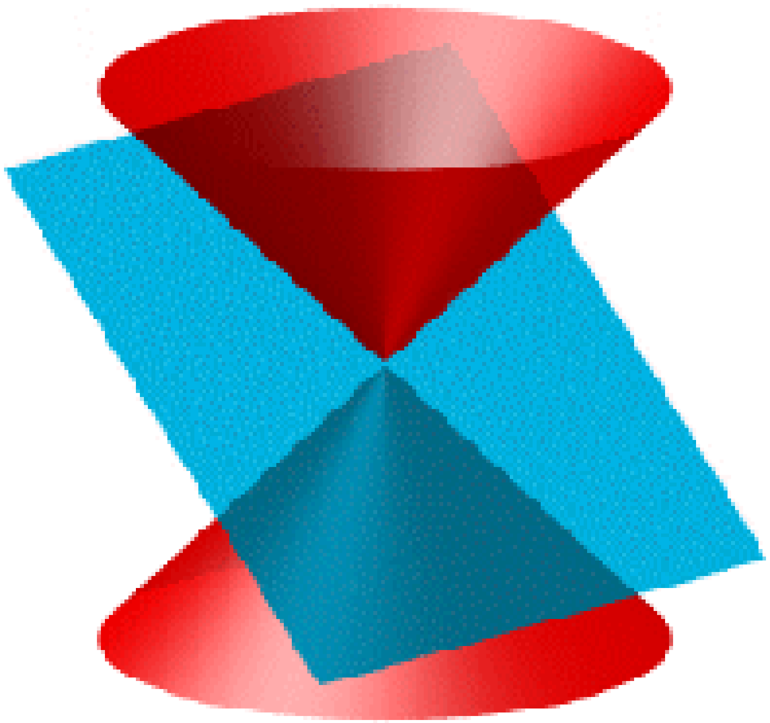
What is light-front quantization?

Xiangdong Ji, UMD

ILCAC seminar, Jan, 18, 2023



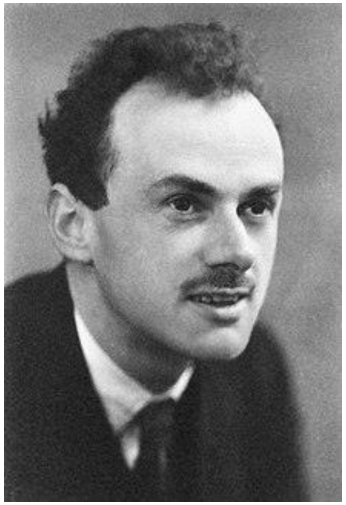
Outline

- The standard definition
- LFQ as Infinite-momentum frame (IMF) limit of instant quantization
  - Super-renormalizable theories
  - Theories with non-trivial UV divergences
  - Analogy with critical phenomena
- LFQCD as an EFT of Euclidean QCD (LaMET)
- Some examples
- Summary

The standard definition

Relativistic dynamics

- Three forms: (Dirac, 1949)
  - Instant: ordinary dynamics
  - Point: used in heavy-ion collision
  - Light-front: light-traveler dynamics?



REVIEWS OF MODERN PHYSICS VOLUME 21, NUMBER 3 JULY, 1949

Forms of Relativistic Dynamics

P. A. M. Dirac
St. John's College, Cambridge, England

For the purposes of atomic theory it is necessary to combine the restricted principle of relativity with the Hamiltonian formulation of dynamics. This combination leads to the appearance of ten fundamental quantities for each dynamical system, namely the total energy, the total momentum and the 6-vector which has three components equal to the total angular momentum. The usual form of dynamics expresses everything in terms of dynamical variables at one instant of time, which results in specially simple expressions for six or these ten, namely the components of momentum and of angular momentum. There are other forms for relativistic dynamics in which others of the ten are specially simple, corresponding to various sub-groups of the inhomogeneous Lorentz group. These forms are investigated and applied to a system of particles in interaction and to the electromagnetic field.

Front-form (light-front) dynamics

Light-front travelling in $z(x^3)$ direction x^+=const

- Front-form coordinates

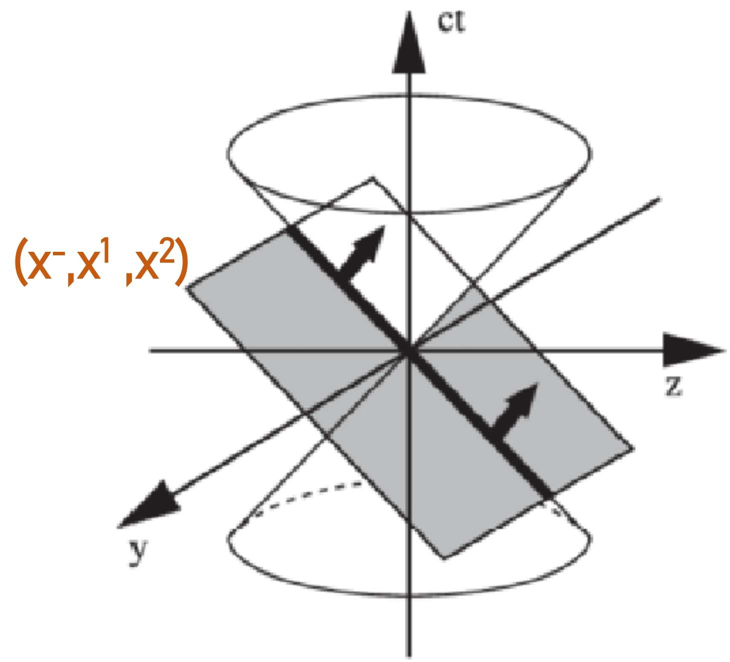
$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3)$$

$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3)$$

$$x_\perp = (x^1, x^2)$$

- The front-front dynamics is determined by x^+ evolution through the "Light-front Hamiltonian"

$$P^- = \frac{1}{\sqrt{2}}(P^0 - P^3)$$



P.A.M. Dirac,
Rev. Mod. Phys. 21, 1949

LF Quantum Field Theory

- Postulating commutators at equal LF time
- Expand the fields in terms of creation and annihilation operators at a particular LF time
- LF momentum & Hamiltonian eigen-quation

$$\hat{P}^-|\Psi_n\rangle = \frac{M_n^2}{2P^+}|\Psi_n\rangle$$

$$|P\rangle = \sum_{m,n}^{\infty} \int \Pi_{i=1}^{n}[dx_i d^2\vec{k}_{i\perp}]\Pi_{i=1}^{m}[d^2\vec{k}_{i\perp}]$$
$$\times\delta(x_1 + ... + x_n - 1)\delta^2(\vec{k}_{1\perp} + ... + \vec{k}_{n+m\perp})$$
$$\phi(x_1, \vec{k}_{1\perp}, ..., 0, \vec{k}_{n+m\perp})a^\dagger_{x_1,\vec{k}_{1\perp}} ... a^\dagger_{x_n,\vec{k}_{n\perp}}|\mathbf{0}_{nm}\rangle$$

Brodsky, Pauli & Pinsky, Phys. Rep. 301, 1998

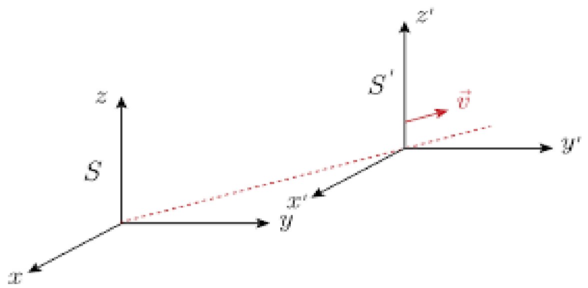
LFQ as Infinite-momentum frame (IMF) limit of instant quantization

Infinite momentum frame (IMF) & QFT

- Fubini & Furlan, Physics, 1, 1965

  Infinite momentum frame (IMF), or light-travelling proton, introduced as a slick "math trick" to derive useful sum rules & light-cone algebra.

