On the definition of spatial densities in composite systems

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Outline

- Stating the problem;
- The charge density in ZAMF;
- The charge density in moving frames;
- Spatial densities corresponding to EMT in ZAMF;
- Spatial densities corresponding to EMT in moving frames;
- Summary;

Talk based on:

E.Epelbaum, J.Gegelia, N.Lange., U.-G.Meißner., M.V.Polyakov,
Phys. Rev. Lett. **129**, 012001 (2022), [arXiv:2201.02565 [hep-ph]].

J.Y.Panteleeva, E.Epelbaum, J.Gegelia, U.-G.Meißner,
[arXiv:2211.09596 [hep-ph]].

Generalization to spin-1/2 and spin-3/2 systems (not covered in talk):

J.Y.Panteleeva, E.Epelbaum, J.Gegelia, U.-G.Meißner,
Phys. Rev. D **106**, no.5, 056019 (2022), [arXiv:2205.15061 [hep-ph]].

C. E. Carlson, "Comments and extensions of a recent suggestion for a relativistic charge density definition," [arXiv:2208.00826 [hep-ph]].

H.Alharazin, B.-D.Sun, E.Epelbaum, J.Gegelia, U.-G.Meißner,
JHEP **02**, 163 (2023), [arXiv:2212.11505 [hep-ph]].

Stating the problem

Charge density of a nucleon is traditionally defined as the three-dimensional Fourier transform of its electric FF in the Breit frame.

R. Hofstadter, F. Bumiller, and M. R. Yearian, Rev. Mod. Phys. **30**, 482 (1958).
F. J. Ernst, R. G. Sachs and K. C. Wali, Phys. Rev. **119**, 1105-1114 (1960).
R. G. Sachs, Phys. Rev. **126**, 2256-2260 (1962).

Similar relations have been suggested for Fourier transforms of gravitational FFs and various local distributions in

M. V. Polyakov and A. G. Shuvaev, [arXiv:hep-ph/0207153 [hep-ph]].
M. V. Polyakov, Phys. Lett. B **555**, 57 (2003).
M. V. Polyakov and P. Schweitzer, Int. J. Mod. Phys. A **33** (2018) no.26, 1830025.

For systems whose intrinsic size is comparable with the Compton wavelength this definition of spatial densities was criticized in

M. Burkardt, Phys. Rev. D **62** (2000), 071503(R), [erratum: Phys. Rev. D **66** (2002), 119903(E)].
G. A. Miller, Phys. Rev. Lett. **99**, 112001 (2007).
G. A. Miller, Phys. Rev. C **79**, 055204 (2009).
G. A. Miller, Ann. Rev. Nucl. Part. Sci. **60** (2010), 1-25.
R. L. Jaffe, Phys. Rev. D **103** (2021) no.1, 016017.
G. A. Miller, Phys. Rev. C **99**, no.3, 035202 (2019).
A. Freese and G. A. Miller, Phys. Rev. D **103**, 094023 (2021).

Miller pointed out that Sachs's derivation implicitly assumes *delocalized* wave packets, resulting in moments of the charge density governed by the size of the wave packet.

Charge density for spin-0 systems was studied by Jaffe in relationship to $\Delta^2 = 6F'(0)$, the size of the wave packet R and the Compton wavelength $1/m$.

He argued that the interpretation of the Fourier transformed FFs as charge densities is not valid for systems with $\Delta \sim 1/m$.

Definition of spatial distributions has attracted much attention.

To give examples ...

The light-front approach allows one to define purely intrinsic spatial densities, which have probabilistic interpretation

M. Burkardt, Phys. Rev. D **62** (2000), 071503(R).
G. A. Miller, Phys. Rev. Lett. **99**, 112001 (2007).
G. A. Miller, Phys. Rev. C **79**, 055204 (2009).
G. A. Miller, Ann. Rev. Nucl. Part. Sci. **60** (2010), 1-25.
Y. Guo, X. Ji and K. Shiells, Nucl. Phys. B **969**, 115440 (2021).

These densities are obtained only as two-dimensional distributions.

Alternatively, the phase-space approach allows one to define fully relativistic and unambiguous three-dimensional spatial densities.

C. Lorcé, Phys. Rev. Lett. **125**, no.23, 232002 (2020).
C. Lorcé, P. Schweitzer, K. Tezgin, Phys. Rev. D **106**, no.1, 1 (2022).
C. Lorcé, H. Moutarde, A. P. Trawiński, Eur. Phys. J. C **79**, no.1, 89 (2019).

