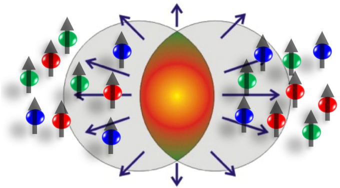
Relativistic spin hydrodynamics

Xu-Guang Huang

Fudan University, Shanghai



Reimei Workshop on Polarization Phenomena and Lorentz Symmetry Violation in Dense Matter"

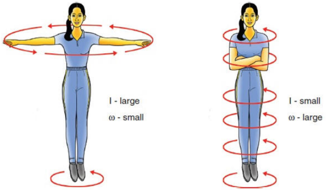
October 07, 2022 @ Yonsei University (online)

Outline

Introduction

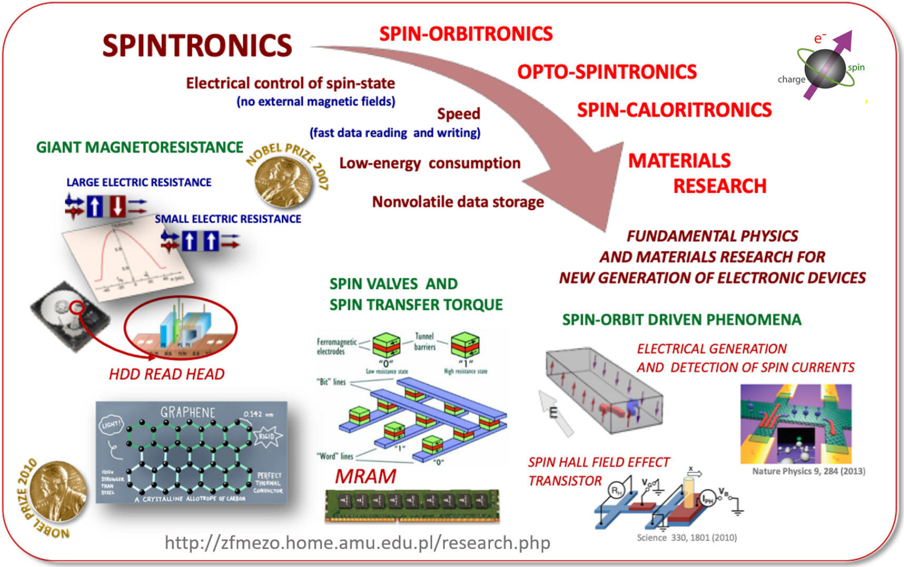
What is spin hydrodynamics?

- What is the hydrodynamics for angular momentum conservation?



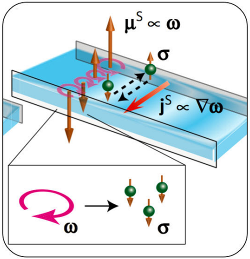
- For spinless fluid, no additional independent hydro variables. For spinful fluid, we can find additional independent (quasi-)hydro variables (chosen as spin) — Spin hydrodynamics.
- My talk is based on the following works:
  - K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, arXiv: 1901.06615
  - M. Hongo, X.-G. Huang, M. Stephanov, M. Kaminski, and H.-U. Yee, arXiv: 2107.14231; 2201.12390
  - Y.-C. Liu and X.-G. Huang, arXiv: 2109.15301
  - Z. Cao, K. Hattori, M. Hongo, X.-G. Huang, and H. Taya, arXiv: 2205.08051
- Spin hydro is interesting in theory and can be applied to describe a number of systems.

