Effective quarks and gluons forming hadrons in the Fock space

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ILCAC Seminar, March 20th, 2024

Junta de Andalucía
Consejería de Universidad, Investigación e Innovación
Unión Europea
Fondo Europeo de Desarrollo Regional
"Una manera de hacer Europa"
Andalucía se mueve con Europa

Motivation

1. Hadrons in terms of quarks and gluons and their dynamics

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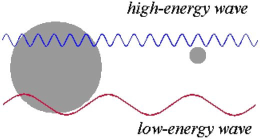
2. QCD at different scales: from asymptotic freedom to confinement



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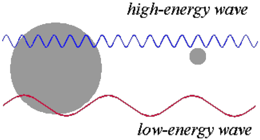


3. Hamiltonian of QCD is complicated
   → It needs regularization
   → How to deal with an infinite number of particles??
   → Starting from QCD, derive an effective Hamiltonian

Hadronic states in QCD

Every state in QCD is a superposition of infinitely many Fock components

$$|\Psi_{\text{meson}}\rangle = c_1|q\bar{q}\rangle + c_2|q\bar{q}g\rangle + c_3|q\bar{q}gg\rangle + c_4|q\bar{q}q\bar{q}gg\rangle + \dots$$

$$P_{|q\bar{q}\rangle} = |\langle q\bar{q}|\Psi_{\text{meson}}\rangle|^2 = c_1^2$$

Masses are given by the eigenvalues of Hamiltonian operator acting on the Fock space

$$H_{QCD}|\Psi_{\text{meson}}\rangle = E_n|\Psi_{\text{meson}}\rangle$$

E_n are the energy levels of the system, i.e. **masses of hadrons**

Problem:

In practice it is not feasible!

The method of calculation

Renormalization Group Procedure for Effective Particles

Main ingredients of the method

Hamiltonian approach

Connection with non-rel. physics: Schrödinger equation
Space of operators
Compatible with quantum simulations

Front-form dynamics

Connection with high-energy physics
Helps with vacuum problem
Helps with changes for reference frames

Renormalization GROUP

Connection between low- and high-energy physics
Reduce d.o.f.: define *effective particles*

The method of calculation

Originally Similarity Renormalization Group
[S.D. Głazek, K.G. Wilson, PRD49, PRD57]

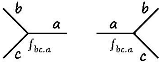
Lagrangian density of QCD $\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu}$

1. **Canonical Hamiltonian** Use front-form dynamics:
   - $\mathcal{L}_{\mathrm{QCD}} \to \mathcal{T}^{\mu\nu}_{\mathrm{QCD}} \to H_{\mathrm{QCD}} = \int_{\mathrm{x}^+=0} \mathcal{H}_{\mathrm{QCD}}(\mathrm{x})d\mathrm{x}, \quad A^+ = 0$

     $k^+ = k^0 + k^3, \quad k^- = k^0 - k^3, \quad \vec{k}^\perp = (k^1, k^2); \quad x = k^+/P^+,$

   x
2. **Regularization** Interaction vertices and counterterms:
   - UV- and small-x divergences

$$\mathcal{M}^2_{bc} = \frac{\kappa^2_{bc} + m^2}{x_b x_c} \qquad f_{bc.a,t_r} = e^{-t_r\left(\mathcal{M}^2_{bc} - m^2\right)^2}$$



3. **Renormalization**
   Scale parameter t introduced by RGPEP

$$H_0 \quad \to \quad H_t \ = \ U_t H_0 U_t^\dagger$$

The method of calculation: RGPEP

Renormalization group procedure for effective particles

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Change of basis through a similarity transformation

$$H_t = U_t H_0 U_t^\dagger \,, \qquad U_t|\psi_0\rangle = |\psi_t\rangle \,, \quad H_0|\psi_0\rangle = H_t|\psi_t\rangle = E|\psi_0\rangle$$

U_t depends on an *energy-scale* parameter t, and preserves the eigenvalues

H_t satisfies

$$\frac{dH_t}{dt} = [G_t, H_t], \text{ where } G_t \text{ is a generator}$$

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Generator:

$$U_t = T \exp\left(-\int_0^t d\tau \, G_\tau\right) \ , \qquad G_t = [\mathcal{H}_f, \mathcal{H}_{Pt}]$$