- What a QFT (scalar) look like when boosted to IMF?



S. Weinberg, Dynamics at infinite momentum
Phys. Rev. 150 (1966) 1313

Weinberg's QFT rules in IMF

- All kinematic infinities (γ_∞ -factor) can be removed from the calculations, resulting a "new" set of rules for Hamiltonian perturbation theory.
- Weinberg was not aware of Dirac's work. However, what he had showed was (formally)

  QFT in IMF = Dirac's front-form dynamics,
  "provided all UV divergences are ignored."
- The finding was very quickly confirmed by

  Susskind (1968), Bardakci, and Halpern (1968)
  Chang and Ma (1969), Kogut & Soper (1970): QED

IMF limit & UV divergences

- How does UV divergences affect IMF=LFQ?
- Not a problem for super-renormalizable (SR) theories:
  - All non-trivial interactions are irrelevant operators, or, all couplings have positive mass dimensions.
  - The number of counter-terms needed to render the Green functions finite decreases as the number of loops increases.
- Many interesting 1+1D QFT, but not Gross-Neveu etc models which are not SR.

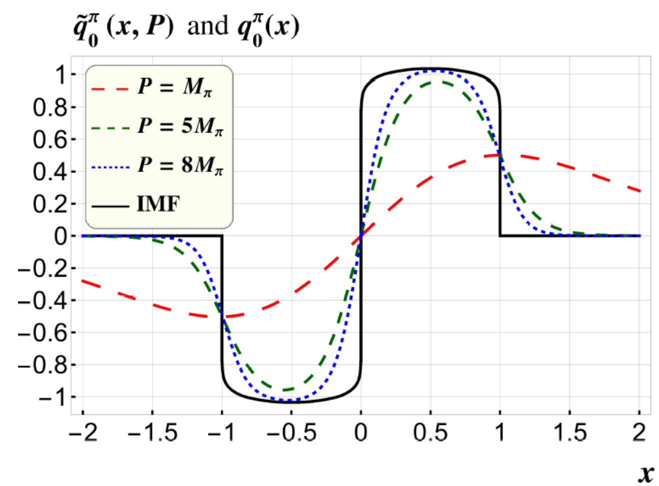
t' Hooft model

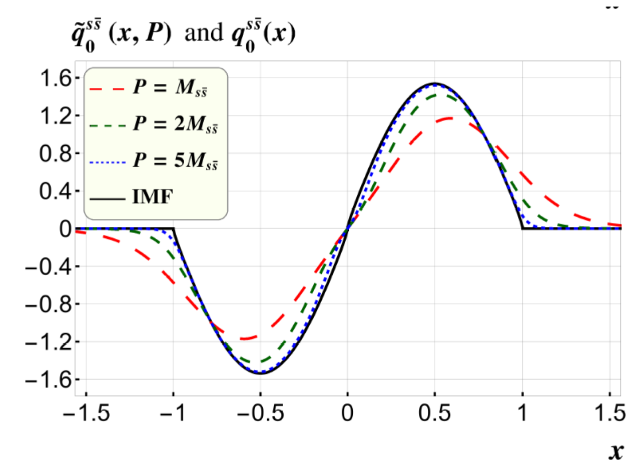
- 1+1 D QCD with $N_c = \infty$

  Can be solved exactly at any finite P^z. (Y. Jia PRD 2018) et al.)
- Mom dis. Calculated at various mom:

  $p_\pi^z = m_\pi, 5m_\pi, 8m_\pi \ldots$

  $p_\phi^z = m_\phi, 2m_\phi, 5m_\phi \ldots$
- LFQ is recovered from instant form as a smooth limit of $p^z \to \infty$





Critical (renormalizable) theories?

- A dimensionless integral depending on P^z in 3+1D QCD

$$\int^{\Lambda_{UV}} d^4k \frac{1}{(n \cdot k)(\gamma \cdot P - \gamma \cdot k) \;\; k^2}$$

- Integral is UV divergent, Λ_{UV} shall be larger than any physics scales (P^z)
  - A log dependence $ln(P^z/\Lambda_{UV})$!
  - Naïve $P^z \to \infty$ limit does not exist, $P^z = \infty$ is singular or not analytical.

Take the integral to LF

- LFQ is obtained by taking $P^z \to \infty$ under the integral sign (after integrating k^0)
- The integral becomes formally independent of P^z
- The UV behavior of the integral is now different
  - There is a light-cone singularity originated from $ln(P^z/\Lambda_{UV})$ after taking $P^z \to \infty$
- Spurious LC singularities
  - Time-ordered graphs are potentially more divergent because of Lorentz symmetry breaking
  - For gauge-invariant quantities, choice of LC gauge may lead to additional LC divergences.

What is LFQ?

- It depends on how to regularize LC div.
- If one uses regulators that respect basic symmetries (Lorentz, gauge), there will be no spurious LC singularities.
- Define LFQ:
  - The LC divergence can be regulated with standard UV regulators (like DIM REG).
  - LFQ is an EFT of the original theory, in the same sense HQET is an EFT of heavy quark physics.

Double-log dependences

- Consider the WF renormalization of a quark in axial gauge: nA=0.
- The WF renormalization constant Z(p) has a double log dependence

$$Z = 1 + \alpha \ln^2 P^z + \ \dots$$

which is due to soft & collinear physics

- In LFQ, this leads to a new type of LC divergences which can again be regulated in the standard UV regularization, leading to double poles per loop in DR.

So, when $P^z \to \infty$ is not well defined…

- We define the limit by taking $P^z \to \infty$ under the integral sign.
- Regularize the LC divergences with standard UV method. And the result is defined as LFQ, and is an also EFT of the original theory.

LFQ ≡ EFT of large P limit

- PS: All spurious LC divergences are effectively regulated without any essential symmetry breaking.