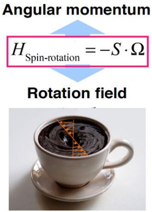
Spintronics in solid materials



Spintronics in liquid materials

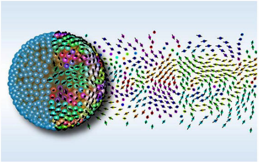
- Spin hydrodynamic generation:





Takahashi et al. Nature Physics 12, 52 (2016)

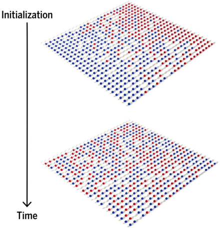
- Ferromagnetic fluid:



Wu et al. PNAS 118, e2017355118 (2021)

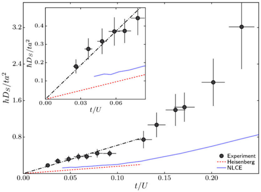
Spin transport in cold atomic gases

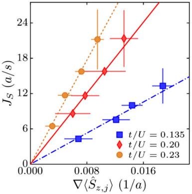
- One example on optical lattice:



$$\hat{H} = -t \sum_{\langle i,j\rangle,\sigma} \left(\hat{c}^{\dagger}_{\sigma,i} \hat{c}_{\sigma,j} + h.c. \right) + U \sum_i \hat{n}_{\uparrow,i} \hat{n}_{\downarrow,i}$$
$$-\mu_{\uparrow} \sum_i \hat{n}_{\uparrow,i} - \mu_{\downarrow} \sum_i \hat{n}_{\downarrow,i} + \Delta_{\uparrow} \sum_i i_x \hat{n}_{\uparrow,i} + \Delta_{\downarrow} \sum_i i_x \hat{n}_{\downarrow,i}$$

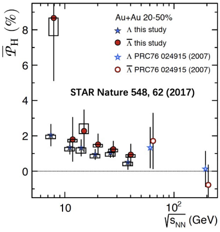
Nichols et al. Science 363, 383 (2018)

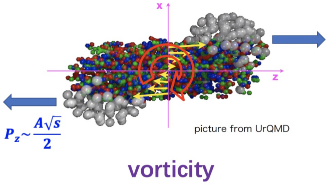




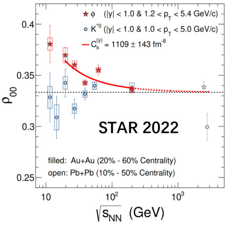
Spin polarization in heavy ion collisions

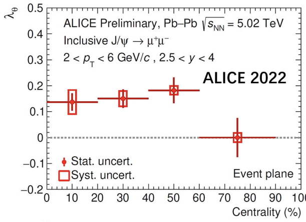
- Spin polarization of spin-1/2 hyperons:





- Spin alignment of spin-1 mesons $\phi, K^{*0}, J/\psi$:

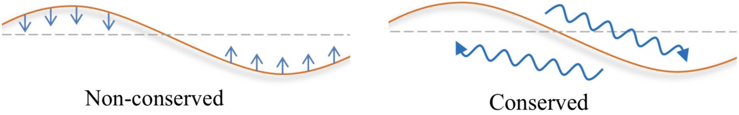




Construction of spin hydrodynamics

Hydrodynamics

- Long-time large-distance effective theory of conserved densities (hydrodynamic modes).
  - Non-hydro modes relax at a finite time scale $\tau = 1/\Gamma$.
    Hydro modes relax at $\tau_{\rm hydro} = 1/\omega_{\rm hydro}(k) \to \infty$ when $k \to 0$.
  - Hydrodynamics is constructed using spatial derivative expansion.
  - Typical hydro modes: energy density, momentum density, baryon charge density, $\cdots$.



- For example, hydro equations for energy and momentum densities:

$$\text{Energy-momentum conservation:} \quad \partial_\mu \Theta^{\mu\nu}(x) = 0$$

with energy-momentum tensor $\Theta^{\mu\nu}$ expanded order by order in derivative giving the constitutive relations,

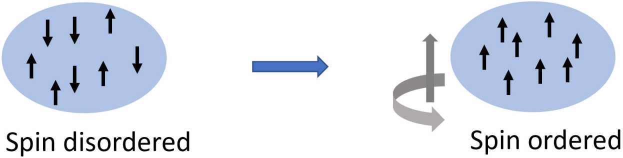
$$\Theta^{\mu\nu} = \underbrace{\epsilon u^\mu u^\nu + p\Delta^{\mu\nu}}_{\text{Ideal hydro}} \underbrace{-\zeta\theta\Delta^{\mu\nu} - 2\eta\partial_\perp^{\langle\mu}u^{\nu\rangle}}_{\text{1st-order viscous hydro}} + O(\partial^2)$$

Can spin be a true hydro mode?