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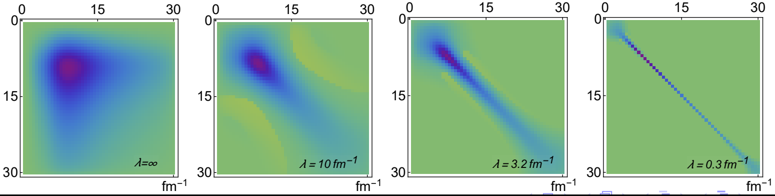
Example: [MGR, Arriola, PLB 800 (2020), APP Sup.14 (2021)]

Evolution of a model potential $V(p, p')$, $\quad t = 1/\lambda^4$



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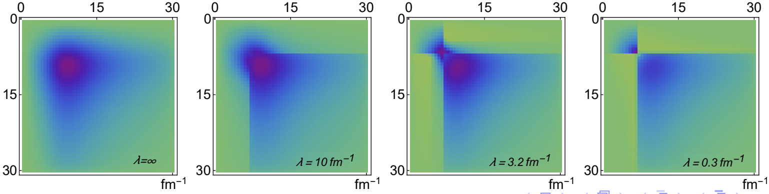


The method of calculation

Effective quanta

Creation and annihilation operators become *effective*

$$a^\dagger|0\rangle = |g\rangle \quad \to \quad a^\dagger_\lambda|0\rangle = |g_\lambda\rangle \qquad a^\dagger_\lambda = U_\lambda \, a^\dagger \, U^\dagger_\lambda$$

They create/annihilate particles of type $\lambda = t^{-1/4}$ or of *size* $s = 1/\lambda$

$[\lambda]$ = energy $\sim$ scale $\qquad [s = 1/\lambda]$ = length $\sim$ size $\qquad t = s^4$

Hadrons in terms effective particles

Hadrons can be described in terms of effective particles of type λ

$$|\Psi_{\text{meson}\,\lambda}\rangle = c_{1,\lambda}|q\bar{q}\rangle_\lambda + c_{2,\lambda}|q\bar{q}g\rangle_\lambda + c_{3,\lambda}|q\bar{q}gg\rangle_\lambda + c_{4,\lambda}|q\bar{q}q\bar{q}gg\rangle_\lambda + \ldots$$

and describe from asymptotic freedom to bound states

The method of calculation

Effective quanta

Creation and annihilation operators become *effective*

$$a^\dagger|0\rangle = |g\rangle \quad \to \quad a_s^\dagger|0\rangle = |g_s\rangle \qquad a_s^\dagger = U_s\, a^\dagger\, U_s^\dagger$$

They create/annihilate particles of type λ or of *size* $s = 1/\lambda$

$$\boxed{[\lambda] = \text{energy} \;\sim \text{scale} \qquad [s = 1/\lambda] = \text{length} \;\sim \text{size} \qquad t = s^4}$$

Hadrons in terms effective particles

Hadrons can be described in terms of effective particles of size s

$$|\Psi_{\text{meson }s}\rangle = c_{1,s}|q\bar{q}\rangle_s + c_{2,s}|q\bar{q}g\rangle_s + c_{3,s}|q\bar{q}gg\rangle_s + c_{4,s}|q\bar{q}q\bar{q}gg\rangle_s + \ldots$$

and describe from asymptotic freedom to bound states

RGPEP

Solve the RGPEP equation perturbatively

$$H'_\lambda \quad = \quad [[H_f, H_{P\lambda}], H_\lambda] \qquad a_\lambda = U_\lambda \, a \, U_\lambda^\dagger$$

perturbatively, order by order

$$H_\lambda \quad = \quad H_f + gH_{1,\lambda} + g^2 H_{2,\lambda} + g^3 H_{3,\lambda} + ...$$

