

Going to the light-front with Contour Deformations

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Light Cone 2022 – September 21st 2022

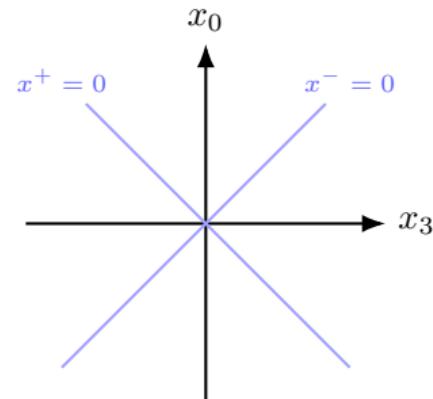
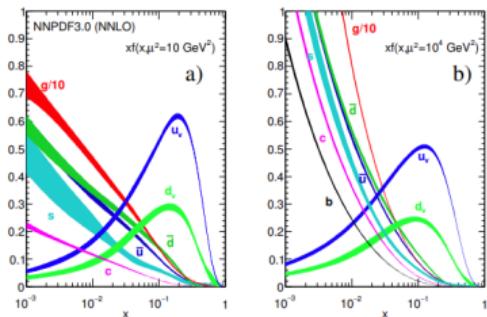


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OE,FCT-Portugal, CERN/FIS-PAR/0023/2021

Hadrons on the Light Front

Focus: Hadrons on the light front, $x^+ = 0$.

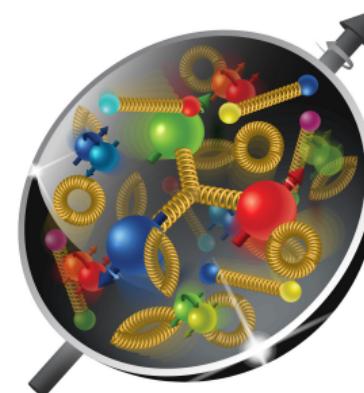
- Natural frame for defining parton distribution functions: PDFs, TMDs, ...



- Future: COMPASS/AMBER @ CERN
EIC @ Brookhaven National Laboratory.

(AMBER: arXiv:1808.00848)

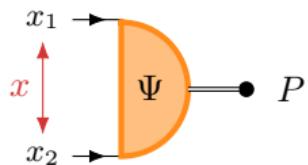
(EIC: Eur. Phys. J. A 52.9 (2016))



Hadronic quantities

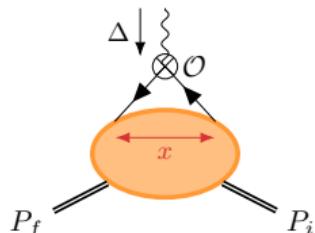
- Bethe-Salpeter Wavefunction

$$\langle 0 | T\Phi(x)\Phi(0) | P \rangle$$



- Generic Correlator

$$\langle P_f | T\Phi(x)\mathcal{O}\Phi(0) | P_i \rangle$$



► With $x^+ = x^0 + x^3$, $x^- = x^0 - x^3$, $\vec{x}_\perp = \{x^1, x^2\}$.

BSWF
Bethe-Salpeter Wavefunction

$$\langle 0 | T\bar{\psi}(x)\mathcal{O}\psi(0) | P \rangle$$

$$\int dq^-$$

LFWF
Light-Front Wavefunction

$$\int d^2q_\perp$$

PDA
Parton distribution amplitude

$$\mathcal{G}(x, P, \Delta = 0)$$

$$\langle P | T\bar{\psi}(x)\mathcal{O}\psi(0) | P \rangle$$

$$\int dq^-$$

TMD
Transverse Momentum Distribution

$$\int d^2q_\perp$$

PDF
Parton Distribution Function

$$\mathcal{G}(x, P, \Delta)$$

$$\langle P_f | T\bar{\psi}(x)\mathcal{O}\psi(0) | P_i \rangle$$

$$\int dq^-$$

GTMD
Generalized Transverse Momentum Distribution

$$\int d^2q_\perp$$

GPD
Generalized Parton Distribution

(Lorce, Pasquini, Vanderhaeghen; 2011)