$P_z = \infty$ is a critical point

- Due to Lorentz boost, the longitudinal correlation functions in the coordinate space

$$C(\lambda) \sim \exp(-\lambda/\xi_{\mathrm{corr}})$$

Longitudinal correlation length: $\xi_{\mathrm{corr}} \sim \frac{P^z}{M} \to \infty$,

  - ➢ The correlation functions decay algebraically

$$C(\lambda) \sim \lambda^{-1+\alpha}$$

  - ➢ Corresponding to small-x behavior of PDFs

$$f(x) \sim x^{-\alpha}$$

LFQCD as an EFT of Euclidean QCD

LFQ for QCD

- Light-front wave-function (amplitudes)
- PDF, DA, TMDPDF, GPDs, etc…
- All of these can be obtained from large momentum limit of time-independent correlation functions in instant form of QCD

  Large momentum effective theory or LaMET

- Time-independent correlation functions can be computed from Euclidean formulation of QCD, such as lattice QCD or instanton calculus.
- Many LaMET works in PDF, DA, TMDPDF, GPDs

Main steps for LFWF through LaMET

1. Defining LFWF amplitudes as gauge-invariant matrix elements.
2. Perturbative LFQ without LF Hamiltonian
3. Using off-LC regulators and lattice QCD to finish nonperturbative calculations (again without LF-Hamiltonian), matching to on LC scheme.

Step 1: LFWF amplitudes as gauge-invariant matrix elements

- Consider a hadron state with Fock states expansion on pert. QCD vacuum,

$$|P\rangle = \sum_{n=1}^{\infty} \int d\Gamma_n \psi_n(x_i, \vec{k}_{i\perp}) \prod a_i^\dagger(x_i, \vec{k}_{i\perp})|0\rangle$$

- The LFWF amplitudes

$$\psi_n(x_i, \vec{k}_{i\perp}) = \langle 0| \prod a_i(x_i, \vec{k}_{i\perp})|P\rangle$$

we can replace the pert vacuum by QCD vacuum if one ignores the zero modes.

Gauge-invariance

- Introduce guage-invariant LF fields

$$\Phi_i^{\pm}(\xi) = W_n^{\pm}(\xi)\phi(\xi) \ ,$$

with light-like gauge-link $W_n^{\pm}(\xi)$ defined as

$$W_n^{\pm}(\xi) = \mathcal{P}\mathrm{exp}\left[-ig\int_0^{\pm\infty} d\lambda n \cdot A(\xi+\lambda n)\right]$$

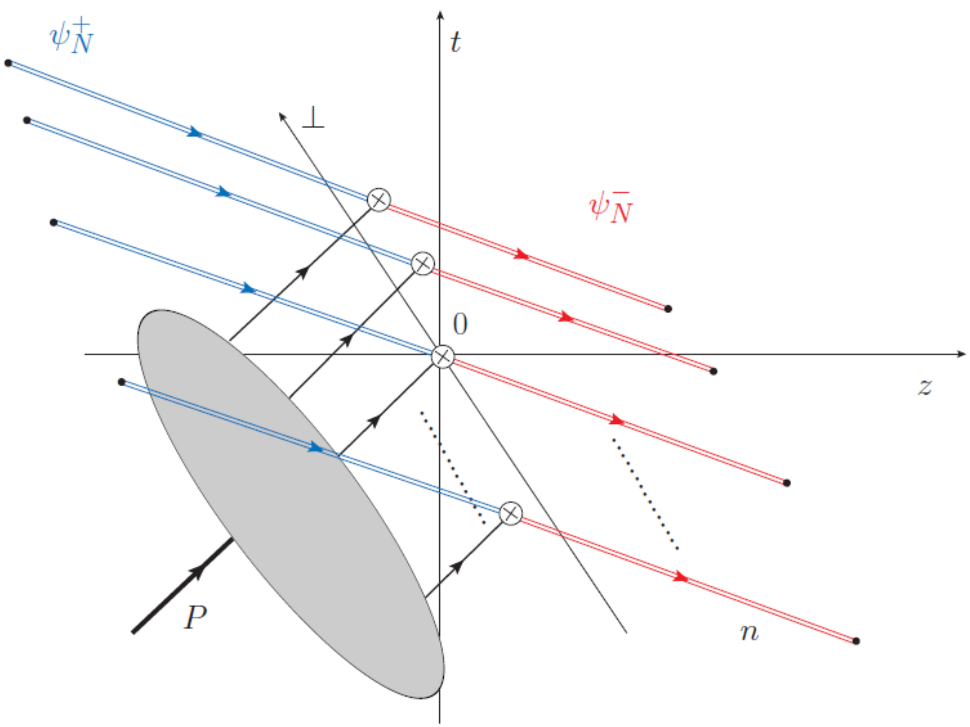
- Gauge-invariant LFWF amplitude (covariant)

$$\psi_N^{\pm 0}(x_i, \vec{b}_{i\perp}, \mu) = \int \prod_{i=1}^{N} d\lambda_i e^{i\lambda_i x_i} \times e^{+i\lambda_0 x_0}$$

$$\times \langle 0|\mathcal{P}_N \prod_{i=1}^{N} \Phi_i^{\pm}(\lambda_i n + \vec{b}_{i\perp})\Phi_0^{\pm}(\lambda_0 n + \vec{b}_{0\perp})|P\rangle \ .$$

where $|0\rangle$ is the QCD vacuum.

Gauge-invariant LFWF amplitudes



❖ In singular gauges, such as the light-cone gauge $A^+ = 0$, one has to choose the link connection at $\xi^- = \pm\infty$, which are related to zero modes.

Step2: Perturbative LFQ without LF Hamiltonian

- Standard approach to get the LFWF amplitudes is through LC-time-ordered pert. theory in light-cone gauge.

$$
\begin{aligned}
U|\Phi\rangle = \sqrt{Z_\Phi}\Big\{ |\Phi\rangle &+ \sum_{n_1}{}' \frac{|n_1\rangle\langle n_1|H_{\rm int}(0)|\Phi\rangle}{p^- - p^-_{n_1} + i\epsilon} \\
&+ \sum_{n_1 n_2}{}' \frac{|n_1\rangle\langle n_1|H_{\rm int}(0)|n_2\rangle\langle n_2|H_{\rm int}(0)|\Phi\rangle}{(p^- - p^-_{n_1} + i\epsilon)(p^- - p^-_{n_2} + i\epsilon)} \\
&+ \cdots \Big\},
\end{aligned}
$$

where H_{int} is the interactions part of P^-

However, it contains a lot of divergences.

(Zhang & Harindranath, 1993)

Computation with gauge-invariant amplitudes

$$\psi_N^{\pm 0}(x_i, \vec{b}_{i\perp}, \mu) = \int \prod_{i=1}^{N} d\lambda_i e^{i\lambda_i x_i} \times e^{+i\lambda_0 x_0}$$
$$\times \langle 0|\mathcal{P}_N \prod_{i=1}^{N} \Phi_i^{\pm}(\lambda_i n + \vec{b}_{i\perp})\Phi_0^{\pm}(\lambda_0 n + \vec{b}_{0\perp})|P\rangle \ .$$

- The calculation can be done in covariant gauge and covariant perturbation theory.
- The only new divergences come from the light-cone gauge link.
- This has been called rapidity divergence arising from zero modes.