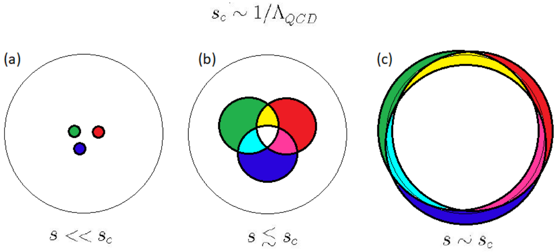
$$\begin{aligned}
\mathcal{H}'_f &= 0 \,, \\
g\mathcal{H}'_{\lambda\,1} &= [[\mathcal{H}_f, g\mathcal{H}_{1P\lambda}], \mathcal{H}_f] \,, \\
g^2\mathcal{H}'_{\lambda\,2} &= [[\mathcal{H}_f, g^2\mathcal{H}_{2P\lambda}], \mathcal{H}_f] + [[\mathcal{H}_f, g\mathcal{H}_{1P\lambda}], g\mathcal{H}_{1\lambda}] \,, \\
g^3\mathcal{H}'_{\lambda\,3} &= [[\mathcal{H}_f, g^3\mathcal{H}_{3P\lambda}], \mathcal{H}_f] + [[\mathcal{H}_f, g^2\mathcal{H}_{2P\lambda}], g\mathcal{H}_{1\lambda}] + [[\mathcal{H}_f, g\mathcal{H}_{1Pt\lambda}], g^2\mathcal{H}_{2\lambda}] \,, \\
&\vdots
\end{aligned}$$

→ Integration produces functions with *form factors*

$$e^{-(\mathcal{M}_a^2 - \mathcal{M}_b^2)^2/\lambda^4}$$

The concept of effective particles

Effective particles of type λ can change their relative motion kinetic energy through a single effective interaction by no more than about λ

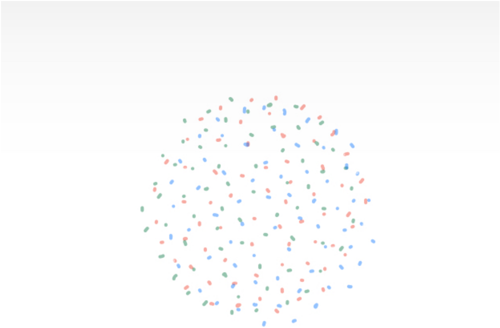


$s = 1/\lambda$

$f_\lambda = e^{-(\mathcal{M}_1^2 - \mathcal{M}_2^2)^2/\lambda^4}$

Figure adapted from Patryk Kubiczek

The concept of effective particles



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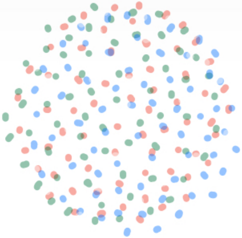

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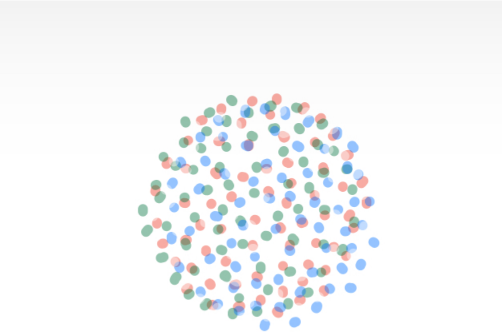

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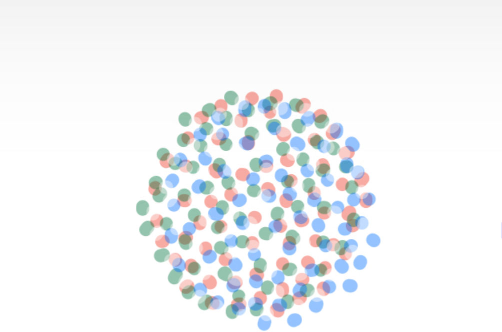