- But, spin is not conserved, only total angular momentum (AM) is:

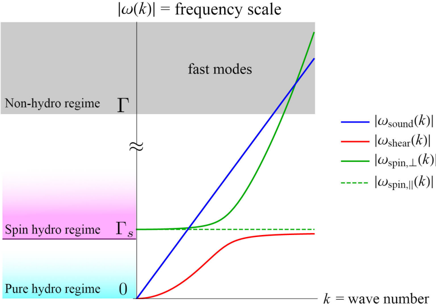
$$\partial_\mu J^{\mu\nu\rho} = 0,$$
$$J^{\mu\nu\rho} = \underbrace{x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu}}_{\text{orbital AM}} + \underbrace{\Sigma^{\mu\nu\rho}}_{\text{spin AM}}$$
$$\Rightarrow \qquad \partial_\mu \Sigma^{\mu\nu\rho}(x) = \Theta^{\rho\nu} - \Theta^{\nu\rho}$$

- Thus spin is a true hydro mode (conserved quantity) only when $\Theta^{\mu\nu}$ is symmetric.
  - In general, not possible. The anti-symmetric part of $\Theta^{\mu\nu}$ is a torque acting on spin.
  - Such torque is spin-orbit coupling (SOC). For example, for Dirac fermions, SOC $\propto 1/m$ and thus vanishes at heavy fermion limit.
- The transfer of AM between spin part and orbital part is generally dissipative.



The spin hydro regime

- When spin relaxation rate $\Gamma_s \ll$ relaxation rate Γ of other micro modes (Hongo, XGH, Kaminski, Stephanov, and Yee 2107.14231):

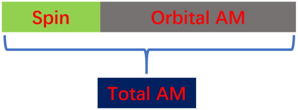


- An extended hydro framework for pure hydro modes and slow spin modes $\Rightarrow$ Relativistic dissipative spin hydrodynamics

$=$ **Pure hydro** $+$ **Slow spin**

Ambiguity in defining spin current

- The definition of spin current $\Sigma^{\mu\nu\rho}$ is ambiguous.



- Pseudo-gauge transformation: Transformations that preserve total AM and the conservation laws (Becattini, Florkowski, and Speranza 1807.10994)

$$\Sigma^{\mu\nu\rho} \to \Sigma^{\mu\nu\rho} - \Phi^{\mu\nu\rho},$$
$$\Theta^{\mu\nu} \to \Theta^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda\mu\nu} - \Phi^{\mu\lambda\nu} - \Phi^{\nu\lambda\mu}\right)$$

- Formulation of spin hydro depends on the choice of pseudo-gauge:
  - Non-anti-symmetric gauge, $\Sigma^{\mu\nu\rho} = u^\mu \sigma^{\nu\rho} + \cdots$
    (Florkowski *et al.* 1705.00587, Montenegro *et al.* 1701.08263, Hattori *et al.* 1901.06615, Fukushima and Pu 2010.01608, Li *et al.* 2011.12318, Gallegos *et al.* 2101.04759, Gallegos *et al.* 2203.05044, She *et al.* 2105.04060, Hu 2209.10979, $\cdots$)
  - Anti-symmetric gauge $\Sigma^{\mu\nu\rho} = \varepsilon^{\mu\nu\rho\gamma}\Sigma_\gamma$
    (Hongo *et al.* 2107.14231, Bhadury *et al.* 2002.03937, Hongo *et al.* 2201.12390, Cao *et al.* 2205.08051, $\cdots$)

Construction of spin hydro

- Step 1: Identify the conservation laws (symmetries)

$$\text{Energy-momentum conservation:} \quad \partial_\mu \Theta^{\mu\nu}(x) = 0$$
$$\text{Agular momentum conservation:} \quad \partial_\mu \Sigma^{\mu\nu\rho}(x) = \Theta^{\rho\nu} - \Theta^{\nu\rho}$$