Bethe-Salpeter Wavefunction

- The Bethe-Salpeter Wavefunction (BSWF) appears as the residue of a correlation function $G(p)$:

$$\Psi(x, P) = \langle 0 | T\phi(0)\phi(x) | P \rangle$$

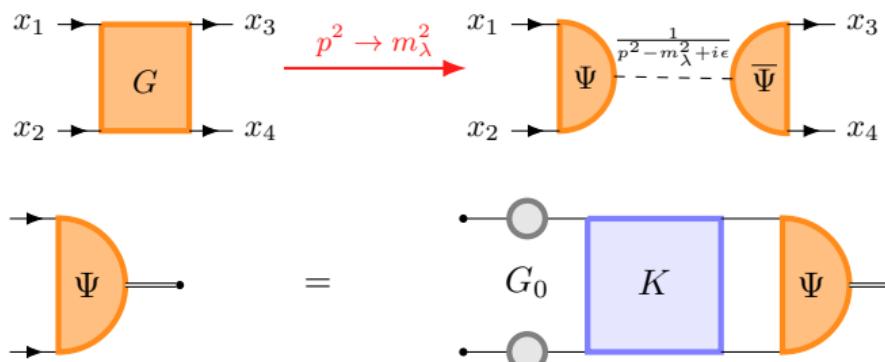
$$\Psi(k, P) = \int d^4x e^{-ik \cdot x} \Psi(x, P)$$

- Determined by the Bethe-Salpeter Equation:

$$\Psi = \mathbf{G}_0 \mathbf{K} \Psi$$

G₀ product of the dressed propagators;

K interaction kernel between the two particles.



Light-Front Wavefunction

- ▶ **Light-Front Wavefunction (LFWF):** Fourier transform of the BSWF at $x = \lambda n + \vec{x}_\perp$, with n along the light front.

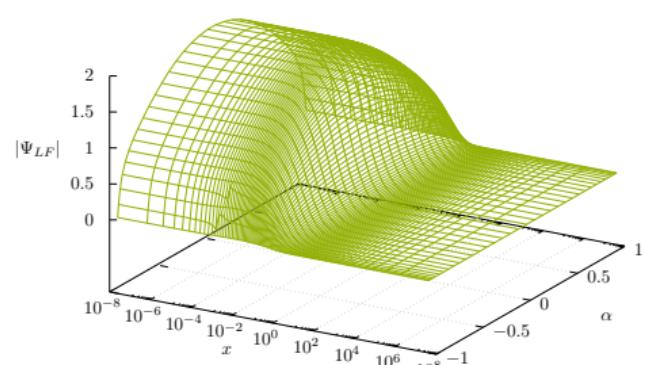
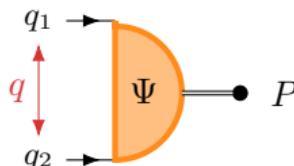
$$\Psi_{LF}(q^+, \vec{q}_\perp, P) = \\ = \mathcal{N} \int dq^- \Psi(q^-, q^+, \vec{q}_\perp, P),$$

$\xi = \frac{q_1^+}{P^+} = 2\alpha - 1$ is the longitudinal momentum fraction

$q = k + \frac{\alpha}{2}P$ is the relative momentum (with $k^+ = 0$).

- ▶ **Parton Distribution Amplitude (PDA):** Integration of the Ψ_{LF} over q_\perp .

$$\phi(\alpha, P) = \int d^2 q_\perp \Psi_{LF} \left(\frac{\alpha}{2} P^+, \vec{q}_\perp, P \right).$$

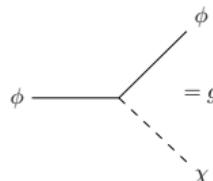


Scalar toy model

Scalar toy model:

ϕ of mass m

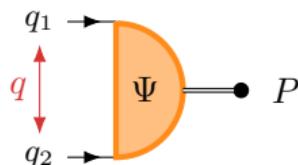
χ of mass μ



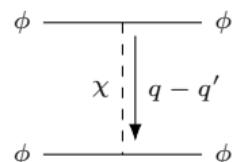
► \mathbf{G}_0 — We use tree level propagators