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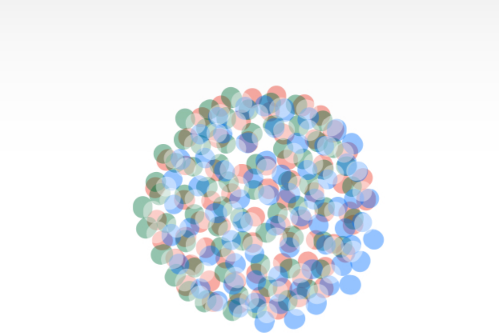

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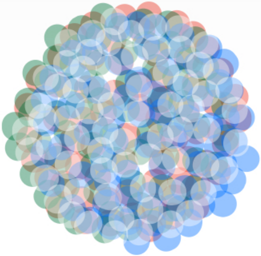

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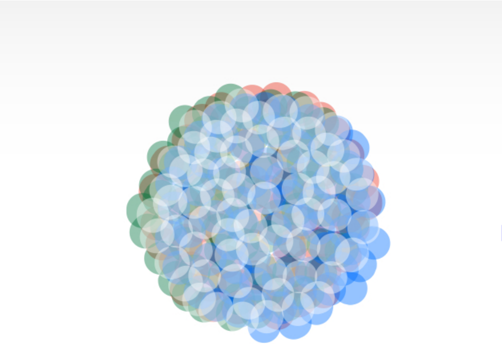

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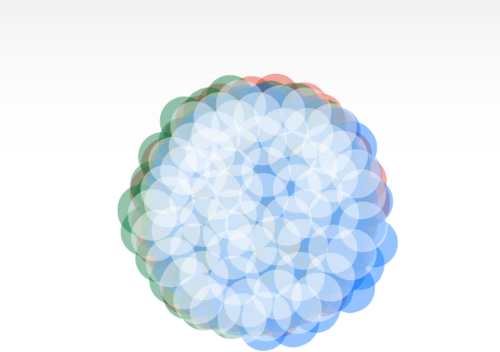

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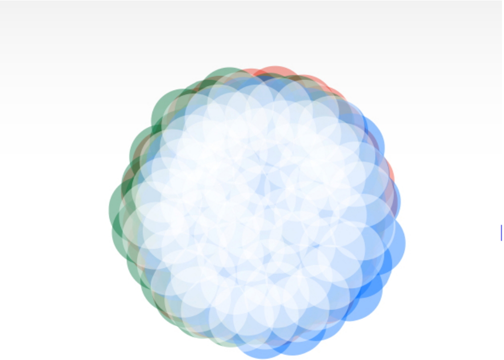

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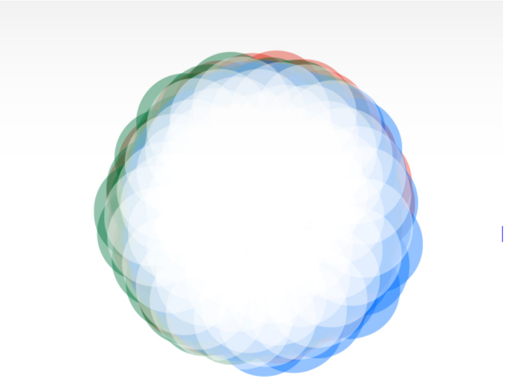

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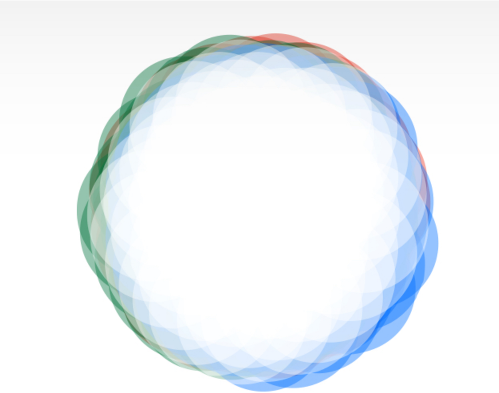