- Step 2: Choose a pseudo-gauge (*e.g.*, anti-symmetric gauge)
- Step 3: Identify the (quasi-)hydro modes
  - Seven (quasi-)hydro modes: ϵ, u^a, σ_a (or $\sigma_{ab} = -\varepsilon_{abcd} u^c \sigma^d$) with constraints $u^2 = -1$, $\sigma^a u_a = 0$.
  - First law of local thermodynamics: $T ds = d\epsilon - \mu^a d\sigma_a$.
  - Conjugate variables: inverse temperature $\beta \equiv \frac{\partial s}{\partial \epsilon}$, spin chemical potential $\mu^a = -T \frac{\partial s}{\partial \sigma_a}$ (or $\mu^{ab} = -2T \frac{\partial s}{\partial \sigma_{ab}}$).
- Step 4: Power counting schemes
  - Scheme I:

$$\{\beta, u^a\} = O(\partial^0) \quad \text{and} \quad \{\mu^a, \sigma_a, \omega_{\mu\nu} \equiv \nabla_{[\mu} u_{\nu]}\} = O(\partial)$$

  - Scheme II:

$$\{\beta, u^a, \mu^a, \sigma_a, \omega_{\mu\nu} \equiv \nabla_{[\mu} u_{\nu]}\} = O(\partial^0)$$

Construction of spin hydro

- Step 5: Tensor decomposition (Landau-Lifshitz frame)

$$\begin{aligned}\Theta^{\mu}{}_{a} &= \epsilon u^{\mu} u_{a} + p\Delta^{\mu}_{a} + u^{\mu}\delta q_{a} - \delta q^{\mu} u_{a} + \delta\Theta^{\mu}{}_{a} \\ \Sigma^{\mu}{}_{ab} &= \varepsilon^{\mu}{}_{abc}(\sigma^{c} + \delta\sigma u^{c}) \\ s^{\mu} &= s u^{\mu} + \delta s^{\mu}\end{aligned}$$

- Step 7: Entropy production [$O(\partial)$ terms give Gibbs-Duhem relation]

$$\partial_{\mu} s^{\mu} = -\delta\Theta^{\mu}{}_{a}\big|_{(s)} \nabla_{\mu}\beta^{a} - \delta\Theta^{\mu}{}_{a}\big|_{(a)} (\nabla_{\mu}\beta^{a} - \beta\mu_{\mu}{}^{a}) + O(\partial^{3})$$

- Step 6: Second law of thermodynamics $\partial_{\mu} s^{\mu} \geq 0$
  $\Rightarrow$ First-order constitutive relations ($\Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}$):

$$\delta\Theta^{\mu}{}_{a}\big|_{(s)} = -\left[\eta\left((\Delta^{\mu\nu}\Delta_{ab} + \Delta^{\mu}_{b}\Delta^{\nu}_{a}) - \frac{2}{3}\Delta^{\mu}_{a}\Delta^{\nu}_{b}\right) + \zeta\Delta^{\mu}_{a}\Delta^{\nu}_{b}\right]\nabla_{\nu} u^{b}$$

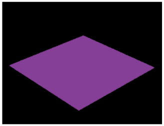
$$\delta\Theta^{\mu}{}_{a}\big|_{(a)} = -\frac{1}{2}\eta_{s}(\Delta^{\mu\nu}\Delta_{ab} - \Delta^{\mu}_{b}\Delta^{\nu}_{a})(\omega_{\nu}{}^{b} - \mu_{\nu}{}^{b})$$

  with $\eta \geq 0$ shear, $\zeta \geq 0$ bulk, and $\eta_{s} \geq 0$ rotational viscosities.

- The conservation-law equations turn to spin hydro equations. An equation of state $p = p(\epsilon, \sigma_{a})$ should be input to close the equations.