Regularization of LC rapidity divergences

- Delta-regulator :

$$e^{i\int dx^- A^+(x^-)} \to e^{i\int dx^- A^+(x^-)} e^{-\delta^- |x^-|}.$$

M.G. Echevarria, I.Scimemi, and A. Vladimirov, PRD 2016

- LF length regulator

$$e^{i\int dx^- A^+(x^-)} \to e^{i\int_0^{L^-} dx^- A^+(x^-)}$$

A. Vladimirov, PRL 2020

- Exponential regulator

Defined through final-state cuts, not applicable for LFWFs.

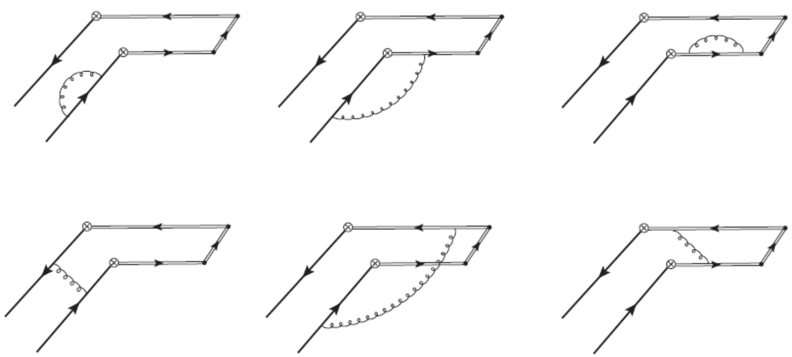
Y.Li, D.Neill, and H.X.Zhu, NPB 2020

....

Example: Meson LFWF

- Leading amplitudes

$$\psi_{\bar{q}q}^{\pm 0}(x, b_\perp, \mu, \delta^-) = \frac{1}{2}\int \frac{d\lambda}{2\pi} e^{i x_r \lambda}$$
$$\times \langle 0|\overline{\Psi}_n^{\pm}(\lambda n/2 + \vec{b}_\perp)\gamma^+\gamma^5\Psi_n^{\pm}(-\lambda n/2)|P\rangle\Big|_{\delta^-}$$



$$\psi_{\bar{q}q}^{\pm 0}(x, b_\perp, \mu, \delta^-)$$
$$= \frac{\alpha_s C_F}{2\pi}\left[F(x, x_0, b_\perp, \mu)\right]_+ + \frac{\alpha_s C_F}{2\pi}\delta\left(x - x_0\right)$$
$$\times \left\{L_b\left(\frac{3}{2} + \ln\frac{-(\delta^-)^2 \mp i0}{x_0\bar{x}_0}\right) + \frac{1}{2}\right\} ,$$

where $C_F = (N_c^2 - 1)/2N_c$ and $L_b = \ln\frac{\mu^2 b_\perp^2}{4e^{-2\gamma_E}}$ and

$$F(x, x_0, b_\perp, \mu)$$
$$= \left[-\left(\frac{1}{\epsilon_{\mathrm{IR}}} + L_b\right)\left(\frac{x}{x_0(x_0 - x)} + \frac{x}{x_0}\right) + \frac{x}{x_0}\right]$$
$$\times \theta(x)\theta(x_0 - x) + (x \to \bar{x}, x_0 \to \bar{x}_0) ,$$

Soft Functions with rapidity divergence (zero modes)

- Define the soft functions

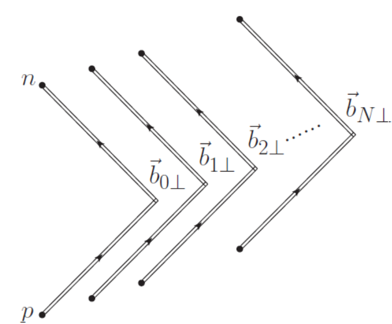
$$S^{\pm}_{N,\mathcal{R}}(\vec{b}_{i\perp}, \mu, \delta^+, \delta^-) = \langle 0|\mathcal{T}\mathcal{P}_N \prod_{i=0}^{N} \mathcal{C}^{\pm}(\vec{b}_{i\perp}, \delta^+, \delta^-)|0\rangle \ ,$$

where C contains conjugating Wilson lines

$$\mathcal{C}^{\pm}(\vec{b}_{\perp}, \delta^+, \delta^-) = W^{\pm}_{n}(\vec{b}_{\perp})|_{\delta^-} W^{\dagger}_{p}(\vec{b}_{\perp})|_{\delta^+} \ ,$$

where W_p is defined as

$$W^{\pm}_{p}(\vec{b}_{\perp}) = \mathcal{P}\exp\left[-ig\int_{0}^{\pm\infty} d\lambda' p\cdot A(\lambda' p + \vec{b}_{\perp})\right] \ .$$



Renormalization of rapidity divergence

- Rapidity renormalized WF amplitudes

$$\psi_N^{\pm}(x_i, \vec{b}_{i\perp}, \mu, \zeta) = \lim_{\delta^- \to 0} \frac{\psi_N^{\pm 0}(x_i, \vec{b}_{i\perp}, \mu, \delta^-)}{\sqrt{S_{N,\mathcal{R}}^{\pm}(\vec{b}_{i\perp}, \mu, \delta^- e^{2y_n}, \delta^-)}}$$

- Rapidity evolution equation (Collins-Soper eq.)

$$2\zeta \frac{d}{d\zeta} \ln \psi_N^{\pm}(x_i, \vec{b}_{i\perp}, \mu, \zeta) = K_N(\vec{b}_{i\perp}, \mu) \ .$$

Kernel K is non-perturbative at large $\vec{b}_{\perp i}$

Rapidity renormalized meson WF amplitudes

- One-loop result, potentially calculable to all-order in pQCD

$$
\begin{aligned}
&\psi^{\pm}_{\bar{q}q}(x, b_\perp, \mu, \zeta) \qquad\qquad\qquad\qquad\qquad (\\
&= \frac{\alpha_s C_F}{2\pi}\left[F(x, x_0, b_\perp, \mu)\right]_+ + \frac{\alpha_s C_F}{2\pi}\delta(x - x_0) \\
&\times \left\{ -\frac{L_b^2}{2} + L_b\left(\frac{3}{2} + \ln\frac{\mu^2}{\pm\sqrt{\zeta\bar{\zeta}} - i0}\right) + \frac{1}{2} - \frac{\pi^2}{12} \right\}
\end{aligned}
$$

- Evolutions

The renormalized WF amplitude satisfies the rapidity (momentum) evolution equation

$$2\zeta\frac{d}{d\zeta}\ln\psi^{\pm}_{\bar{q}q}(x, b_\perp, \mu, \zeta) = K_1(b_\perp, \mu) \ ,$$

and the RGE:

$$
\begin{aligned}
&\mu^2\frac{d}{d\mu^2}\ln\psi^{\pm}_{\bar{q}q}(x, b_\perp, \mu, \zeta) \\
&= \frac{1}{2}\Gamma_{\rm cusp}(\alpha_s)\ln\frac{\mu^2}{\pm\sqrt{\zeta\bar{\zeta}} - i0} - \gamma_H(\alpha_s) \ .
\end{aligned}
$$

Step 3: Computing non-pert. LFWF through lattice QCD & matching

- Light-momentum effect theory (LaMET) allows light-front correlations be connected to Euclidean correlations: IR physics is independent of frames)
- Differences in UV can be matched through pert QCD.
- Rapidity divergences are related to large-Pz divergences.
- Standard lattice QCD method can be used to compute Euclidean correlations.