$$\mathbf{G}_0 = \frac{1}{q_1^2 + m^2} \frac{1}{q_2^2 + m^2}$$

► \mathbf{K} — We assume a single scalar exchange

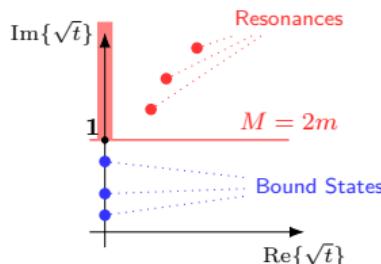


$$\mathbf{K} = \frac{g^2}{(q - q')^2 + \mu^2}$$



► The BSWF is a function of the kinematic invariants:

$$-M^2 = \frac{P^2}{4m^2} = \textcolor{red}{t} \quad \frac{k^2}{m^2} = \textcolor{blue}{x} \quad \textcolor{green}{\omega} = \hat{k} \cdot \hat{P}$$



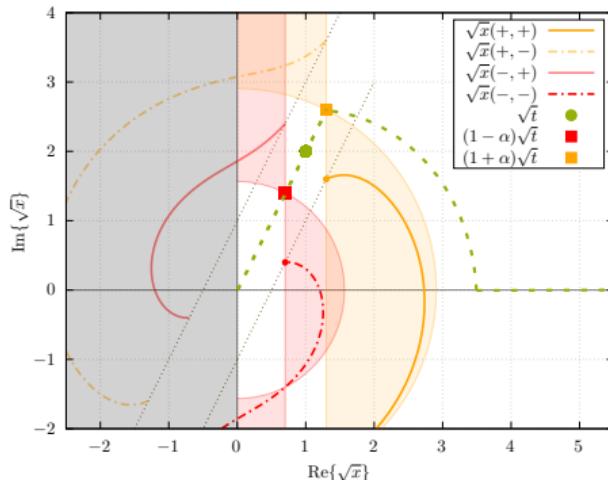
$$\begin{aligned} \psi(\textcolor{blue}{x}, \textcolor{green}{\omega}, \textcolor{red}{t}, \alpha) &= \frac{m^4}{(2\pi)^3} \frac{1}{2} \int_0^\infty dx' x' \int_{-1}^1 d\omega' \sqrt{1 - \omega'^2} \mathbf{G}_0(x', \omega', \textcolor{red}{t}, \alpha) \\ &\times \int_{-1}^1 dy \mathbf{K}(\textcolor{blue}{x}, \textcolor{green}{\omega}, x', \omega', y) \psi(x', \omega', \textcolor{red}{t}, \alpha) \end{aligned}$$

Analytic Structure

$$\mathbf{G}_0 = \frac{1}{q_1^2 + m^2} \frac{1}{q_2^2 + m^2}$$

- ▶ Poles when $q_{1/2}^2 = -m^2$
- ▶ Integration in $\omega \implies$ branch cuts in complex x' plane:

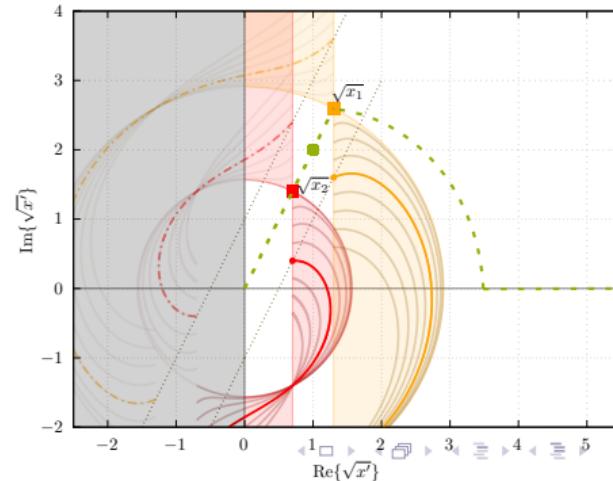
$$\sqrt{x}_\pm^\lambda = \mp(1 \pm \alpha)\sqrt{t} \left[\omega + i\lambda \sqrt{1 - \omega^2 + \frac{1}{(1 \pm \alpha)^2 t}} \right]$$



$$\mathbf{K} = \frac{g^2}{(q - q')^2 + \mu^2}$$

- ▶ Poles when $(q - q')^2 = -\mu^2$.
- ▶ Branch cuts in complex x' plane:

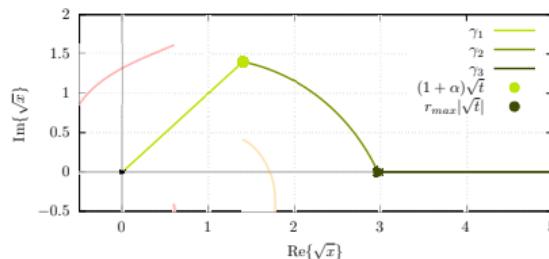
$$\sqrt{x'} = \sqrt{x} \left(\Omega \pm i \sqrt{1 - \Omega^2 + \frac{\beta^2}{x}} \right)$$



Integration Path

► Constraints:

1. Must go through $(1 + |\alpha|)\sqrt{t}$.
2. $\text{Re}\{\sqrt{x}\}$ and $|\sqrt{x}|$ must always increase.

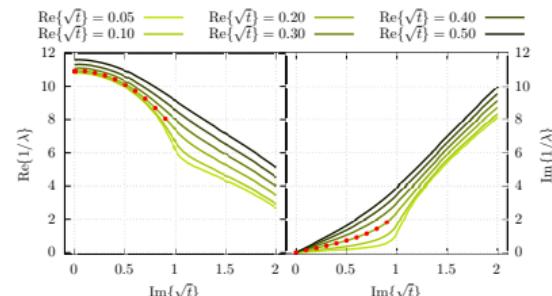
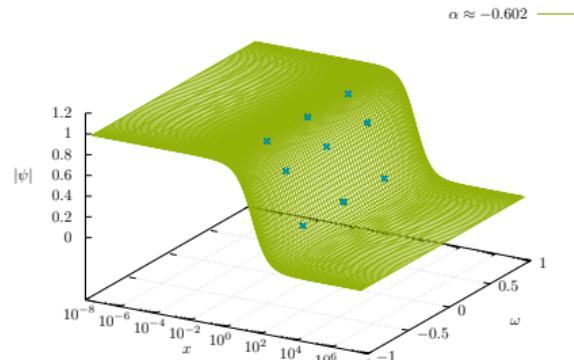


γ_1 — Line from the origin to $(1 + |\alpha|)\sqrt{t}$.

γ_2 — Return to the real axis, with increasing radius.

γ_3 — $\sqrt{x} \rightarrow \infty$ on the real axis.

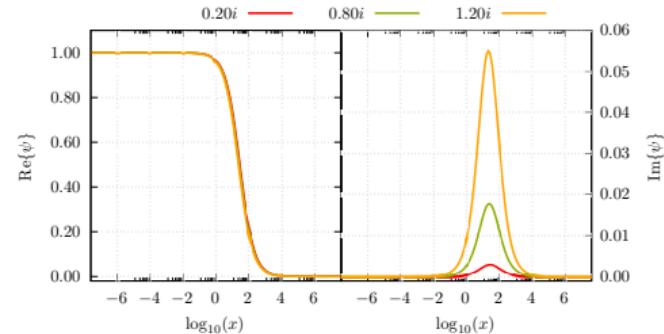
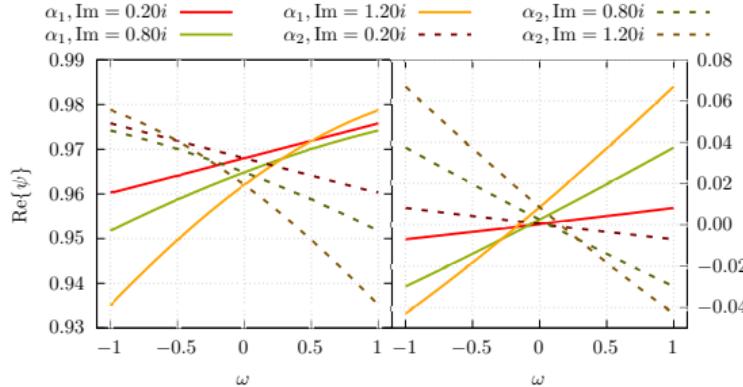
- If the cuts don't cross the real axis \implies no deformation needed.