Linearized spin hydrodynamics

- Perturbation about global static thermal equilibrium

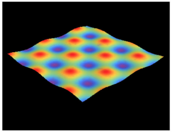


$\epsilon = \epsilon_0$

$u^\mu = (1, \mathbf{0})$

$\sigma^a = 0$

perturb



$\epsilon = \epsilon_0 + \delta\epsilon$

$u^\mu = (1, \mathbf{0}) + \delta u^\mu$

$\sigma^a = 0 + \delta\sigma^a$

$$\partial_0 \delta\epsilon + \partial_i \delta\pi^i = 0,$$

$$\partial_0 \delta\pi_i + c_s^2 \partial_i \delta\epsilon - \gamma_\parallel \partial_i \partial^j \delta\pi_j - (\gamma_\perp + \gamma_s)(\delta_i^j \nabla^2 - \partial_i \partial^j)\delta\pi_j + \frac{1}{2}\Gamma_s \varepsilon_{0ijk} \partial^j \delta\sigma^k = 0,$$

$$\partial_0 \delta\sigma_i + \Gamma_s \delta\sigma_i + 2\gamma_s \varepsilon_{0ijk} \partial^j \delta\pi^k = 0$$

where we introduced a set of static/kinetic coefficients as

$$c_s^2 \equiv \frac{\partial p}{\partial \epsilon}, \quad \gamma_\parallel \equiv \frac{1}{\epsilon_0 + p_0}\left(\zeta + \frac{4}{3}\eta\right), \quad \gamma_\perp \equiv \frac{\eta}{\epsilon_0 + p_0},$$

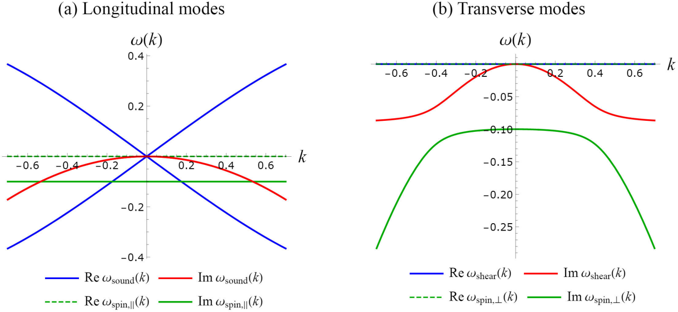
$$\chi_s \delta_{ij} \equiv \frac{\partial \sigma_i}{\partial \mu^j}, \quad \gamma_s \equiv \frac{\eta_s}{2(\epsilon_0 + p_0)}, \quad \Gamma_s \equiv \frac{2\eta_s}{\chi_s}$$

- By diagonalizing these coupled linear equations, one obtains the dispersion relations of (quasi-)hydro modes.

Linearized spin hydrodynamics

- Dispersion relations (Hattori *et al.* 1901.06615, Hongo *et al.* 2107.14231)

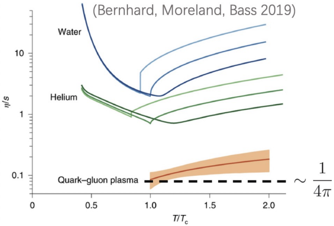
$$\begin{cases} \bullet \text{ One pair of sound modes}: \ \omega_{\text{sound}}(\boldsymbol{k}) = \pm c_s|\boldsymbol{k}| - \frac{i}{2}\gamma_{\|}\boldsymbol{k}^2 + O(\boldsymbol{k}^3), \\ \bullet \text{ One longitudinal spin mode}: \ \omega_{\text{spin},\|}(\boldsymbol{k}) = -i\Gamma_s, \\ \bullet \text{ Two shear modes}: \ \omega_{\text{shear}}(\boldsymbol{k}) = -i\gamma_\perp \boldsymbol{k}^2 + O(\boldsymbol{k}^4), \\ \bullet \text{ Two transverse spin modes}: \ \omega_{\text{spin},\perp}(\boldsymbol{k}) = -i\Gamma_s - i\gamma_s\boldsymbol{k}^2 + O(\boldsymbol{k}^4). \end{cases}$$

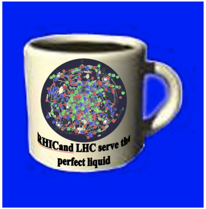


- Mode mixing between shear and transverse spin mode: One gradient can affect two modes.