These densities depend on both coordinates and momenta and do not have a strict probabilistic interpretation.

We revisited the definition of the charge density and other spatial densities ...

Disclaimer:

$$\frac{\textit{What I know about internal structure of hadrons}}{\textit{What I do not know}} \approx 0\,!$$

I present definition of spatial densities via sharply localized states.

We use spherically symmetric wave packets, corresponding to zero averaged momentum frame - ZAMF.

Why spherically symmetric Packets?

Short answer: Because our space is rotationally invariant!

We put our system in a specially designed state which reveasls its 3D structure.

Constructing a three-dimensional image of an object by putting together all possible two-dimensional cuts, the coefficients of magnification of all cuts must be the same.

Otherwise the three-dimensional image will be distorted.

Within our approach the moments of charge distribution do not depend on the state also for non-symmetric packets!

Localized states

We use normalizable Heisenberg-picture states:

$$|\Phi, \mathbf{X}, s\rangle = \int \frac{d^3p}{\sqrt{2E(2\pi)^3}}\, \phi(s, \mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{X}} |p, s\rangle,$$

where $\mathbf{X}$ is the position of the system, and $|p, s\rangle$ are normalized as

$$\langle p', s'|p, s\rangle = 2E(2\pi)^3 \delta_{s's}\, \delta^{(3)}(\mathbf{p}' - \mathbf{p}), \qquad p = (E, \mathbf{p}).$$

Profile function $\phi(s, \mathbf{p}) = \phi(\mathbf{p}) = \phi(|\mathbf{p}|)$ corresponds to ZAMF and satisfies the condition

$$\int d^3p\, |\phi(s, \mathbf{p})|^2 = 1.$$

It is convenient to define dimensionless profile functions

$$\phi(\mathbf{p}) = R^{3/2}\, \tilde{\phi}(R\mathbf{p}),$$

Sharp localizations of the system correspond to small R.

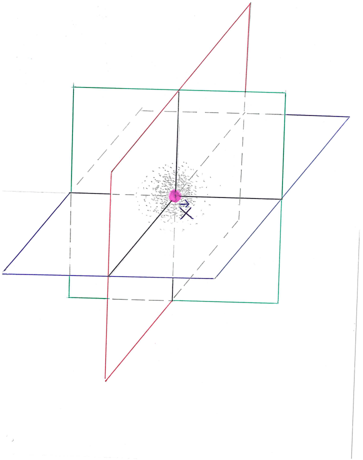

Figure: Symbolic representation of the localized state.

The charge density of a spin-0 system in ZAMF

We start with a spin-0 system (following Jaffe).

We assume that the system is an eigenstate of the charge operator

$$\hat{Q}|p\rangle = Q|p\rangle, \qquad \hat{Q} = \int d^3r\, \hat{\rho}(\mathbf{r}, 0),$$

$\hat{\rho}(\mathbf{r}, 0)$ is the electric charge density at $t = 0$, and we take $Q = 1$.

Matrix elements of $\hat{\rho}(\mathbf{r}, 0)$ for momentum eigenstates:

$$\langle p'|\hat{\rho}(\mathbf{r}, 0)|p\rangle = e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}(E + E')F(q^2)\,,$$

where $F(q^2)$ is the electric FF and $q = p' - p$.

The charge density distribution has the form

$$\langle \Phi, \mathbf{X} | \hat{\rho}(\mathbf{r}, 0) | \Phi, \mathbf{X} \rangle = \int \frac{d^3p\, d^3p' (E + E')}{(2\pi)^3 \sqrt{4EE'}} F\left(q^2\right) \phi^\star(\mathbf{p}')\phi(\mathbf{p}) e^{i\mathbf{q}\cdot(\mathbf{X}+\mathbf{r})},$$

where $q^2 = (E' - E)^2 - \mathbf{q}^2$.

Without loss of generality we choose $\mathbf{X} = 0$.

