
\Phi_{1S} &= \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} \phi_{1S}, \\
\Phi_{2S} &= \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \phi_{2S},
\end{align*}
\]

\[
V_{Q\bar{Q}} = a + br - \frac{4\kappa}{3r} + \frac{2\hat{S}_Q \cdot \hat{S}_{\bar{Q}}}{3m_Qm_{\bar{Q}}} \nabla^2 V_{\text{Coul}}.
\]

\[
\frac{1}{2} \cot^{-1}(2\sqrt{6}) < \theta < \frac{\pi}{4},
\]

constrained by empirical mass gap: \( \Delta M_P > \Delta M_V \)
\[ \langle P' | \bar{q} \Gamma^\mu q | P \rangle = \mathcal{G}^\mu \mathcal{F} \]

\[ \mathcal{F} : \text{Physical observables} \quad \mathcal{G}^\mu : \text{Lorentz factors} \]
BT Construction on Hadronic Matrix Element Calculations

\[ \langle P' | \bar{q} \Gamma^\mu q | P \rangle = \mathcal{O}^\mu \mathcal{F} \]

\( \mathcal{F} \): Physical observables \quad \mathcal{O}^\mu \): Lorentz factors

• Most Previous LFQM: Apply BT \((M \to M_0)\) only to LHS.

\[ \langle P' | \bar{q} \Gamma^\mu q | P \rangle = \mathcal{O}^\mu \mathcal{F} \]

\[ \mathcal{F} = \frac{1}{\mathcal{O}^\mu} \langle P' | \bar{q} \Gamma^\mu q | P \rangle \]
**BT Construction on Hadronic Matrix Element Calculations**

\[ \langle P' | \bar{q} \Gamma^\mu q | P \rangle = \wp^\mu \mathcal{F} \]

\(\mathcal{F}\): Physical observables \(\wp^\mu\): Lorentz factors

- Most Previous LFQM: Apply BT \((M \to M_0)\) only to the matrix element.

\[ \langle P' | \bar{q} \Gamma^\mu q | P \rangle = \wp^\mu \mathcal{F} \]

\[ \mathcal{F} = \frac{1}{\wp^\mu} \langle P' | \bar{q} \Gamma^\mu q | P \rangle \]

Fails to show the independency of \(\mu\) due to the LF zero-modes from the "bad" current.
New development of a “self-consistent” LFQM based on the BT construction.

- In our LFQM: Apply BT \((M \rightarrow M_0)\) equally to both sides

\[
M \rightarrow M_0
\]

\[
\langle P' | \bar{q} \Gamma^\mu q | P \rangle = \delta^\mu \mathcal{F}
\]

\[
\mathcal{F} = \left( P' | \bar{q} \frac{\Gamma^\mu q}{\delta^\mu} | P \right)
\]

becomes independent of the current components!
Pion Form Factor

General structure for $P(P) \to P(P')$ transition:

$$
\langle P'|\bar{q} \gamma^\mu q|P \rangle = \mathcal{\mathfrak{q}}^\mu F(q^2) + q^\mu \frac{(M^2 - M'^2)}{q^2} H(q^2), \quad \mathcal{\mathfrak{q}}^\mu = (P + P')^\mu - q^\mu \frac{(M^2 - M'^2)}{q^2}
$$

$$
q^\mu = (P - P')^\mu
$$
Pion Form Factor

General structure for $P(P) \rightarrow P(P')$ transition:

$$\langle P'|\bar{q} \gamma^\mu q|P \rangle = \mathcal{O}^\mu F(q^2) + q^\mu \frac{(M^2 - M'^2)}{q^2} H(q^2), \quad \mathcal{O}^\mu = (P + P')^\mu - q^\mu \frac{(M^2 - M'^2)}{q^2}$$

$$q^\mu = (P - P')^\mu$$

For elastic process, only gauge invariant form factor $F(q^2)$ survives!

$$\langle P'|\bar{q} \gamma^\mu q|P \rangle = \mathcal{O}^\mu F_P(q^2) \quad \mathcal{O} \cdot q = 0$$
Pion Form Factor

\[ \langle P' | \bar{q} \gamma^\mu q | P \rangle = g^\mu \mathcal{F}_p(q^2), \quad g^\mu = (P + P')^\mu - q^\mu \frac{(M^2 - M'^2)}{q^2} \]