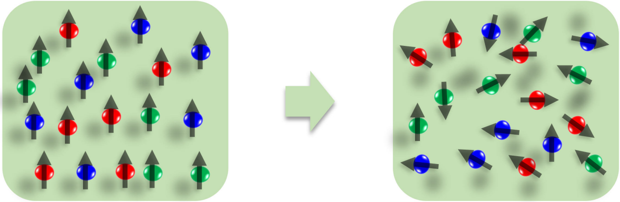
Transport coefficients

- Viscosities (Wilson coefficients) are characteristic parameters of matter. For example, shear viscosity of QGP:





- The new rotational viscosity η_s characterized local spin relaxation:



Kubo formulas for rotational viscosity

- Spin hydrodynamic retarded spin-spin correlator

$$\tilde{G}_{\mathrm{R}}^{\sigma^i\sigma^j}(\omega,\boldsymbol{k}) = \frac{i\chi_s\Gamma_s + \cdots}{\omega + i\Gamma_s + O(\boldsymbol{k}^2)}\delta^{ij}$$

- Recall the scale separation condition:

$$\delta\Theta^{\mu}{}_{a}\Big|_{(a)} = \begin{cases} -(\eta_s)^{\mu}{}_{a}{}^{\nu}{}_{b}(\nabla_\nu u^b - \mu_\nu{}^b) & \text{when} \quad \Gamma_s \ll \omega \ll \Gamma, \\ 0 & \text{when} \quad \omega \ll \Gamma_s \end{cases}$$

  The spin hydrodynamic spin-spin correlator gives:

$$\omega\tilde{G}_{\mathrm{R}}^{\sigma^i\sigma^j}(\omega,\boldsymbol{k}=0) = \frac{i\chi_s\omega\Gamma_s}{\omega + i\Gamma_s}\delta^{ij} \xrightarrow{\Gamma_s\ll\omega\ll\Gamma} 2i\eta_s$$

- Field theoretical Kubo formula for rotational viscosity

$$\eta_s = \frac{1}{2}\lim_{\Gamma_s\ll\omega\ll\Gamma}\omega\,\mathrm{Im}\tilde{G}_{\mathrm{R}}^{\sigma^z\sigma^z}(\omega,\mathbf{0}) = 2\lim_{\Gamma_s\ll\omega\ll\Gamma}\frac{1}{\omega}\mathrm{Im}\tilde{G}_{\mathrm{R}}^{\Theta^{xy}_{(a)}\Theta^{xy}_{(a)}}(\omega,\mathbf{0})$$

- Another Kubo formula at $\omega \to 0$ can also be derived:

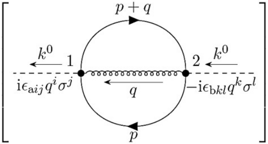
$$\frac{\chi_s^2}{2\eta_s} = \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\tilde{G}_{\mathrm{R}}^{\sigma^z\sigma^z}(\omega,\mathbf{0})$$

Spin relaxation rate at heavy quark limit

- Using spin-spin correlator to calculate η_s is tedious, it is much easier to use source-source correlator at $\Gamma_s \ll \omega \ll \Gamma$
- At heavy quark limit and leading-log approximation

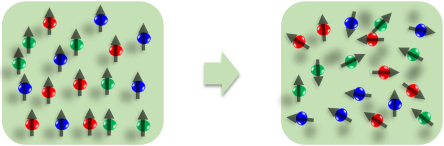
$$\mathcal{L} = -M\psi^\dagger\psi + \mathrm{i}\psi^\dagger D_0\psi - \frac{1}{2M}(\boldsymbol{D}\psi)^\dagger \cdot \boldsymbol{D}\psi + \frac{g}{2M}\psi^\dagger(\boldsymbol{B}\cdot\boldsymbol{\sigma})\psi + \mathcal{L}_{\text{gluon}} + \mathcal{O}(1/M^2)$$

$$\chi_s\Gamma_s = \frac{1}{6T}\delta^{\mathrm{ab}} G_{12}^{\Theta_\mathrm{a}\Theta_\mathrm{b}}(\Gamma_s \ll k^0 \ll \Gamma) = \frac{g^2}{12M^2T}\delta^{\mathrm{ab}} \lim_{k^0\to 0} \mathrm{Tr} \left[\quad \right]$$