Introducing the total and relative momenta via $\mathbf{p} = \mathbf{P} - \mathbf{q}/2$ and $\mathbf{p}' = \mathbf{P} + \mathbf{q}/2$, the charge density is written as

$$\begin{aligned}
\rho_\phi(\mathbf{r}) &\equiv \langle \Phi, \mathbf{0} | \hat{\rho}(\mathbf{r}, 0) | \Phi, \mathbf{0} \rangle \\
&= \int \frac{d^3P\, d^3q}{(2\pi)^3 \sqrt{4EE'}} (E + E')\, F\left[(E - E')^2 - \mathbf{q}^2\right] \\
&\times \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right) \phi^\star\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) e^{i\mathbf{q}\cdot\mathbf{r}}.
\end{aligned}$$

Traditional interpretation of the charge density in terms of $F(-\mathbf{q}^2)$, emerges by first taking the static limit $E = E' = m$ in the integrand,

$$\rho_{\phi,\,\mathrm{naive}}(\mathbf{r}) = \int \frac{d^3P\, d^3q}{(2\pi)^3}\, \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right) \phi^\star\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) F\left(-\mathbf{q}^2\right)\, e^{i\mathbf{q}\cdot\mathbf{r}},$$

and subsequently taking the limit $R \to 0$.

This can be done without specifying the functions $F\left(q^2\right)$ and $\phi(\mathbf{p})$ using the method of dimensional counting

J.Gegelia, G.Japaridze, K.Turashvili, Theor. Math. Phys. **101**, 1313 (1994).

For $F\left(q^2\right)$ decreasing at large q^2 faster than $1/q^2$, the only non-vanishing contribution to $\rho_{\phi,\,\mathrm{naive}}(\mathbf{r})$ in the $R \to 0$ limit is obtained from the region of large $\mathbf{P}$.

The resulting "naive" charge density has the familiar form

$$\begin{aligned} \rho_{\mathrm{naive}}(r) &= \int \frac{d^3\tilde{P}\, d^3q}{(2\pi)^3} F\left(-\mathbf{q}^2\right) |\tilde{\phi}(\tilde{\mathbf{P}})|^2 e^{i\mathbf{q}\cdot\mathbf{r}} \\ &= \int \frac{d^3q}{(2\pi)^3} F\left(-\mathbf{q}^2\right) e^{i\mathbf{q}\cdot\mathbf{r}}. \end{aligned}$$

This expression corresponds to ZAMF in states with $R \gg \frac{1}{m}$.

As argued by Jaffe $\rho_{\mathrm{naive}}(r)$ is valid for the hierarchy of scales $\Delta \gg 1/m$, because we have to take $\Delta \gg R \gg 1/m$.

New definition

The method of dimensional counting allows to take the $R \to 0$ limit without employing the static approximation, resulting in

$$\rho_\phi(\mathbf{r}) = \int \frac{d^3\tilde{P}\, d^3q}{(2\pi)^3}\, F\left[\frac{(\tilde{\mathbf{P}}\cdot\mathbf{q})^2}{\tilde{\mathbf{P}}^2} - \mathbf{q}^2\right] |\tilde{\phi}(\tilde{\mathbf{P}})|^2\, e^{i\mathbf{q}\cdot\mathbf{r}}.$$

Using $\tilde{\phi}(\tilde{\mathbf{P}}) = \tilde{\phi}(|\tilde{\mathbf{P}}|)$ ans switching to spherical coordinates we get

$$\rho(r) = \frac{1}{4\pi} \int d^2\hat{n}\, \delta(r_\parallel) \rho_{\hat{\mathbf{n}}}(r_\perp)\,,$$

where

$$\rho_{\hat{\mathbf{n}}}(r_\perp) = \int \frac{d^2q_\perp}{(2\pi)^2}\, F(-\mathbf{q}_\perp^2)\, e^{i\mathbf{q}_\perp\cdot\mathbf{r}_\perp}\,,$$

$\hat{\mathbf{n}}$ is a unit vector, $\mathbf{a}_\perp = \hat{\mathbf{n}} \times (\mathbf{a} \times \hat{\mathbf{n}})$, $a_\parallel = \mathbf{a}\cdot\hat{\mathbf{n}}$, $a_\perp \equiv |\mathbf{a}_\perp|$.

These expressions establish a geometric interpretation of $\rho(r)$.

The validity of the new definition does not depend on the relation between Δ and $1/m$.

The obtained result for $\rho(r)$ does not depend on the particle's mass.

Therefore $\rho_{\rm naive}(r)$, does *not* emerge from $\rho(r)$ by taking the static limit:

$$\rho_{\rm naive}(r) \neq \lim_{m\to\infty} \rho(r).$$

Mathematically, the reason for this mismatch is the non-commutativity of the $R \to 0$ and $m \to \infty$ limits of $\rho_\phi(\mathbf{r})$.