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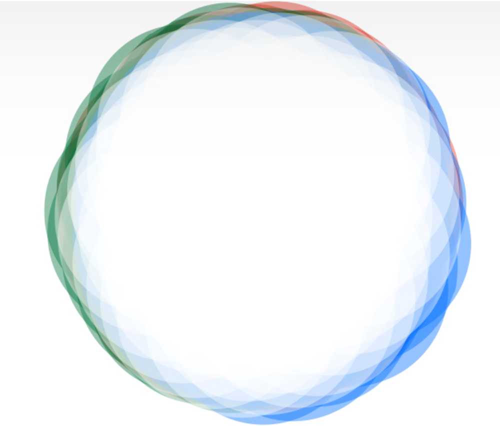


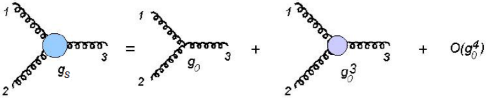
Asymptotic freedom

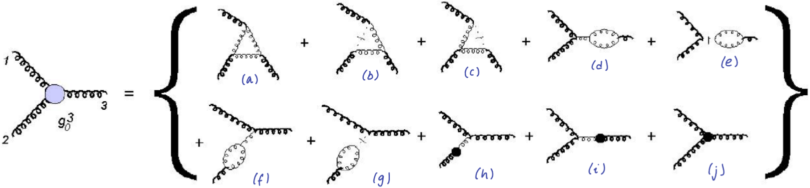
Example of third-order calculation

Example of 3rd-order calculation:

→ **The three-gluon vertex:**

$$Y_\lambda = gH_{1\lambda} + g^3 H_{3\lambda}$$





$$Y_\lambda = \sum_{123} \int [123]\, \tilde{Y}_\lambda(k_1, k_2, k_3, \sigma)\, a^\dagger_{1,\lambda} a^\dagger_{2,\lambda} a_{3,\lambda} + H.c.$$

→ We obtain the running coupling with the correct AF behavior in $\kappa_{12} \to 0$:

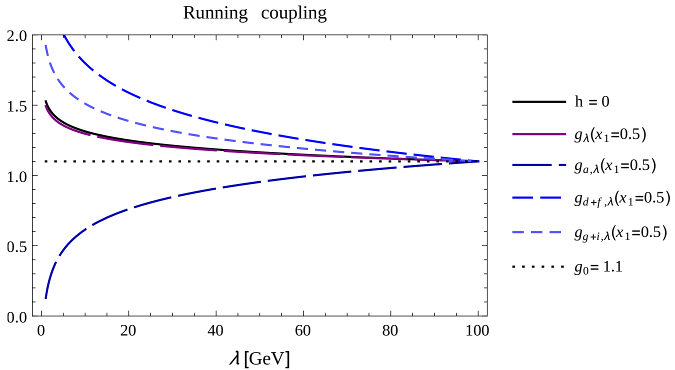
$$\Rightarrow g_\lambda = g_0 - \frac{g_0^3}{48\pi^2} N_c\, 11 \ln \frac{\lambda}{\lambda_0} \,,$$

[MGR, Głazek, PRD 92], [Galvez-Viruet, MGR, PRD 108 (2023)]

Example of 3rd-order calculation:

The running coupling

Cutoff dependences cancel in $m_g \to 0$
even when every contribution diverge in this limit



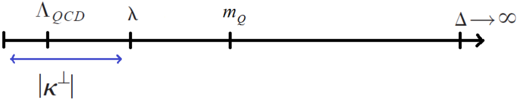
[Galvez-Viruet, MGR, PRD 108 (2023)]

Bound states

Example of second-order calculation

Effective theory for heavy quarks

Assumptions



- ⋆ QCD with only quarks of heavy mass m_b (4.18 GeV/c^2), m_c (1.5 GeV/c^2)
- ⋆ No $Q\bar{Q}$ pair production (too heavy)
- ⋆ 2nd-order perturbative RGPEP:

$$H_{QCD\,\lambda} = H_f + gH_{1,\lambda} + g^2 H_{2,\lambda} \qquad |\Psi_\lambda\rangle = |Q_\lambda \bar{Q}_\lambda\rangle + |Q_\lambda \bar{Q}_\lambda\, g_\lambda\rangle$$