- Spin relaxation rate Γ_s for heavy quark (Hongo *et al.* 2201.12390)

$$\Gamma_s \equiv \frac{2\eta_s}{\chi_s} = \frac{N_c^2-1}{2N_c}\frac{g^2 m_D^2 T}{6\pi M^2}\ln\frac{1}{g}$$



When strong vorticity is present

- Rotating fluid could be at global thermal equilibrium

(at thermal equilibrium)

$$\frac{dN_s}{d\boldsymbol{p}} \sim e^{-(H_0 - \boldsymbol{\omega}\cdot\boldsymbol{S})/T}$$

$$S = \frac{N_\uparrow - N_\downarrow}{2(N_\uparrow + N_\downarrow)} \sim \frac{\omega}{4T}$$

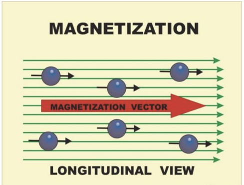
- Power counting scheme II (Cao *et al.* 2205.08051):

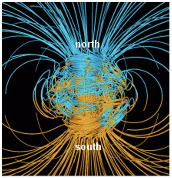
$$\{\beta, u^a, \mu^a, \sigma_a, \omega_{\mu\nu} \equiv \nabla_{[\mu} u_{\nu]}\} = O(\partial^0)$$

- Anisotropy in ideal constitutive relation: Gyrohydrodynamics

$$\Theta^{\mu}{}_{a(0)} = \epsilon u^\mu u_a + p_\perp \Delta^\mu_a + (p_\parallel - p_\perp)\hat{\omega}^\mu \hat{\omega}_a$$

  - Similar to magnetohydrodynamics





When strong vorticity is present

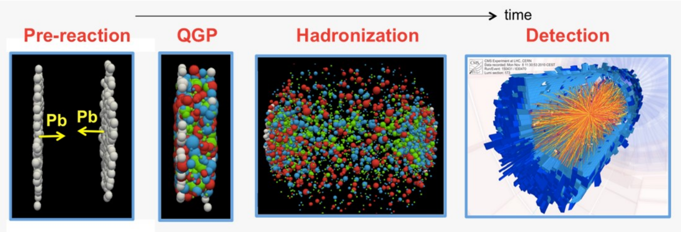
- Anisotropy in dissipative constitutive relation (Cao *et al.* 2205.08051):

$$\delta\Theta^{(\mu\nu)} = -T\eta^{\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)} - T\xi^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} - \beta\mu_{\rho\sigma}),$$

$$\delta\Theta^{[\mu\nu]} = -T\gamma^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} - \beta\mu_{\rho\sigma}) - T\xi'^{\,\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)}$$

  - 14 viscosities: 3 bulk, 4 shear, 3 rotational, and 4 cross viscosities
  - Among them, 7 are Hall-type viscosities.
  - Cross viscosites appears also in liquid crystals.

- When applied to HICs, how to convert velocity, temperature, spin chemical potential into hadron observables?

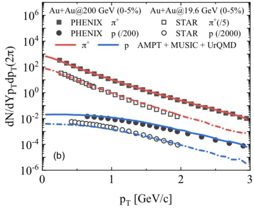


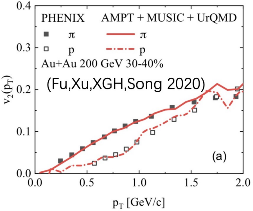
Spin Cooper-Frye formula

Freeze-out of particle number

- Cooper-Frye type formula converts hydro outcomes to momentum space distributions.