(Kusaka, Williams; 1995); (Sauli, Adam, Jr; 2003)

(Karmanov, Carbonell; 2006); (Frederico, Salme, Viviani; 2014)

BSWF Results



- ▶ Small ω and α dependence.
- ▶ Symmetry for the combined transformation $\alpha \rightarrow -\alpha$ and $\omega \rightarrow -\omega$.
- ▶ Approximately a monopole:

$$\psi \approx \frac{1}{q^2 + \gamma}$$

Light-Front Wavefunction

- The LFWF is defined as:

$$\begin{aligned}\Psi_{LF}(\alpha, k_\perp, P) \\ = \mathcal{N} \int dq^- \Psi(q, P)|_{q^+ = \frac{\alpha}{2}P^+, q_\perp = k_\perp}\end{aligned}$$

- In our kinematic variables:

$$q^- = -\frac{2m^2}{P^+} (2\sqrt{x}\sqrt{t}\omega + \alpha t)$$

α and $x = \frac{k^2}{m^2} = \frac{k_\perp^2}{m^2}$ and t are external variables

- Need the BSWF in $\omega \in (-\infty, \infty)$.
- We use the Schlessinger method for analytic continuation:

$$R(\omega) = \frac{f(\omega_1)}{1 + \frac{a_1(\omega - \omega_1)}{1 + \frac{a_2(\omega - \omega_2)}{1 + \frac{a_3(\omega - \omega_3)}{\dots}}}}$$

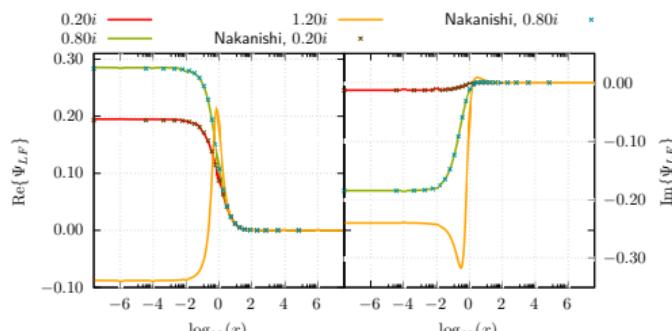
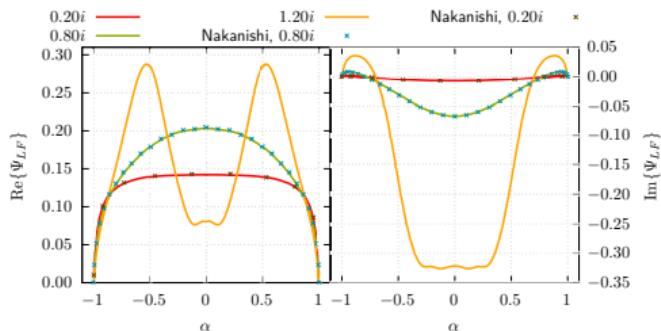
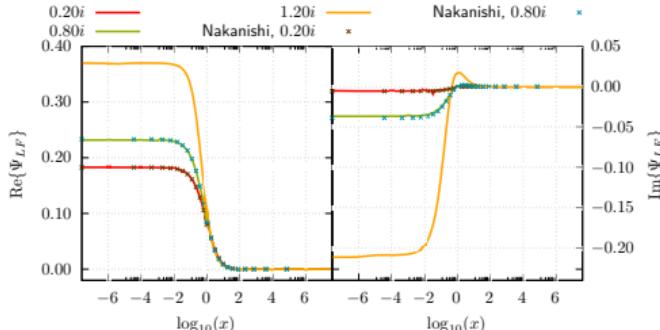
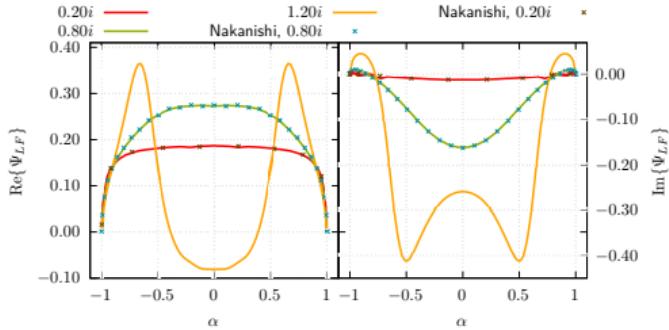
(L. Schlessinger, 1968) (Tripolt et al., 2019) (D. Binosi, R-A. Tripolt; 2019)