In \( q^+ = 0 \) frame,

\[ \langle P' | \bar{q} \gamma^\mu q | P \rangle = \int_0^1 dp_1^+ \int \frac{d^2 k_\perp}{16\pi^3} \phi'(x, k'_\perp)\phi(x, k_\perp) \sum_{\lambda' s} \mathcal{R}_{\lambda_2 \lambda}^+ \left[ \frac{\bar{u}_{\lambda_2}(p_2)}{\sqrt{x_2}} \gamma^\mu \frac{u_{\lambda_1}(p_1)}{\sqrt{x_1}} \right] \mathcal{R}_{\lambda_1 \bar{\lambda}}. \]

\[ F_p^{(\mu)}(Q^2) = \int_0^1 dp_1^+ \int \frac{d^2 k_\perp}{16\pi^3} \phi'(x, k'_\perp)\phi(x, k_\perp) \frac{1}{g^\mu} \sum_{\lambda' s} \mathcal{R}_{\lambda_2 \lambda}^+ \left[ \frac{\bar{u}_{\lambda_2}(p_2)}{\sqrt{x_2}} \gamma^\mu \frac{u_{\lambda_1}(p_1)}{\sqrt{x_1}} \right] \mathcal{R}_{\lambda_1 \bar{\lambda}}. \]
Pion Form Factor

\[ \langle P' | \bar{q} \gamma^\mu q | P \rangle = \phi^\mu F_P(q^2), \quad \phi^\mu = (P + P')^\mu - q^\mu \frac{(M^2 - M'^2)}{q^2} \]

In \( q^+ = 0 \) frame,

\[ \langle P' | \bar{q} \gamma^\mu q | P \rangle = \int_0^1 dp_1^+ \int \frac{d^2k_\perp}{16\pi^3} \phi'(x, k'_\perp) \phi(x, k_\perp) \sum_{\lambda'\lambda} R_{\lambda_2\lambda}^+ \frac{\bar{u}_{\lambda_2}(p_2)}{\sqrt{x_2}} \gamma^\mu \frac{u_{\lambda_1}(p_1)}{\sqrt{x_1}} R_{\lambda_1\bar{\lambda}}. \]

Apply \( M \to M_0 \) only to

Previous LFQM method, which couldn’t resolve the zero-mode.

\[ F_{P}^{(\mu)}(Q^2) = \int_0^1 dp_1^+ \int \frac{d^2k_\perp}{16\pi^3} \phi'(x, k'_\perp) \phi(x, k_\perp) \frac{1}{\phi^\mu} \sum_{\lambda'\lambda} R_{\lambda_2\lambda}^+ \frac{\bar{u}_{\lambda_2}(p_2)}{\sqrt{x_2}} \gamma^\mu \frac{u_{\lambda_1}(p_1)}{\sqrt{x_1}} R_{\lambda_1\bar{\lambda}}. \]

This leads to \( F_{\pi}^{(+)}(Q^2) = F_{\pi}^{(\perp)}(Q^2) \neq F_{\pi}^{(-)}(Q^2) \)
Pion Form Factor

\[ \langle P' | \bar{q} \gamma^\mu q | P \rangle = \Theta^\mu F_p(q^2), \quad \Theta^\mu = (P + P')^\mu - q^\mu \frac{(M^2 - M'^2)}{q^2} \]

In \( q^+ = 0 \) frame,

\[ \langle P' | \bar{q} \gamma^\mu q | P \rangle = \int_0^1 dp_1^+ \int \frac{d^2k_\perp}{16\pi^3} \phi'(x, k_\perp) \phi(x, k_\perp) \sum_{\lambda' s} R^+_{\lambda_2 \bar{\lambda}} \left[ \bar{u}_{\lambda_2}(p_2) \frac{\gamma^\mu u_{\lambda_1}(p_1)}{\sqrt{x_2} \sqrt{x_1}} \right] R_{\lambda_1 \bar{\lambda}}. \]

Apply \( M \rightarrow M_0 \) both to

Our New method, which now resolves the zero-mode.

\[ F_p^{(\mu)}(Q^2) = \int_0^1 dp_1^+ \int \frac{d^2k_\perp}{16\pi^3} \phi'(x, k_\perp) \phi(x, k_\perp) \frac{1}{\Theta^\mu} \sum_{\lambda' s} R^+_{\lambda_2 \bar{\lambda}} \left[ \bar{u}_{\lambda_2}(p_2) \frac{\gamma^\mu u_{\lambda_1}(p_1)}{\sqrt{x_2} \sqrt{x_1}} \right] R_{\lambda_1 \bar{\lambda}}. \]