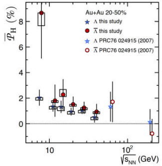
$$N(p) = \int d\Xi_\mu \frac{p^\mu}{E_p} f(T(x), u^\mu(x), \mu(x))$$

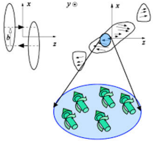




- We need a similar formula to connect spin hydro with obervables.

$$\bar{S}^\mu(p) \qquad \Leftarrow \qquad T(x), u^\alpha(x), \mu(x), \color{red}{\mu_{\alpha\beta}(x)}$$





Freeze-out of spin polarization

- Such a formula at local equilibrium can be obtained via e.g. kinetic theory or local Gibbs density operator with same type of pseudo-gauge as spin hydro

$$\hat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left\{ -\int d\Xi_\mu(y)\left[\hat{\Theta}^{\mu\nu}(y)\beta_\nu(y) - \frac{1}{2}\hat{\Sigma}^{\mu\rho\sigma}(y)\mu_{\rho\sigma}(y)\right]\right\}$$

- Spin Cooper-Frye formula for Dirac fermions at local equilibrium (Buzzegoli 2109.12084, Liu and Huang 2109.15301)

$$\bar{S}_\mu(p) = \bar{S}_{5\mu}(p) - \frac{1}{8\int d\Xi\cdot p\ \ n_F}\int d\Xi\cdot p\frac{n_F(1-n_F)}{E_p} \times\left\{\epsilon_{\mu\nu\alpha\beta}p^\nu\mu^{\alpha\beta} + 2\frac{\varepsilon_{\mu\nu\rho\sigma}p^\rho n^\sigma}{p\cdot n}\left[p_\lambda(\xi^{\nu\lambda}+\Delta\mu^{\nu\lambda}) + \partial^\nu\alpha\right]\right\}$$

  - Here, $\xi_{\mu\nu} = \partial_{(\mu}\beta_{\nu)}$ is thermal shear and $\Delta\mu_{\alpha\beta} = \mu_{\alpha\beta} + \partial_{[\mu}\beta_{\nu]}$ is the difference between spin chemical potential and thermal vorticity.

Freeze-out of spin polarization

- Spin Cooper-Frye formula for Dirac fermions

$$\bar{S}_\mu(p) = \bar{S}_{5\mu}(p) - \frac{1}{8\int d\Xi\cdot p\ \ n_F}\int d\Xi\cdot p\frac{n_F(1-n_F)}{E_p}$$
$$\times\left\{\epsilon_{\mu\nu\alpha\beta}p^\nu\mu^{\alpha\beta} + 2\frac{\varepsilon_{\mu\nu\rho\sigma}p^\rho n^\sigma}{p\cdot n}\left[p_\lambda(\xi^{\nu\lambda}+\Delta\mu^{\nu\lambda})+\partial^\nu\alpha\right]\right\}$$

  - $\bar{S}_5^\mu$ is the polarization induced by finite chirality (Liu *et al.* 2002.03753, Shi *et al.* 2008.08618, Buzzegoli *et al.* 2009.13449, Gao 2105.08293)
  - When $\Delta\mu_{\alpha\beta}=0$, namely, when spin chemical potential is given by thermal vorticity. It goes to previous results (Liu and Yin 2103.09200, Becattini *et al.* 2103.10917)
  - When global equilibrium is reached $\Delta\mu_{\alpha\beta}=0=\xi_{\alpha\beta}$, it goes to previous results (Becattini *et al.* 1303.3431, Fang *et al.* 1604.04036, Liu *et al.* 2002.03753)
  - It is accurate at $O(\partial)$.
  - n^μ is a unit frame vector to specify helicity.
  - Out of local equilibrium, collisions induce additional contribution (Lin and Wang 2206.12573)
- With this formula, we can convert spin hydro into momentum space spin polarization.

Summary

Summary

- Spin polarization and spin transport are common in a number of physical systems.
- It is possible to formulate a (quasi-)hydrodynamic theory for spin transports.
- The first-order dissipative spin hydrodynamics has been constructed.
- The Cooper-Frye type spin polarization formula is obtained.

- Numerical spin hydrodynamics.
- Spin Cooper-Frye formula for vector mesons.
- Higher-order and causal spin hydrodynamics.
- Anomalous spin hydrodynamics.
- ......

Thank you!