- $\{a_i\}$ obtained by imposing $R(\omega_i) = f(\omega_i)$

Definition of the LFWF

$$\Psi_{LF}(\alpha, x, t) = \mathcal{N} \frac{2\sqrt{x}\sqrt{t}}{i\pi} \int_{-\infty}^{\infty} d\omega \Psi(x, \omega, t, \alpha)$$

LFWF: α and x dependance

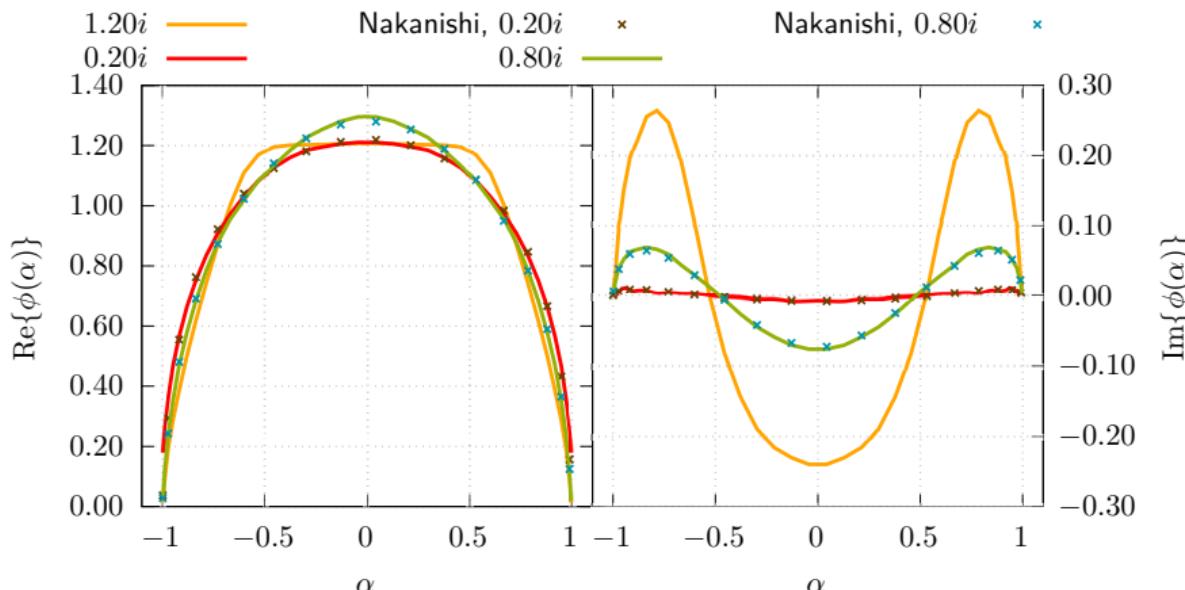


- ▶ Symmetric in α .
- ▶ Vanishes at $\alpha = \pm 1$.

PDA: α dependance

- In our variables:

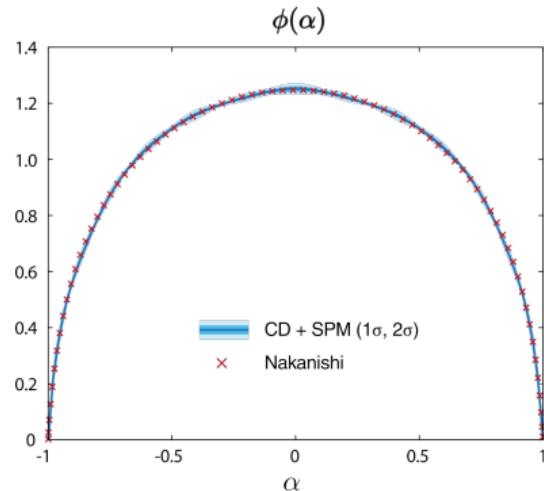
$$\phi(\alpha) = \int_0^\infty dx \Psi_{LF}(x, \alpha, t)$$



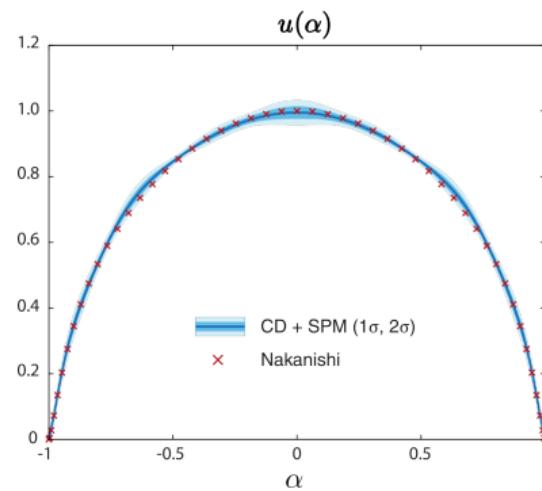
- Symmetric in α .
- Vanishes at $\alpha = \pm 1$.