Quasi LFWF amplitudes

- Euclidean WF amplitudes

$$\widetilde{\psi}_N^{\pm}(x_i, \vec{b}_{i\perp}, \mu, \zeta_z) = \lim_{L\to\infty} \int d\lambda_i e^{-i\lambda_i x_i - i\lambda_0 x_0}$$

$$\frac{\langle 0|\mathcal{P}_N \prod_{i=1}^N \Phi_i^{\pm}(\lambda_i n_z + \vec{b}_{i\perp}; L)\Phi_0^{\pm}(\lambda_0 n_z; L)|P\rangle}{\sqrt{Z_E(2L, \vec{b}_{i\perp}, \mu)}}$$

- Gauge-invariant fields

$$\Phi_i^{\pm}(\xi; L) = \mathcal{P}\exp\left[ig \int_0^{\mp L \pm \xi^z} d\lambda A^z(\xi + \lambda n_z)\right]\phi(\xi)$$

Regularizing RD through off-light-cone soft function

- Define two off-light-cone vectors

$$p \to p_Y = p - e^{-2Y}(P^+)^2\bar{n},\ n \to n_{Y'} = n - e^{-2Y'}\frac{\bar{p}}{(P^+)^2}$$

- Soft functions

$$S_N^{\pm}(\vec{b}_{i\perp}, \mu, Y, Y') = \frac{\langle 0|\mathcal{P}_N \mathcal{T} \prod_{i=0}^{N} \mathcal{C}^{\pm}(\vec{b}_{i\perp}, Y, Y')|0\rangle}{\sqrt{Z_E(Y)}\sqrt{Z_E(Y')}}$$

$$\mathcal{C}^{\pm}(\vec{b}_\perp, Y, Y') = W^{\pm}_{n_{Y'}}(\vec{b}_\perp) W^{\dagger}_{p_Y}(\vec{b}_\perp)\ ,$$

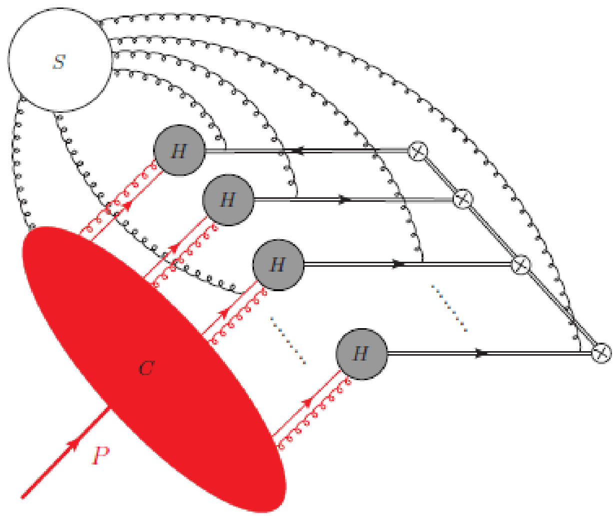
where the off-light-cone gauge-links W_{p_Y} and $W_{n'_Y}$ defined as

$$W_{p_Y}(\vec{b}_\perp) = \mathcal{P}\exp\left[-ig\int_0^{-\infty} d\lambda' p_Y \cdot A(\lambda' p_Y + \vec{b}_\perp)\right]$$

and

$$W^{\pm}_{n_{Y'}}(\vec{b}_\perp) = \mathcal{P}\exp\left[-ig\int_0^{\pm\infty} d\lambda n_{Y'} \cdot A(\lambda n_{Y'} + \vec{b}_\perp)\right]$$

Matching Formula



$$\widetilde{\psi}^{\pm}_{N}(x_i, \vec{b}_{i\perp}, \mu, \zeta_z)\sqrt{S_{rN}(\vec{b}_{i\perp}, \mu)} = e^{\ln\frac{\mp\zeta_z - i0}{\zeta} K_N(\vec{b}_{i\perp},\mu)}$$

$$\times H^{\pm}_{N}\left(\zeta_{z,i}/\mu^2\right)\psi^{\pm}_{N}(x_i, \vec{b}_{i\perp}, \mu, \zeta) + \ldots \, ,$$

Examples

Transverse Momentum Dependent Wave Functions from Lattice QCD

(Lattice Parton Collaboration (LPC))

Min-Huan Chu,[1,2] Jin-Chen He,[1,3] Jun Hua,[4,5,*] Jian Liang,[4,5] Xiangdong Ji,[3] Andreas Schäfer,[6] Hai-Tao Shu,[6,†] Yushan Su,[3] Wei Wang,[1,7] Ji-Hao Wang,[8,9] Yi-Bo Yang,[8,9,10,11] Jun Zeng,[1] Jian-Hui Zhang,[12,13] and Qi-An Zhang[14]

[1] *INPAC, Shanghai Key Laboratory for Particle Physics and Cosmology, Key Laboratory for Particle Astrophysics and Cosmology (MOE), School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China*

[2] *Yang Yuanqing Scientific Computering Center, Tsung-Dao Lee Institute, Shanghai Jiao Tong University, Shanghai 200240, China*

[3] *Department of Physics, University of Maryland, College Park, MD 20742, USA*

[4] *Guangdong Provincial Key Laboratory of Nuclear Science, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, China*

[5] *Guangdong-Hong Kong Joint Laboratory of Quantum Matter, Southern Nuclear Science Computing Center, South China Normal University, Guangzhou 510006, China*

[6] *Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*

[7] *Southern Center for Nuclear-Science Theory (SCNT), Institute of Modern Physics, Chinese Academy of Sciences, Huizhou 516000, Guangdong Province, China*

[8] *CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

[9] *School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study, UCAS, Hangzhou 310024, China*

[10] *International Centre for Theoretical Physics Asia-Pacific, Beijing/Hangzhou, China*

[11] *School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

[12] *School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen 518172, China*

[13] *Center of Advanced Quantum Studies, Department of Physics, Beijing Normal University, Beijing 100875, China*

[14] *School of Physics, Beihang University, Beijing 102206, China*

We present a first lattice QCD calculation of the transverse momentum dependent wave functions (TMDWFs) in large-momentum effective theory. Numerical simulations are based on 2+1+1 flavors of highly improved staggered quarks action with lattice spacing a=0.121 fm from MILC Collaboration, and another 2 +1 flavor clover fermions and tree-level Symanzik gauge action configuration generated by CLS Collaboration with a=0.098 fm. We present the result for soft function that incorporates the one-loop perturbative contributions and a coherent normalization. Based on the obtained soft function, we simulate the equal-time quasi-TMDWFs on the lattice, and extract the physical TMDWFs. A comparison with the phenomenological parameterization is made and consistent behaviors between the two lattice ensembles and phenomenological model are found. Our studies provide crucial *ab initio* theory inputs for making precise predictions for exclusive processes under QCD factorization.