Then we get \( F_{\pi}^{(+)}(Q^2) = F_{\pi}^{(\perp)}(Q^2) = F_{\pi}^{(-)}(Q^2) \)
\[ F_{\pi}^{SLF(\mu)}(Q^2) = \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} \frac{\phi(x, k_\perp)\phi'(x, k'_\perp)}{\sqrt{k_\perp^2 + m^2} \sqrt{k'_\perp^2 + m^2}} O^{(\mu)}_{LFQM} \]

TABLE II: The operators \( O^{(\mu)}_{LFQM} \) and their helicity contributions to the pion form factor in the standard LFQM.

<table>
<thead>
<tr>
<th>( F_\pi^{(\mu)} )</th>
<th>( O^{(\mu)}_{LFQM} )</th>
<th>( \mathcal{H}^{(\mu)}_{(\uparrow\rightarrow\uparrow)+(\downarrow\rightarrow\downarrow)} )</th>
<th>( \mathcal{H}^{(\mu)}_{(\uparrow\rightarrow\downarrow)+(\downarrow\rightarrow\uparrow)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_\pi^{(+)} )</td>
<td>( k_\perp \cdot k'_\perp + m^2 )</td>
<td>( k_\perp \cdot k'_\perp + m^2 )</td>
<td>0</td>
</tr>
<tr>
<td>( F_\pi^{(\perp)} )</td>
<td>( k_\perp \cdot k'_\perp + m^2 )</td>
<td>( k_\perp \cdot k'_\perp + m^2 )</td>
<td>0</td>
</tr>
<tr>
<td>( F_\pi^{(-)} )</td>
<td>( 2(1-x)q_\perp^2 M_0^2 (k_\perp \cdot k'<em>\perp + m^2 + q</em>\perp \cdot k'_\perp) )</td>
<td>( 2q_\perp^2 {(k_\perp \cdot k'<em>\perp + m^2)(k</em>\perp^2 + k_\perp \cdot q_\perp + m^2) + (1-x)(k_\perp \times q_\perp)^2} )</td>
<td>( 2q_\perp^2 {(1-x)m^2 q_\perp^2} )</td>
</tr>
</tbody>
</table>

\[ k'_\perp = k_\perp + (1-x)q_\perp. \]

"The first proof of the pion form factor's independence from current components in the LFQM!"
The pion decay constant defined by the local operator with $\pi(\bar{\mu})$ is zero.

In our recent research \cite{Volmer et al.}, we have shown in Ref. \cite{Amendolia et al.} that the final form of the full operator differs from those of Horn et al. and Tadevosyan et al.

For the minus component of the current, if the physical pion mass is used in the Lorentz factor, the contribution to $P_m$ yields identical results, specifically

$$A_m = P_m Q^2 f^Q.$$ 

For unequal quark mass the axial vector current is given by

$$\frac{d}{dQ^2} X_m Q^2 f^Q = \frac{d}{dQ^2} X_m Q^2 f^Q,$$

where $m(Q^2)$ is the quark mass.

We have shown in Ref. \cite{Amendolia et al.} that the DA provides information about the probability amplitudes for unequal quark mass, the pion distribution amplitude (DA) can be obtained from the DA.

The right panel of Fig. 3 showing the full result (solid line), independent of the current components. The dashed and dotted lines represent the valence and helicity flip contributions from $J^\pm$. The final result of the independent manner. To provide a comprehensive understanding, we obtain the pseudoscalar meson decay constant within our standard LFQM in a process-independent and current component-independent framework.

$$f^L_{\pi} = 130 \text{ MeV}$$

(Exp.$=131 \text{ MeV}$)

$$r^L_{\pi} = 0.654 \text{ fm}$$

(Exp.$=0.659(4) \text{ fm}$)

$$F_{\pi}^{(+)}(Q^2) = F_{\pi}^{(-)}(Q^2) = F_{\pi}^{(-)}(Q^2)$$
4. Unpolarized TMDs of pion \[ C. \text{Lorce}, B. \text{Pasquini}, P. \text{Schweitzer} \quad \text{EPJC}76,415(2016) \]

\[
\int \frac{[dz]}{2(2\pi)^3} e^{ip\cdot z} \langle P|\bar{\psi}(0)\gamma^+\psi(z)|P\rangle|_{z^+ = 0} = f_1^q(x, p_T),
\]