We thank Cedric Lorcé for pointing out that similar results for spin-0 systems have been published long ago in

G. N. Fleming, *Charge Distributions from Relativistic Form Factors.* Physical Reality & Math. Descrip., 357 (1974).

The dependence of $\rho(r)$ on $F(-\mathbf{q}_\perp^2)$ rather than on $F(-\mathbf{q}^2)$ affects the radial profile of the charge density.

We compare $\rho(r)$ and $\rho_{\rm naive}(r)$ for a charged and a neutral particles.

We employ form factors

$F_{\rm p}(q^2) = G_{\rm D}(q^2) = (1 - q^2/\Lambda^2)^{-2}$

with $\Lambda^2 = 0.71$ GeV^2,

and

$F_{\rm n}(q^2) = A\tau/(1 + B\tau)\, G_{\rm D}(q^2)$,

where $\tau = -q^2/(4m_p^2)$, $A = 1.70$, $B = 3.30$.

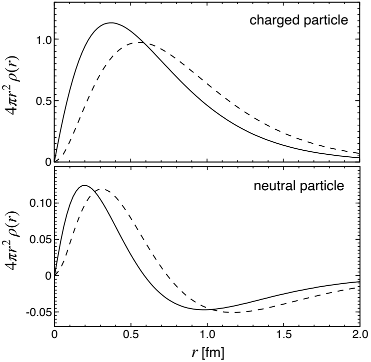

Figure: Radial charge density distributions $4\pi r^2 \rho(r)$ (solid lines) and $4\pi r^2 \rho_{\rm naive}(r)$ (dashed lines) for a charged and a neutral particles.

The charge density in moving frames

To consider the same system from the point of view of a moving frame we replace $\phi(\mathbf{p})$ with

$$\phi_{\mathbf{v}}(\mathbf{p}) = \sqrt{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{p}}{E}\right)}\,\phi\left[\mathbf{p}_\perp + \gamma(\mathbf{p}_\parallel - \mathbf{v}E)\right],$$

where $\mathbf{v}$ denotes the boost velocity and $\gamma = (1 - v^2)^{-1/2}$, $\mathbf{p}_\parallel = (\mathbf{p}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$, $\mathbf{p}_\perp = \mathbf{p} - \mathbf{p}_\parallel$.

S. E. Hoffmann, [arXiv:1804.00548 [quant-ph]].

Analogously to ZAMF we evaluate the $R \to 0$ limit and obtain

$$\begin{aligned}
\rho_{\mathbf{v}}(\mathbf{r}) &= \int \frac{d^3q}{(2\pi)^3}\, \bar{F}\left(q_{\parallel}, q_{\perp}\right)\, e^{i\mathbf{q}\cdot\mathbf{r}}\,, \\
\bar{F}(q_{\parallel}, q_{\perp}) &= \frac{1}{4\pi}\int_{-1}^{+1} d\eta \int_0^{2\pi} d\phi \\
&\times\; F\left\{\frac{\left[\sqrt{1-\eta^2}\cos\phi\, q_{\perp} + \gamma(\eta+v)q_{\parallel}\right]^2}{\gamma^2(1+v\eta)^2} - \mathbf{q}^2\right\}.
\end{aligned}$$

where $q_{\parallel} \equiv \hat{\mathbf{v}}\cdot\mathbf{q}$ and $q_{\perp} \equiv |\mathbf{q}_{\perp}|$.

In the IMF with $v \to 1$ and $\gamma \to \infty$, the charge density turns into the usual two-dimensional distribution in the transverse plane

$$\rho_{\rm IMF}(\mathbf{r}) = \delta(r_{\|})\, \rho_{\rm IMF}(r_\perp)$$

with

$$\rho_{\rm IMF}(r_\perp) = \int \frac{d^2 q_\perp}{(2\pi)^2}\, F\left(-\mathbf{q}_\perp^2\right) e^{i\mathbf{q}_\perp \cdot \mathbf{r}_\perp}\,.$$

The interpretation of the obtained results follows from the comparison of ZAMF and IMF.

The ZAMF expression $\rho(r)$ is given by a continuous (isotropic) superposition of the two-dimensional "images" of the system, $\rho_{\mathrm{IMF}}(\mathbf{r})$, corresponding to all possible IMFs.

The full image of a three-dimensional object can be reconstructed by putting together all possible two-dimensional projections.