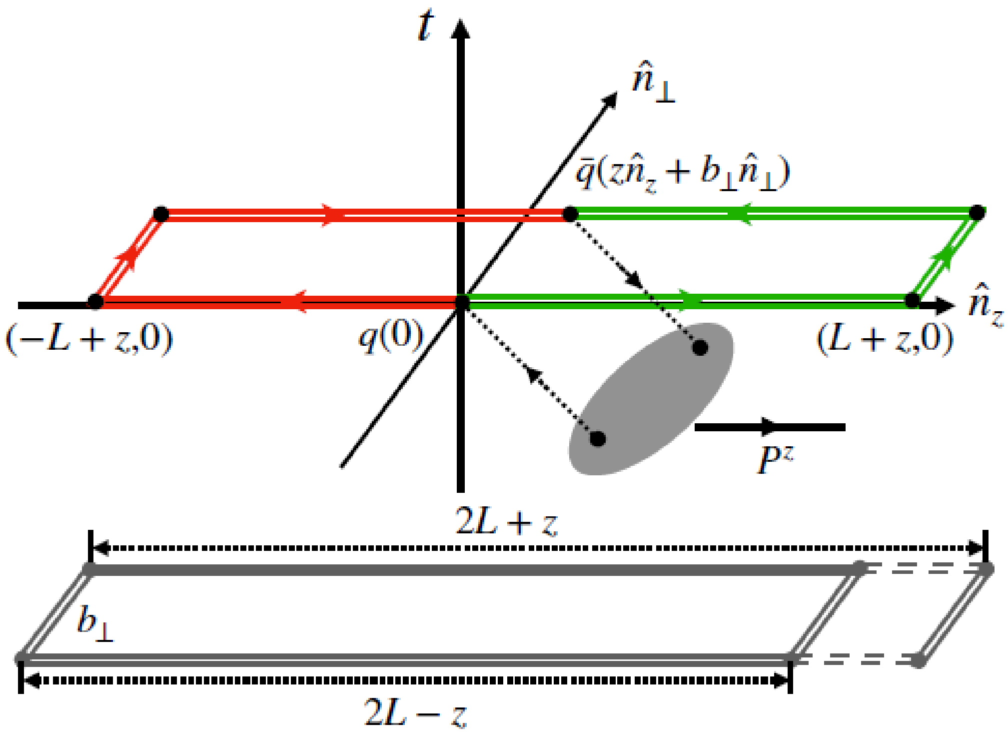


FIG. 1. Illustration of quasi-TMDWF in coordinate space with a staple-shaped gauge-link inside. As the green and red double lines represent the gauge-link in $\tilde{\Psi}^+(z, b_\perp, \mu, \zeta^z)$ and $\tilde{\Psi}^-(z, b_\perp, \mu, \zeta^z)$, a corresponding staple-shaped Wilson loop $Z_E(2L \pm z, b_\perp, \mu)$ is constructed to cancel the linear and cusp divergences.

TABLE I. The numerical simulation setup. On each ensemble, we put 8/4 source slices in time direction.

Ensemble	a(fm)	$L^3 \times T$	m_π^{sea}	m_π^{val}	Measure
a12m310	0.121	$24^3 \times 64$	310 MeV	670 MeV	1053×8
X650	0.098	$48^3 \times 48$	333 MeV	662 MeV	911×4

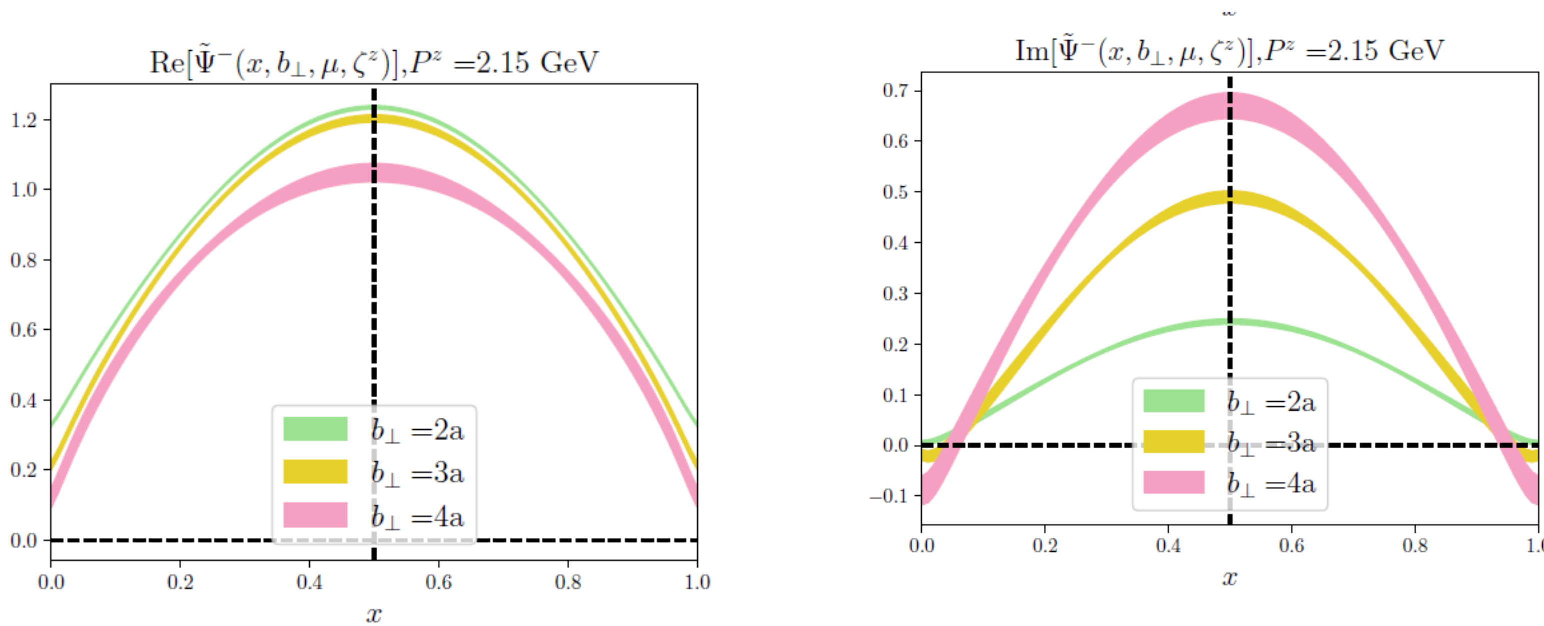


FIG. 2. The real part (upper panel) and the imaginary part (lower panel) of quasi-TMDWF in momentum space, with hadron momentum $P^z = 2.15$ GeV on MILC ensemble.