- ⋆ $k/m_Q \to 0$ simplifies the equations

Structure of the eigenvalue problem

Gluon-mass ansatz

$$\left[\begin{array}{ccc} \ldots & \ldots & \ldots \\ \ldots & H_f + g^2 H_{2,\lambda} & g H_{1,\lambda} \\ \ldots & g H_{1,\lambda} & H_f + g^2 H_{2,\lambda} \end{array}\right] \left[\begin{array}{l} \ldots \\ |Q_\lambda \bar{Q}_\lambda G_\lambda\rangle \\ |Q_\lambda \bar{Q}_\lambda\rangle \end{array}\right] = E \left[\begin{array}{l} \ldots \\ |Q_\lambda \bar{Q}_\lambda G_\lambda\rangle \\ |Q_\lambda \bar{Q}_\lambda\rangle \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc} H_f + g^2 H_2 + m_G^2 & g H_1 \\ g H_1 & H_f + g^2 H_2 \end{array}\right] \left[\begin{array}{c} |Q\bar{Q}G\rangle \\ |Q\bar{Q}\rangle \end{array}\right] = E \left[\begin{array}{c} |Q\bar{Q}G\rangle \\ |Q\bar{Q}\rangle \end{array}\right]$$

Reduction to the $|Q_\lambda \bar{Q}_\lambda\rangle$ component

We follow [Wilson PRD 2 (1970) 1438]

$$H_{Q\bar{Q}\,\mathrm{eff},\lambda} |Q_\lambda \bar{Q}_\lambda\rangle = E |Q_\lambda \bar{Q}_\lambda\rangle$$

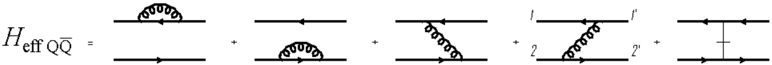
Remember: $m_g \neq m_G$

m_g: canonical gluon mass ($m_g > 0,\ m_g \to 0$)

m_G: Gluon-mass ansatz ($m_G \neq 0$)

The effective eigenvalue equation

$$H_{Q\bar{Q}\,\mathrm{eff}\,\lambda}|Q_\lambda\bar{Q}_\lambda\rangle = E|Q_\lambda\bar{Q}_\lambda\rangle$$



mass terms gluon exch. terms inst. int.

$$|Q_t\bar{Q}_t\rangle \quad = \quad \int [1'2']\, P^+\, \tilde{\delta}(P - k_1' - k_2')\, \psi_{1'2'}(\kappa^{\perp}_{1'2'}, x_{1'})\, \frac{\delta_{c_{1'}c_{2'}}}{\sqrt{3}}\, b^{\dagger}_{1',\lambda} d^{\dagger}_{2',\lambda}|0\rangle$$

mass terms $\rightarrow$ $+$ logarithmic divergence
potential terms $\rightarrow$: $-$ logarithmic divergence

The effective eigenvalue equation

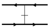
The eigenvalue equation in the NR limit (in the limit $m_g \to \infty$) is

$$\left[\frac{|\vec{k}_{12}|^2}{m} - B + \frac{\delta m_t^2}{2m} + \frac{\delta m_t^2}{2m}\right] \psi_{12}(\kappa_{12}^{\perp}, x_1) + \int \frac{d^3\vec{k}_{1'2'}}{(2\pi)^3} V_{Q\bar{Q}}(\vec{k}_{12} - \vec{k}_{1'2'})\, \psi_{1'2'}(\kappa_{1'2'}^{\perp}, x_{2'}) = 0$$

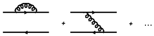
where

$$V_{Q\bar{Q}}(\vec{q}) \quad = \quad V_{C,BF}(\vec{q}) + W(\vec{q})$$

with



$$\Rightarrow\; V_{C,BF}(\vec{q}) \quad = \quad -\frac{4}{3}\frac{4\pi\alpha}{|\vec{q}|^2} \;+\; BF$$



$$\Rightarrow\; W(\vec{q}) \quad = \quad \frac{4}{3}\, 4\pi\alpha \left[\frac{1}{\vec{q}^2} - \frac{1}{q_z^2}\right] \frac{m_G^2}{m_G^2 + \vec{q}^2}\, e^{-2m^2 \frac{|\vec{q}^2|^2}{q_z^2\,\lambda^4}}$$