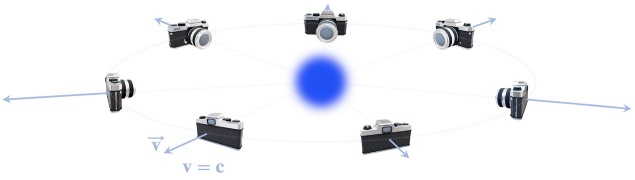

Figure: 3D image as a composition of 2D pictures

EMT spatial densities

Spatial densities of EMT differ from the ones of the e.m. current.

A localized state is a superposition of eigenstates of the charge operator, which is also an eigenstate of the charge operator with the same eigenvalue.

A packet which is a superposition of one-particle eigenstates of the four-momentum operator with different four-momenta is not an eigenstate of the same operator.

This makes matrix elements of EMT more complicated.

Matrix element of EMT in a localized state using the parametrization in terms of form factors $\Theta_1(q^2)$ and $\Theta_2(q^2)$:

$$
\begin{aligned}
t_\phi^{\mu\nu}(\mathbf{r}) &= \langle \Phi, \mathbf{0} | \hat{T}^{\mu\nu}(\mathbf{r}, 0) | \Phi, \mathbf{0} \rangle \\
&= \int \frac{d^3P\, d^3q}{(2\pi)^3 \sqrt{4EE'}} \left[\left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \Theta_1\left(q^2\right) + 2P^\mu P^\nu \Theta_2\left(q^2\right) \right] \\
&\times \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right) \phi^\star\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) e^{-i\mathbf{q}\cdot\mathbf{r}} \, .
\end{aligned}
$$

Static approximation

Local densities in terms of FFs in the Breit frame emerge by expanding the integrand in $1/m$ and localizing the wave packet.

$$
\begin{aligned}
t^{00}_{\text{naive}}(\mathbf{r}) &= \int \frac{d^3q}{(2\pi)^3}\, m\,\Theta_2\left(-\mathbf{q}^2\right)\, e^{-i\mathbf{q}\cdot\mathbf{r}}\,, \\
t^{0i}_{\text{naive}}(\mathbf{r}) &= 0\,, \\
t^{ij}_{\text{naive}}(\mathbf{r}) &= \frac{1}{mR^2}\int d^3\tilde{P}\,\tilde{P}^i\tilde{P}^j|\tilde{\phi}(\tilde{\mathbf{P}})|^2 \int \frac{d^3q}{(2\pi)^3}\,\Theta_2\left(-\mathbf{q}^2\right) e^{-i\mathbf{q}\cdot\mathbf{r}} \\
&+ \frac{1}{2m}\int \frac{d^3q}{(2\pi)^3}\left(\mathbf{q}^2\delta^{ij} - q^iq^j\right)\Theta_1\left(-\mathbf{q}^2\right) e^{-i\mathbf{q}\cdot\mathbf{r}}\,,
\end{aligned}
$$

In t^{ij}_{naive} we kept terms up to different orders in R !

t^{00}_{naive} and the second term of t^{ij}_{naive} coincide with the spatial densities defined via the gravitational FFs in the Breit frame.

Both of these terms do not depend on the packet.

Spatial densities in sharply localized states ($R \to 0$):

$$t^{\mu\nu}(\mathbf{r}) = N_{\phi,R} \int \frac{d^2\hat{\tilde{P}}\, d^3q}{(2\pi)^3}\, \hat{\tilde{P}}^\mu \hat{\tilde{P}}^\nu\, \Theta_2\left[-\mathbf{q}_\perp^2\right] e^{-i\mathbf{q}\cdot\mathbf{r}}$$
$$- N_{\phi,R,2} \int \frac{d^2\hat{\tilde{P}}\, d^3q}{(2\pi)^3} \left(\tilde{q}^\mu \tilde{q}^\nu + \mathbf{q}_\perp^2 g^{\mu\nu}\right) \Theta_1\left[-\mathbf{q}_\perp^2\right] e^{-i\mathbf{q}\cdot\mathbf{r}},$$