\[
\int \frac{[dz]}{2(2\pi)^3} e^{ip\cdot z} \langle P|\bar{\psi}(0)\gamma_T^j\psi(z)|P\rangle|_{z^+ = 0} = \frac{p_T}{p^+} f_3^q(x, p_T),
\]

\[
\int \frac{[dz]}{2(2\pi)^3} e^{ip\cdot z} \langle P|\bar{\psi}(0)\gamma^{-}\psi(z)|P\rangle|_{z^+ = 0} = \left(\frac{m_\pi}{p^+}\right)^2 f_4^q(x, p_T),
\]

which are related with the forward matrix elements \( \langle P|\bar{q} \gamma^\mu q|P\rangle \) as

\[
2p^+ \int dx f_1^q(x) = \langle P|\bar{\psi}(0)\gamma^+\psi(0)|P\rangle,
\]

\[
2p_T \int dx f_3^q(x) = \langle P|\bar{\psi}(0)\gamma_T^j\psi(0)|P\rangle,
\]

\[
4p^- \int dx f_4^q(x) = \langle P|\bar{\psi}(0)\gamma^-\psi(0)|P\rangle,
\]

\[
\begin{align*}
\int \frac{[dz]}{2(2\pi)^3} e^{ip \cdot z} \langle P | \bar{\psi}(0) \gamma^+ \psi(z) | P \rangle |_{z^+ = 0} = f_1^q(x, p_T), \\
\int \frac{[dz]}{2(2\pi)^3} e^{ip \cdot z} \langle P | \bar{\psi}(0) \gamma_T^j \psi(z) | P \rangle |_{z^+ = 0} = \frac{p_T^j}{p_T} f_3^q(x, p_T), \\
\int \frac{[dz]}{2(2\pi)^3} e^{ip \cdot z} \langle P | \bar{\psi}(0) \gamma^- \psi(z) | P \rangle |_{z^+ = 0} = \left( \frac{m_\pi}{p_T} \right)^2 f_4^q(x, p_T),
\end{align*}
\]

which are related with the forward matrix elements \( \langle P | \bar{q} \gamma^\mu q | P \rangle \) as

\[
\begin{align*}
2P^+ \int dx f_1^q(x) &= \langle P | \bar{\psi}(0) \gamma^+ \psi(0) | P \rangle, \\
2p_T \int dx f_3^q(x) &= \langle P | \bar{\psi}(0) \gamma_T^j \psi(0) | P \rangle, \\
4P^- \int dx f_4^q(x) &= \langle P | \bar{\psi}(0) \gamma^- \psi(0) | P \rangle, \\
f(x) &= \int d^2 p_T f(x, p_T).
\end{align*}
\]
4. Unpolarized TMDs of pion

C. Lorce, B. Pasquini, P. Schweitzer  EPJC76,415(2016)

\[ 2P^+ \int dx f_1^q(x) = \langle P|\bar{\psi}(0)\gamma^+\psi(0)|P\rangle, \]
\[ 2p_T \int dx f_3^q(x) = \langle P|\bar{\psi}(0)\gamma^\perp\psi(0)|P\rangle, \]
\[ 4P^- \int dx f_4^q(x) = \langle P|\bar{\psi}(0)\gamma^-\psi(0)|P\rangle, \]

**Sum rules :**

In free quark model:

\[ \int dx \ f_1^q(x) = 1 \quad x f_3^q(x, p_T) = f_1^q(x, p_T) \]
\[ 2 \int dx \ f_4^q(x) = 1 \]

\[ 2P^+ \int dx f_1^q(x) = \langle P|\bar{\psi}(0)\gamma^+\psi(0)|P \rangle, \]
\[ 2p_T \int dx f_3^q(x) = \langle P|\bar{\psi}(0)\gamma^T\psi(0)|P \rangle, \]
\[ 4P^- \int dx f_4^q(x) = \langle P|\bar{\psi}(0)\gamma^-\psi(0)|P \rangle, \]

Sum rules :

\[ \int dx f_1^q(x) = 1 \]
\[ xf_3^q(x, p_T) = f_1^q(x, p_T) \]
\[ 2 \int dx f_4^q(x) = 1 \]

However...

\[ 2 \int dx f_4^q(x) \neq 1 \]

Fails to meet the sum rule due to the absence of a zero mode!
Derivation of Sum rules