Remarks: If $m_G^2 = 0$, $W = 0 \Rightarrow$ QED
If m_G^2 large $\Rightarrow$ the possible divergence $\vec{q}^2 \to \infty$ cancels

Remarks

- ⋆ Eigenvalue equation for a single particle

$$H_{\mathrm{eff}Q}|Q\rangle = \infty|Q\rangle$$

$$H_{\mathrm{eff}\bar{Q}}|\bar{Q}\rangle = \infty|\bar{Q}\rangle$$

- ⋆ The divergence is canceled when the quarks are bound
  $\rightarrow$ result compatible with *confinement*

The effective eigenvalue equation

Coulomb + Harmonic Oscillator

$$\left[\frac{\vec{k}^2}{m} - B\right]\psi(\vec{k}) + \int \frac{d^3q}{(2\pi)^3} V_{C,\,BF}(\vec{q})\,\psi(\vec{k}-\vec{q}) - \frac{4}{3}\frac{\alpha}{2\pi} b^{-3} \sum_i \tau_i \frac{\partial^2}{dk_i^2}\psi(\vec{k}) = 0$$

$$b = \frac{\sqrt{2m}}{\lambda_0^2}$$

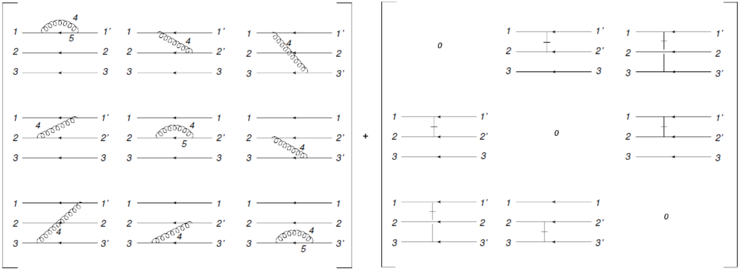
The gluon-mass Ansatz term yields an additional interaction:
harmonic oscillator

Position space

$$\left[2m - \frac{\Delta_{\vec{r}}}{m} - \frac{4}{3}\alpha\left(\frac{1}{r} + BF\right) + \frac{1}{2}\,\tilde{\kappa}\,r^2\right]\psi(\vec{r}) \quad = \quad (2m+B)\,\psi(\vec{r}) \;=\; M\,\psi(\vec{r})\;.$$

[Serafin, Gomez-Rocha, More, Głazek, EPJ C78 (2018)],
and [2023, in preparation]

The effective eigenvalue equation. Baryons

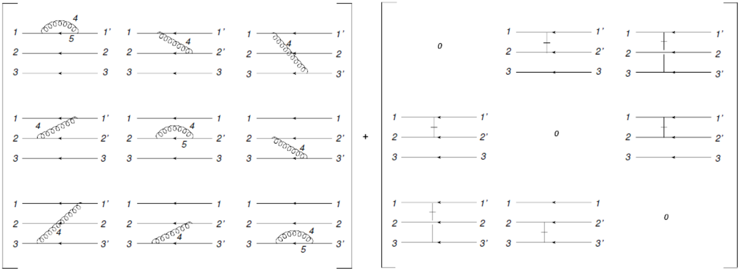


Eigenvalue equation for three quarks

$$H_{\text{eff}\,t}|3Q_t\rangle = \frac{M^2 + (P^{\perp})^2}{P^+}|3Q_t\rangle$$

$$|3Q_t\rangle = \int_{123} P^+ \tilde{\delta}_{P.123}\, \psi_t(123)\, \frac{\epsilon^{c_1 c_2 c_3}}{\sqrt{6}} b^{\dagger}_{t\,1} b^{\dagger}_{t\,2} b^{\dagger}_{t\,3}|0\rangle$$