where $\tilde{q}^\mu = (\hat{\tilde{\mathbf{P}}} \cdot \mathbf{q}, \mathbf{q})$, $\tilde{P}^\mu = (\tilde{P}, \tilde{\mathbf{P}})$, $\hat{\tilde{P}}^\mu = \left(1, \frac{\tilde{\mathbf{P}}}{\tilde{P}}\right)$, $\tilde{P} = |\tilde{\mathbf{P}}|$, $\mathbf{q}_\perp = \hat{\tilde{\mathbf{P}}} \times \left(\mathbf{q} \times \hat{\tilde{\mathbf{P}}}\right)$, $\mathbf{q}_\perp^2 \equiv -\tilde{q}^2$ and

$$N_{\phi,R} = \frac{1}{R} \int d\tilde{P}\tilde{P}^3 |\tilde{\phi}(|\tilde{\mathbf{P}}|)|^2,$$
$$N_{\phi,R,2} = \frac{R}{2} \int d\tilde{P}\tilde{P} |\tilde{\phi}(|\tilde{\mathbf{P}}|)|^2.$$

The dependence on the profile function remains in $N_{\phi,R}$ and $N_{\phi,R,2}$.

Gravitational densities in moving frames

Analogously to the case of EM current we replace $\phi(\mathbf{p})$ with

$$\phi_{\mathbf{v}}(\mathbf{p}) = \sqrt{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{p}}{E}\right)}\,\phi[\mathbf{p}_\perp + \gamma(\mathbf{p}_\parallel - \mathbf{v}E)]$$

and obtain in the $R \to 0$ limit:

$$\begin{aligned} t_{\mathbf{v}}^{\mu\nu}(\mathbf{r}) &= N_{\phi,R}\int \frac{d^3q}{(2\pi)^3}\,\bar{t}^{\mu\nu}(q_\parallel, q_\perp)\, e^{-i\mathbf{q}\cdot\mathbf{r}} \\ &+ N_{\phi,R,2}\int \frac{d^3q}{(2\pi)^3}\,\bar{t}_2^{\mu\nu}(q_\parallel, q_\perp)\, e^{-i\mathbf{q}\cdot\mathbf{r}}\,, \end{aligned}$$

with

$$\bar{t}^{\mu\nu}(q_{\|}, q_{\perp}) = \int_{-1}^{+1} d\eta \int_{0}^{2\pi} d\phi \, \frac{\Omega^{\mu}\Omega^{\nu}}{\gamma(1+v\eta)} \Theta_2 \left[\bar{q}^2\right] ,$$

$$\bar{t}_2^{\mu\nu}(q_{\|}, q_{\perp}) = -\int_{-1}^{+1} d\eta \int_{0}^{2\pi} d\phi \, \frac{1}{\gamma(1+v\eta)} \left[\bar{q}^{\mu}\bar{q}^{\nu} - \bar{q}^2 g^{\mu\nu}\right] \Theta_1 \left[\bar{q}^2\right] ,$$

where $\Omega^{\mu} = (\gamma(1+v\eta), \hat{\omega}_{\perp} + \gamma(\hat{\omega}_{\|} + \mathbf{v}))$,

$$\bar{q}^{\mu} = ([\sqrt{1-\eta^2}\cos\phi \, q_{\perp} + \gamma(\eta + v)q_{\|}]/(\gamma(1+v\eta)), \mathbf{q})$$

$$\hat{\omega} = (\sqrt{1-\eta^2}\cos\phi, \sqrt{1-\eta^2}\sin\phi, \eta).$$

In the infinite momentum frame (IMF) with $v \to 1$, $\gamma \to \infty$, we obtain

$$\bar{t}^{\,\mu\nu}(q_{\|}, q_{\perp}) = 4\pi\,\gamma\,\hat{v}^{\mu}\,\hat{v}^{\nu}\,\Theta_2\left[-\mathbf{q}_{\perp}^2\right],$$

$$\bar{t}_2^{\,\mu\nu}(q_{\|}, q_{\perp}) = -\frac{2\pi}{\gamma}\,\alpha\,\left[\mathbf{q}_{\perp}^2 g^{\mu\nu} + q_v^{\mu} q_v^{\nu}\right]\Theta_1\left[-\mathbf{q}_{\perp}^2\right], \qquad (1)$$

where $\hat{v}^{\mu} = (1, \hat{\mathbf{v}})$, $q_v^{\mu} = (q_{\|}, \mathbf{q})$ and

$$\alpha = \lim_{v \to 1} \int_{-1}^{+1} \frac{d\eta}{1 + v\eta}. \qquad (2)$$

$\alpha/\gamma \sim \sqrt{1 - v}\,\ln(1 - v)$ when the $v \to 1$.