C. Lorce, B. Pasquini, P. Schweitzer

\[ \langle P' | J^\mu | P \rangle = \bar{\phi}^\mu F_\pi(q^2) \]

\[ \bar{\phi}^\mu = (P + P')^\mu \]

\[ 1 = F_\pi(Q^2 = 0) = \lim_{Q \to 0} \frac{\langle P' | J^\mu | P \rangle}{\bar{\phi}^\mu} \]
Derivation of Sum rules

C. Lorce, B. Pasquini, P. Schweitzer

EPJC76,415(2016)

\[ \langle P' | J^\mu | P \rangle = \phi^{\mu} F_{\pi} (q^2) \]

\[ \phi^{\mu} = (P + P')^{\mu} \]

\[ 1 = F_{\pi} (Q^2 = 0) = \lim_{Q \to 0} \frac{\langle P' | J^\mu | P \rangle}{\phi^{\mu}} \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^+ | P \rangle}{\phi^{+}} = \int dx \ f_1^q (x) \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^- | P \rangle}{\phi^{-}} = 2 \int dx \ f_4^q (x) \]
Derivation of Sum rules

C. Lorce, B. Pasquini, P. Schweitzer

EPJC76,415(2016)

\[ \langle P' | J^\mu | P \rangle = \overline{\psi} \mu F_\pi (q^2) \]

\[ \overline{\psi} \mu = (P + P')^\mu \]

\[ 1 = F_\pi (Q^2 = 0) = \lim_{Q \to 0} \frac{\langle P' | J^\mu | P \rangle}{\overline{\psi} \mu} \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^+ | P \rangle}{\overline{\psi}^+} = \int dx \ f_1^q (x) = 1 \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^- | P \rangle}{\overline{\psi}^-} = 2 \int dx \ f_4^q (x) \neq 1 \]
**Derivation of Sum rules**

C. Lorce, B. Pasquini, P. Schweitzer

EPJC76,415(2016)

\[ \langle P' | J^\mu | P \rangle = \bar{\phi}^\mu F_\pi (q^2) \]

\[ \bar{\phi}^\mu = (P + P')^\mu \]

1 = \( F_\pi (Q^2 = 0) = \lim_{Q \to 0} \frac{\langle P' | J^\mu | P \rangle}{\bar{\phi}^\mu} \)

\[ \lim_{Q \to 0} \frac{\langle P' | J^+ | P \rangle}{\bar{\phi}^+} = \int dx \ f_1^q (x) = 1 \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^- | P \rangle}{\bar{\phi}^-} = 2 \int dx \ f_4^q (x) \neq 1 \]

This Work

\[ \langle P' | J^\mu | P \rangle = \phi^\mu F_\pi (q^2) \]

\[ \phi^\mu = (P + P')^\mu - q^\mu \frac{(M^2 - M'^2)}{q^2} \]

1 = \( F_\pi (Q^2 = 0) = \lim_{Q \to 0} \left( \langle P' | \frac{J^\mu}{\phi^\mu} | P \rangle \right) \)
Derivation of Sum rules

This Work

\[ \langle P' | J^\mu | P \rangle = \varphi^\mu F_\pi(q^2) \]

\[ \varphi^\mu = (P + P')^\mu \]

\[ 1 = F_\pi(Q^2 = 0) = \lim_{Q \to 0} \frac{\langle P' | J^\mu | P \rangle}{\varphi^\mu} \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^+ | P \rangle}{\varphi^+} = \int dx \ f_1^q(x) = 1 \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^- | P \rangle}{\varphi^-} = 2 \int dx \ f_4^q(x) \neq 1 \]

\[ \varphi = (P + P')^\mu - q^\mu \frac{(M^2 - M'^2)}{q^2} \]

\[ 1 = F_\pi(Q^2 = 0) = \lim_{Q \to 0} \left( \langle P' | J^\mu | P \rangle \right) \]

\[ \lim_{Q \to 0} \left( \langle P' | J^+ | P \rangle \right) = \int dx \ f_1^q(x) \]

\[ \lim_{Q \to 0} \left( \langle P' | J^- | P \rangle \right) = 2 \int dx \ f_4^q(x) \]
Derivation of Sum rules

\[ \langle P' | J^\mu | P \rangle = \bar{\mathcal{O}}^\mu F_\pi(q^2) \]

\[ \bar{\mathcal{O}}^\mu = (P + P')^\mu \]

\[ 1 = F_\pi(Q^2 = 0) = \lim_{Q \to 0} \frac{\langle P' | J^\mu | P \rangle}{\bar{\mathcal{O}}^\mu} \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^+ | P \rangle}{\bar{\mathcal{O}}^+} = \int dx \: f_1^q(x) = 1 \]

\[ \lim_{Q \to 0} \frac{\langle P' | J^- | P \rangle}{\bar{\mathcal{O}}^-} = 2 \int dx \: f_4^q(x) \neq 1 \]