The effective eigenvalue equation. Baryons



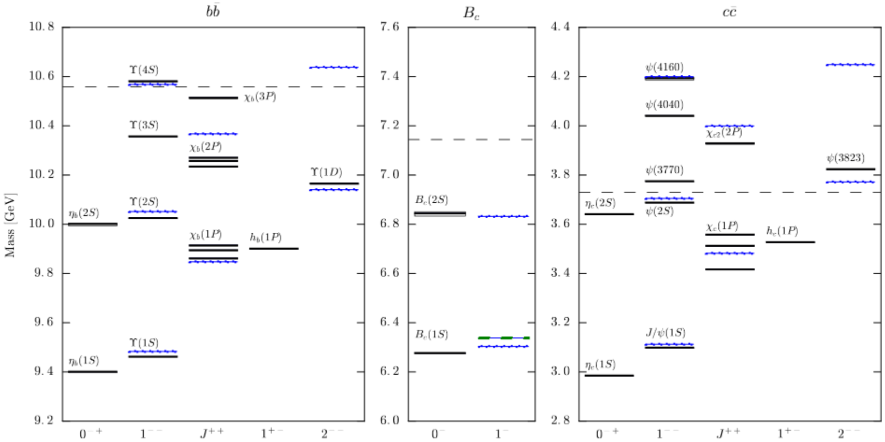
Harmonic oscillator term

$$\int \frac{d^3K'_{12}}{(2\pi)^3} W^{12}\left[\psi(1'2'3) - \psi(123)\right] \approx -w^n \frac{\partial^2\psi(123)}{\partial(K^n_{12})^2}$$

$$\left(\frac{\partial}{\partial \mathbf{K}_{12}}\right)^2 + \left(\frac{\partial}{\partial \mathbf{K}_{23}}\right)^2 + \left(\frac{\partial}{\partial \mathbf{K}_{31}}\right)^2 = \frac{3}{2}\left(\frac{\partial}{\partial \mathbf{K}_{12}}\right)^2 + 2\left(\frac{\partial}{\partial \mathbf{Q}_3}\right)^2$$

$$\omega_{\text{baryon}} = \frac{\sqrt{3}}{2}\sqrt{\frac{\alpha}{18\sqrt{2\pi}}}\,\frac{\lambda^3}{m^2}$$

Some numerical results: heavy mesons

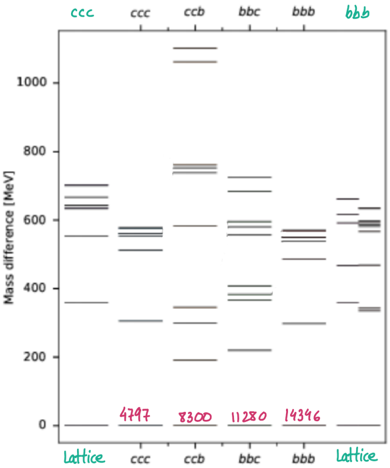


[Serafin, Gomez-Rocha, More, Głazek, EPJ C78 (2018)]

Black: PDG masses, **Blue:** Our calculation

Green: average of many different approaches [MGR, Hilger, Krassnigg, PRD 93 (2016)]

Some numerical results: baryons



Lattice ccc: Padmanath et al.
PRD 90 (2014)

Lattice bbb: Mainel, PRD 85 (2012)
with/without spin int.

The ground state differs from lattice
in 0'6% for ccc
and 0'2% for bbb

Summary and Conclusions

1. RGPEP is a Hamiltonian approach to QCD that connects phenomena at different energy regimes
2. Individual-term divergences cancel each other in physical problems within the formalism
3. Effective potential for quarkonium acquires a simple form in terms of effective particles
4. Even in this crude approximation $\rightarrow$ reasonable spectra

Other applications

For example

- **Scattering:** $\pi\pi$, NN
  MGR, Arriola, PLB 800 (2020), PRD 101 (2020)
- **Tetraquarks:** K. Serafin *et al.* PRD 105 (2022)
- **Proton** Structure in Collisions
  S.D. Glazek, P. Kubiczek [Few Body Syst. 57 (2016) 7, 509-513]
- **Structure functions** for heavy hadrons:
  K. Serafin, PhD Thesis (Warsaw U. 2019).

Acknowledgment

Thank you for your attention