Interpretation

In IMF the matrix elements of EMT in localized states represent two-dimensional distributions.

There are two types of contributions:
1. Depending on the velocity - characterizing the movement of the system as a whole.
2. Contributions which are related to internal properties.

For the $t^{ij}_{\mathbf{v}}(\mathbf{r})$ the contribution generated by $\bar{t}^{ij}$ corresponds to the motion of the system as a whole, while the term generated by $\bar{t}^{ij}_2$ is related to internal characteristics.

Separate interpretation of different contributions has been given previously in

A. Freese and G. A. Miller, Phys. Rev. D **105**, no.1, 014003 (2022), [arXiv:2108.03301 [hep-ph]].

By integrating the IMF expressions over all possible directions one repoduces the corresponding terms in ZAMF.

Thus, the spatial distributions in the ZAMF can be understood as an integral over all directions of the IMF velocity.

In IMF $\bar{t}_2^{ij}$ characterizes the internal structure.
Therefore also in the ZAMF the corresponding term

$$t_2^{ij}(\mathbf{r}) = N_{\phi,R,2} \int \frac{d^2\hat{n}\, d^3q}{(2\pi)^3} \left(-q^i q^j + \mathbf{q}_\perp^2 \delta^{ij}\right) \Theta_1 \left[-\mathbf{q}_\perp^2\right] e^{-i\mathbf{q}\cdot\mathbf{r}}$$

is interpreted as characterizing the distribution of internal forces.

While normalization depends on the profile function the spatial distribution is uniquely determined by the EMT form factor.

We identify the traceless and the trace parts via

$$t_2^{ij}(\mathbf{r}) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\,\delta^{ij}\right) s(r) + \delta^{ij} p(r)\,, \tag{3}$$

Quantities $s(r)$ and $p(r)$ have been interpreted as the shear force and the pressure, respectively.

M. V. Polyakov, Phys. Lett. B **555**, 57 (2003). [hep-ph/0210165].
M. V. Polyakov and P. Schweitzer, Int. J. Mod. Phys. A **33** (2018) no.26, 1830025. [arXiv:1805.06596 [hep-ph]].

Notice that this interpretation has been questioned in
X. Ji and Y. Liu, Phys. Rev. D **106**, no.3, 034028 (2022), [arXiv:2110.14781 [hep-ph]].

Breit-frame expressions correspond to systems in ZAMF in states with characteristic scales of packets much larger than $1/m$.

Such packet is dominated by eigenstates of the energy with $E \approx m$, and therefore it is an approximate eigenstate of the energy operator with the eigenvalue m.

Therefore $t^{00}(\mathbf{r})$ can be interpreted in this case as the spatial distribution of the mass, which is the same as the full energy of the system in the ZAMF in the static approximation.

In sharply localized states $t^{00}(\mathbf{r})$ and $t^{0i}(\mathbf{r})$ can be interpreted as energy and momentum spatial distributions, respectively.

For a spin-0 system in the ZAMF we have:

$$
\begin{aligned}
t^{00}(\mathbf{r}) &= N_{\phi,R} \int \frac{d^2\hat{n}\, d^3q}{(2\pi)^3}\, \Theta_2\left[-\mathbf{q}_\perp^2\right] e^{-i\mathbf{q}\cdot\mathbf{r}} . \\
t^{0i}(\mathbf{r}) &= 0 .
\end{aligned}
$$

Sharp localization of the system requires huge amount of energy, therefore the normalization factor $N_{\phi,R}$ explodes.
However, the functional form of the densities is uniquely determined by the corresponding form factors.

For non-vanishing $\mathbf{X}$, it specifies the center of the spatial densities, therefore $\mathbf{X}$ should be interpreted as the position of the center-of-gravity of the system.

Summary

- We introduced an unambiguous definition of spatial distributions of expectation values of local operators via localized states.
- New definition also applies to systems independently on the Compton wavelength.
- Our results suggest an unconventional $\langle r^2 \rangle = 4F'(0)$ in contrast to the usual relationship $\langle r^2 \rangle_{\text{naive}} = 6F'(0)$ motivated by the Breit frame distribution.
- In case of EMT and gravitational FFs the static approximation leads to spatial densities obtained from FFs in Breit frame.

  Sharply localized states lead to analogous but different results.
- Spatial density related to FF $\Theta_1(q^2)$ is interpreted as characterizing internal forces.