This Work

\[ \langle P' | J^\mu | P \rangle = \mathcal{O}^\mu F_\pi(q^2) \]

\[ \mathcal{O}^\mu = (P + P')^\mu - q^\mu \frac{(M^2 - M'^2)}{q^2} \]

\[ 1 = F_\pi(Q^2 = 0) = \lim_{Q \to 0} \left( \langle P' | \frac{J^\mu}{\mathcal{O}^\mu} | P \rangle \right) \]

\[ \lim_{Q \to 0} \left( \frac{\langle P' | J^+ | P \rangle}{\mathcal{O}^+} \right) = \int dx \: f_1^q(x) = 1 \]

\[ \lim_{Q \to 0} \left( \frac{\langle P' | J^- | P \rangle}{\mathcal{O}^-} \right) = 2 \int dx \: f_4^q(x) = 1 \]
**LF Zero-Mode for twist-4 PDF and its Resolution**

\[ m = 220 \text{ MeV} \]

\[ 2 \int dx \, f_4^q(x) = 1 \]

Zero mode is well taken!

\[ \lim_{Q \to 0} \left\langle P' | \frac{J^-}{\phi^-} | P \right\rangle = 2 \int dx \, f_4^q(x) = 1 \]
Unpolarized TMDs for Pion

\[ f_1^q(x, k_T^2) \]

\[ f_3^q(x, k_T^2) \]

\[ f_4^q(x, k_T^2) \]

\[ x f_1^q(x, k_T^2) \]

\[ x f_3^q(x, k_T^2) \]

\[ x f_4^q(x, k_T^2) \]

\[ x f_3^q(x, k_T^2) = f_1^q(x, k_T^2) \]
Unpolarized PDFs for Pion

\[ \int d^2 k_\perp \] at the initial scale \( \mu_0 = 1 \text{ GeV} \)
4. QCD Evolution of Pion PDFs

Evolved from $\mu_0^2 = 1$ GeV$^2$ to $\mu^2 = 4$ and 27 GeV$^2$

We use the **Higher Order Perturbative Parton Evolution toolkit (HOPPET)** to solve the NNLO DGLAP equation.
Mellin moments: $\langle x^n \rangle = \int_0^1 dx \ x^n f(x)$

| TABLE III. Mellin moments of the pion valence PDF, $f_i^q(x)$, evaluated at the scale $\mu^2 = 4 \text{ GeV}^2$. |
|---|---|---|---|
| $\langle x^u \rangle_{i2}$ | $\langle x^2 u \rangle_{i2}$ | $\langle x^3 u \rangle_{i2}$ | $\langle x^4 u \rangle_{i2}$ |
| This work | 0.236 | 0.101 | 0.055 | 0.033 |
| [83] | 0.2541(26) | 0.094(12) | 0.057(4) | 0.015(12) |
| [84] | 0.2075(106) | 0.163(33) | ... | ... |
| [56] | 0.24(2) | 0.098(10) | 0.049(7) | ... |
| [57] | 0.24(2) | 0.094(13) | 0.047(8) | ... |

[83] B. Joo et al., PRD 100, 114512(2019)
[56] M. Ding et al. PRD 101, 054014(2020)
[57] Z.-F. Cui et al. EPJC 80, 1064(2020)

| TABLE IV. Mellin moments of the pion valence PDF, $f_i^q(x)$, evaluated at the scale $\mu^2 = 27 \text{ GeV}^2$. |
|---|---|---|---|---|
| $\langle x^u \rangle_{i2}$ | $\langle x^2 u \rangle_{i2}$ | $\langle x^3 u \rangle_{i2}$ | $\langle x^4 u \rangle_{i2}$ |
| This work | 0.182 | 0.069 | 0.034 | 0.019 |
| [85] | 0.18(3) | 0.064(10) | 0.030(5) | ... |
| [57] | 0.20(2) | 0.074(10) | 0.035(6) | ... |
| [86] | 0.184 | 0.068 | 0.033 | 0.018 |
| [21] | 0.217(11) | 0.087(5) | 0.045(3) | ... |