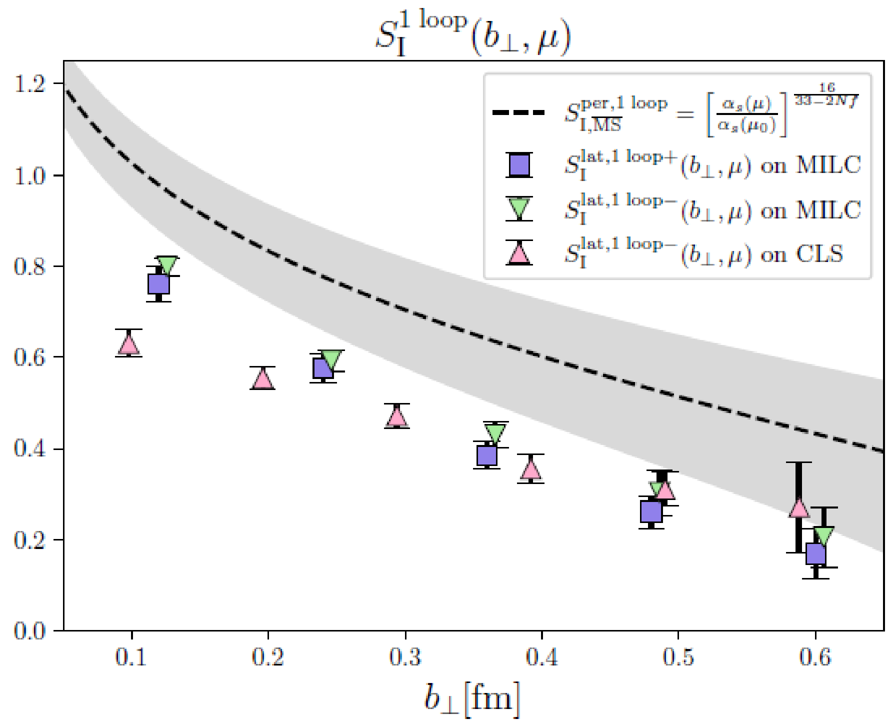
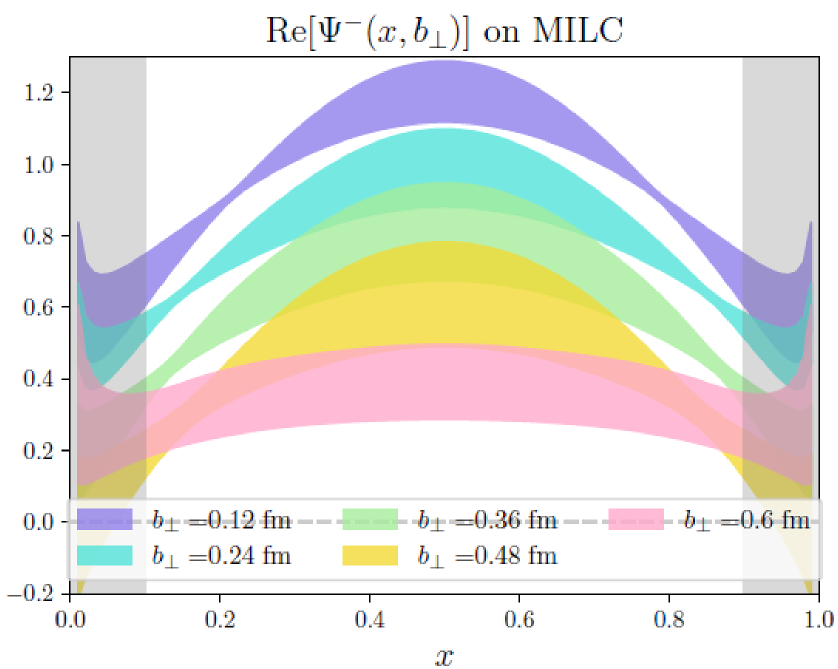
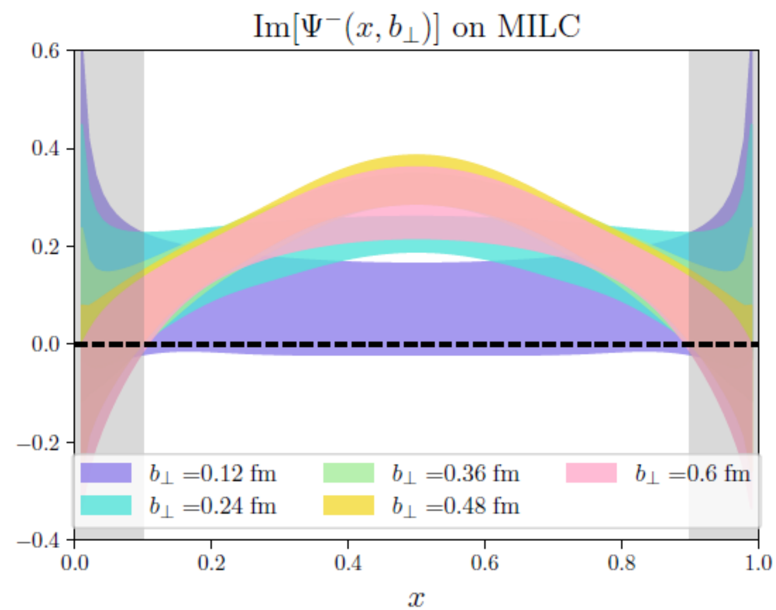


FIG. 3. The one-loop intrinsic soft function as a function of $b_\perp$. The grey band corresponds to the one-loop perturbative result in $\overline{\mathrm{MS}}$ scheme and the band is obtained by $\mu_0 = 1/b_\perp^*$ varying for the scale $b_\perp^* \in \left[1/\sqrt{2}, \sqrt{2}\right] b_\perp$. The label $\pm$ in $S^{\mathrm{lat,1\ loop}\pm}$ represents the lattice results extracted by $\tilde{\Psi}^\pm$.