In Fig. 3, we compare the lowest four moments of the valence quark PDF for the pion. The kinematics of the pion-nucleus-induced DY process determine the initial scale of the kaon PDF, which is in good agreement with the global fit to the data. The gray error band corresponds to the sum of relative uncertainties for each moment. Phenomenological models, such as BLFQ-NJL, give comparable agreement with results from other experimental data, such as from CERN-NA3 and FNAL-E615 experiments. In the lower panel, we illustrate the cross section in the upper panel, and we show these results at different scales.

Adopted from J. Lan et al. (BLFQ Collab.), PRL 122, 172001 (2019)
Mellin moments: $\langle x^n \rangle = \int_0^1 dx \ x^n f(x)$

<table>
<thead>
<tr>
<th>$\langle x \rangle_{t3}^u$</th>
<th>$\langle x^2 \rangle_{t3}^u$</th>
<th>$\langle x^3 \rangle_{t3}^u$</th>
<th>$\langle x^4 \rangle_{t3}^u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^2 = 4 \text{ GeV}^2$</td>
<td>0.471</td>
<td>0.164</td>
<td>0.079</td>
</tr>
<tr>
<td>$\mu^2 = 27 \text{ GeV}^2$</td>
<td>0.365</td>
<td>0.111</td>
<td>0.049</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\langle x \rangle_{t4}^u$</th>
<th>$\langle x^2 \rangle_{t4}^u$</th>
<th>$\langle x^3 \rangle_{t4}^u$</th>
<th>$\langle x^4 \rangle_{t4}^u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^2 = 4 \text{ GeV}^2$</td>
<td>0.069</td>
<td>0.021</td>
<td>0.009</td>
</tr>
<tr>
<td>$\mu^2 = 27 \text{ GeV}^2$</td>
<td>0.053</td>
<td>0.014</td>
<td>0.006</td>
</tr>
</tbody>
</table>
5. Conclusions

• We developed a new method for ensuring self-consistency in the LFQM.

Our LFQM: Noninteracting $Q$ & $\bar{Q}$ representation consistent with the Bakamjian-Thomas (BT) construction!

\[ P^- = p_q^- + p_q^- \text{, i.e. } M^2 \rightarrow M^2_0 \]

\[ \langle 0 | q \Gamma^\mu q | P \rangle = \mathcal{F} \varphi^\mu \]

\( \mathcal{F} \): physical observables (\( \mathcal{F} = f_p, F \cdots \))

\( \varphi^\mu \): Lorentz factors (\( \varphi = P^\mu \cdots \))

\[ \mathcal{F} = \left( 0 \left| \frac{\bar{q} \Gamma^\mu q}{\varphi^\mu} \right| P \right) = \int \int dx \ d^2 k_\perp \cdots \left( \frac{\Gamma^\mu}{\varphi^\mu} \right) \cdots \]

Constrained by BT construction!

This allows one to obtain the physical observables independent of the current components!
Partial Extractions of TMD, PDF, GPD from Pion Form Factor

Form factor: \( F^{(\mu)}(t) \equiv \int \int dx \, dk_\perp \, f^{(\mu)}(x, k_\perp, t) \quad \text{Note) } Q^2 \to -t \)

\( f^{(\mu)}(x, k_\perp, t \to 0) \to \int d\mathbf{k}_\perp \)

\( f_1^q(x, k_\perp) \leftrightarrow f^{(+)}(x, k_\perp, 0) \)

\( 2f_4^q(x, k_\perp) \leftrightarrow f^{(-)}(x, k_\perp, 0) \)

\( H(x, 0, t) = \int d\mathbf{k}_\perp \, f^{(+)}(x, k_\perp, t) \quad \text{GPD at } \zeta = 0 \)

GPD at \( \zeta = 0 \)

PDFs

\( f_1^q(x) : \text{twist-2 PDF} \)

\( f_4^q(x) : \text{twist-4 PDF} \)

\( H(x, 0, 0) \)
Backup Slides
2. Why Light-Front?

**Instant form** \((x^0 = ct = 0)\)

\[
a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}
\]

**Front form** \((x^+ = x^0 + x^3 = 0)\)

\[
a \cdot b = \frac{1}{2}(a^+ b^- + a^- b^+) - \vec{a}_\perp \cdot \vec{b}_\perp
\]

<table>
<thead>
<tr>
<th>Hamiltonian</th>
<th>(p^0)</th>
<th>(p^- = p^0 - p^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Momentum</td>
<td>(p_\perp = (p^1, p^2))</td>
<td>(p_\perp = p^0 + p^3)</td>
</tr>
<tr>
<td>Energy-Momentum Dispersion</td>
<td>(p^0 = \sqrt{M^2 + \vec{p}^2})</td>
<td>(p^- = \frac{M^2 + p^2_{\perp}}{p^+})</td>
</tr>
</tbody>
</table>

Irrational vs. Rational
• Advantage of LFD in the calculation of Form Factors:
  Equal-\(t\) vs Equal Light-front \(\tau\) formulations

Equal \(t\) (Instant form)
• Advantage of LFD in the calculation of Form Factors:
  Equal-\(t\) vs Equal Light-front \(\tau\) formulations

\[ k^0 = \sqrt{m^2 + \vec{k}^2} \]

Equal \(t\) (Instant form)

\[ \vec{k}_1 \quad \vec{k}_2 \quad \text{Allowed!} \]

\[ \vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \]
• Advantage of LFD in the calculation of Form Factors:
  \textit{Equal-}t \textit{ vs Equal Light-front }\tau \textit{ formulations}

\begin{center}
\begin{align*}
\text{Equal }\tau \text{ (Front form)}
\end{align*}
\end{center}
• Advantage of LFD in the calculation of Form Factors:
  Equal-\( t \) vs Equal Light-front \( \tau \) formulations

$$k = \frac{m^2 + k_{\perp}^2}{k^+}$$

Equal \( \tau \) (Front form)

\[
\tau = t + \frac{z}{c} = k_1^+ + k_2^+ + k_3^+= 0
\]
• Advantage of LFD in the calculation of Form Factors:

**Equal-\( t \) vs Equal Light-front \( \tau \) formulations**

Equal \( \tau \) (Front form)

\[ = \]

LF nonvalence
(higher Fock state)

LF valence
Decay Constant and DAs

\[ \langle 0|\bar{q}(0)\gamma^\mu\gamma_5q(0)|P\rangle = i f_\pi P^\mu \]

BT

\[ f_\pi = \left\langle 0|\frac{\bar{q}\gamma^\mu\gamma_5q}{P^\mu}|P\right\rangle \]

with \( M \rightarrow M_0 \)
Decay Constant and DAs

\[ \langle 0 | \bar{q}(0) \gamma^\mu \gamma_5 q(0) | P \rangle = i f_\pi P^\mu \]

\[ f_\pi = \left\langle 0 | \frac{\bar{q}(0) \gamma^\mu \gamma_5 q(0)}{P^\mu} | P \right\rangle \]

with \( M \rightarrow M_0 \)

\[ f_\pi = \sqrt{2N_c} \int_0^1 dx \int \frac{d^2 k_\perp}{16\pi^3} \frac{\phi(x, k_\perp)}{\sqrt{m^2 + k_\perp^2}} (2m) \]

\[ \phi_\pi(x, \mu_0) = \frac{\sqrt{2N_c}}{f_\pi} \int \frac{\mu_0^2 d^2 k_\perp}{16\pi^3} \frac{\phi(x, k_\perp)}{\sqrt{m^2 + k_\perp^2}} (2m) \]

independent of "\( \mu \)!"
Decay Constant and DAs

\[ \langle 0 | \bar{q}(0) \gamma^\mu \gamma_5 q(0) | P \rangle = i f_\pi P^\mu \]

with \( M \rightarrow M_0 \)

\[ f_\pi = \frac{1}{\sqrt{2N_c}} \int_0^1 dx \int \frac{d^2 k_\perp}{16\pi^3} \frac{\phi(x, k_\perp)}{\sqrt{m^2 + k_\perp^2}} (2m) \]

\[ \phi_\pi(x, \mu_0) = \frac{1}{f_\pi} \int \frac{d^2 k_\perp}{16\pi^3} \phi(x, k_\perp) \frac{\phi(x, k_\perp)}{\sqrt{m^2 + k_\perp^2}} (2m) \]

independent of “\( \mu \)”!

See PRD 107, 053003(23); PRD108, 013006(23) by A. Arifi, HMC and CRJ for the analysis of higher-twist DAs.