Lowest fock state LFWF





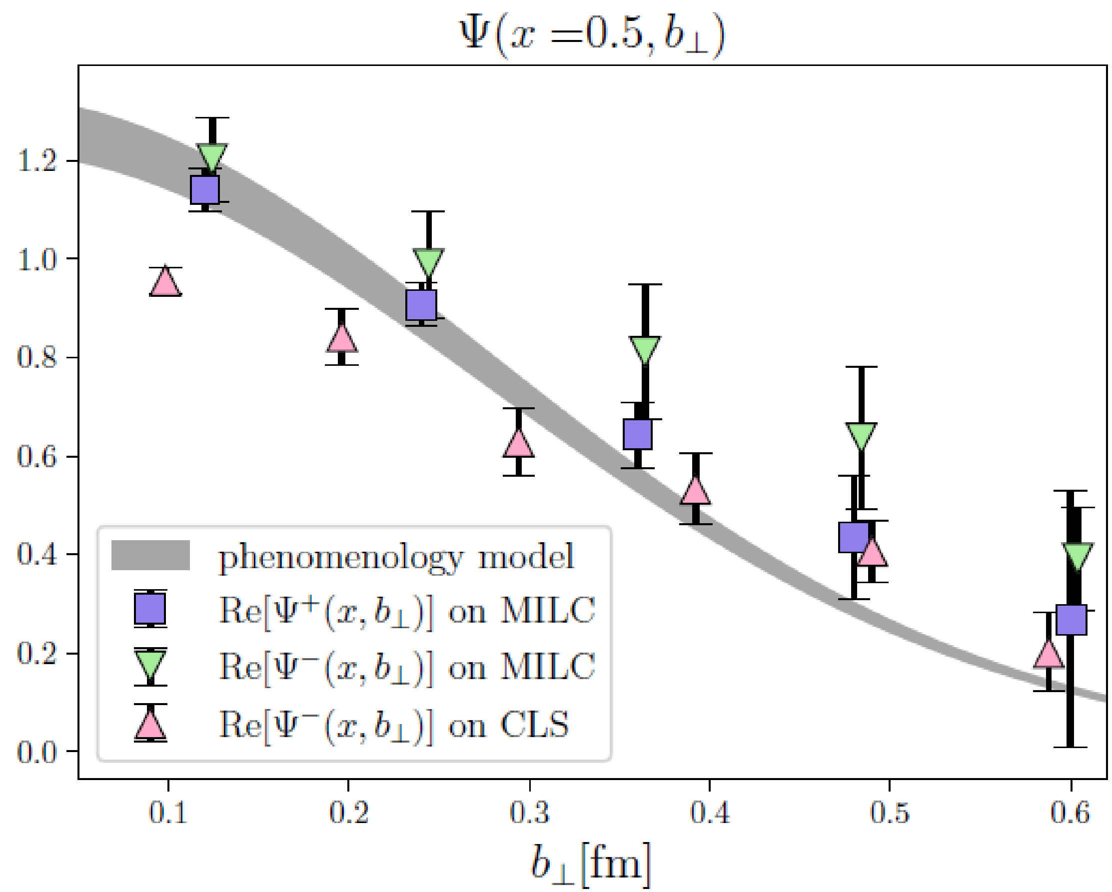


FIG. 5. Comparison of the transverse momentum distribution in our results with $\{\zeta, \mu\} = \{(6\ \mathrm{GeV})^2, 2\ \mathrm{GeV}\}$ and phenomenological model at $x = 0.5$ point.

TMDPDF

- LPC (J.C.He et al, 2211.02340)

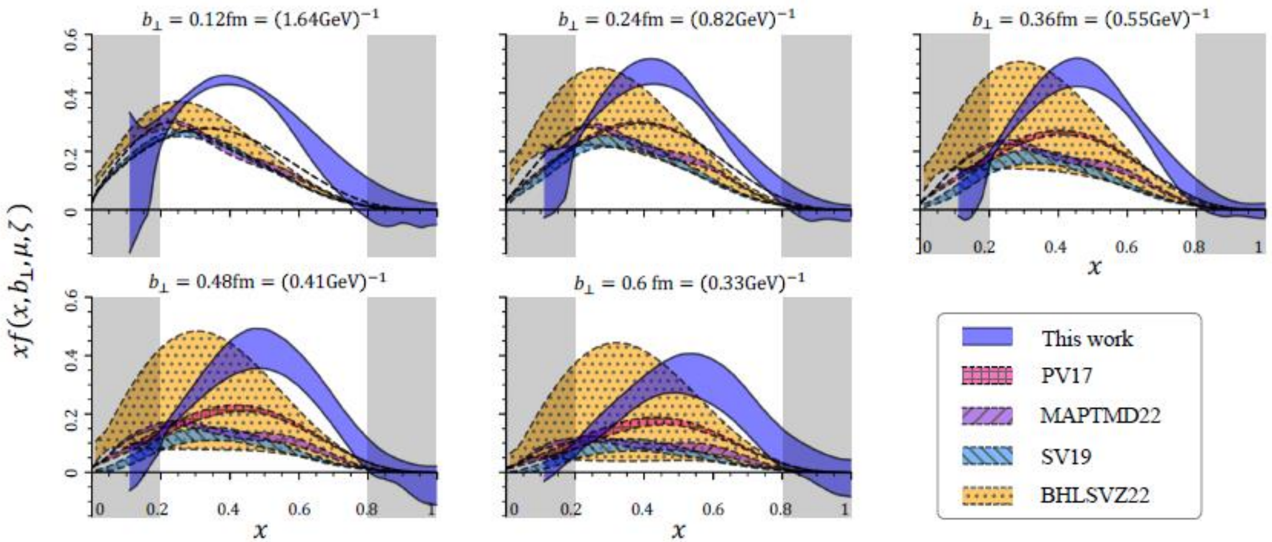


FIG. 5. Our final results for isovector unpolarized TMDPDFs $xf(x, b_\perp, \mu, \zeta)$ at renormalization scale $\mu = 2$ GeV and rapidity scale $\sqrt{\zeta} = 2$ GeV, extrapolated to physical pion mass 135 MeV and infinite momentum limit $P^z \to \infty$, compared with PV17 [6], MAPTMD22 [9], SV19 [7] and BHLSVZ22 [8] global fits (slashed bands). The colored bands denote our results with both statistical and systematic uncertainties, the shaded grey regions imply the endpoint regions where LaMET predictions are not reliable.

Physical effects of zero mode (Ji, 2003.04478)

- While zero modes are difficult to control in the QCD Hamiltonian and LFWFs directly, it is possible to calculate directly through laMET in physical observables: Mass, scalar charge, ...
- Sivers function

$$f^{\mathrm{TMD}}(x, k_\perp, S_\perp) \sim (S_\perp \times \vec{k}_\perp)^z f_{1T}^\perp(x, k_\perp) + \ldots$$

$$f(x, \vec{k}_\perp) = \int \frac{d\lambda}{2\pi} \frac{d^2\vec{b}_\perp}{(2\pi)^2} e^{-i\lambda x + i\vec{k}_\perp \cdot \vec{b}_\perp} \times \langle P|\bar{\psi}(\lambda n + \vec{b}_\perp)\gamma^+ \mathcal{W}_n(\lambda n + \vec{b}_\perp)\psi(0)|P\rangle \ ,$$

where $\mathcal{W}_n(\lambda n + \vec{b}_\perp)$ is the staple-shaped gauge link,

$$\mathcal{W}_n(\xi) = W_n^\dagger(\xi) W_\perp^\dagger(\xi_\perp) W_\perp(0) W_n(0) \ ,$$

a zero-mode effect!

A well-defined procedure has been developed in LaMET to calculate this.

Conclusions

- Directly solving LFQ version of 3+1 QCD is very very hard, if not impossible. It corresponds to a critical point.
- If considered as an effective theory, it can be obtained from Euclidean QCD in the IMF limit.
- Lattice QCD can be used to calculated all the relevant LF quantities, such as LFWF, PDF, DA, TMDPDF, & GPD through EFT matching and running.