PDA: back on the \sqrt{t} imaginary axis

- ▶ Use SPM to get to $\sqrt{t} = bi$, $b \in \mathbb{R}$ (cuts prevent direct evaluation!).
- ▶ Expand PDA in Chebyshev-U: $\phi(\alpha) = (1 - \alpha^2) \sum \phi_n U_n(\alpha)$.
- ▶ Analytic continuation of ϕ_n with SPM.



$$\phi(\alpha) \propto \int dx \Psi_{LF}(x, \alpha)$$



$$u(\alpha) \propto \int dx |\Psi_{LF}(x, \alpha)|^2$$

- ▶ **Input data:** $N \in [10, 50]$ different points starting with $\text{Re } \sqrt{t} = 0.1$, with steps in N of 2.

Unequal masses

- ▶ Consider two ϕ of different masses:

$$m_1 = m(1 + \varepsilon) \quad m_2 = m(1 - \varepsilon)$$

$$\frac{m_1}{m_2} = \frac{1 + \varepsilon}{1 - \varepsilon} \quad 2m = m_1 + m_2$$

- ▶ $\varepsilon \in [-1, 1]$ sets the ratio of the masses
- ▶ \mathbf{G}_0 is now:

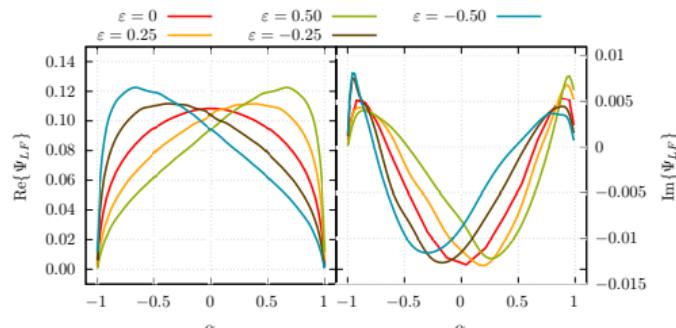
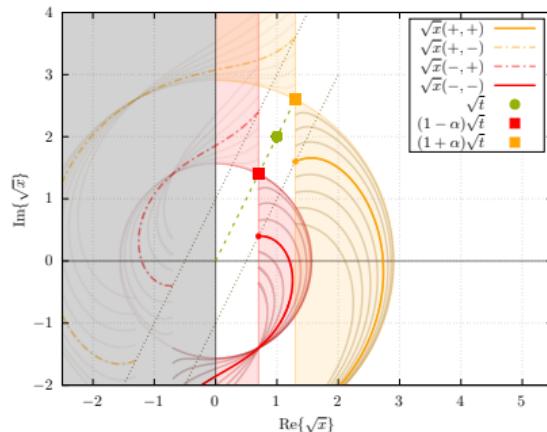
$$\mathbf{G}_0 = \frac{1}{q_1^2 + m_1^2} \frac{1}{q_2^2 + m_2^2}$$

- ▶ Cuts in x :

$$\sqrt{x}_{\pm}^{\lambda} = \mp(1 \pm \alpha)\sqrt{t}$$

$$\times \left[\omega + i\lambda \sqrt{1 - \omega^2 + \frac{1}{t}} \left(\frac{1 \pm \varepsilon}{1 \pm \alpha} \right)^2 \right]$$

- ▶ Integration path still works
- ▶ ε adds skewness



Complex Conjugate Masses

- Also consider complex conjugate mass poles:

$$D_\phi(q, m) = \frac{1}{2} \left(\frac{1}{q^2 + m^2} + \frac{1}{q^2 + (m^*)^2} \right)$$

- $m^2 \rightarrow m^2(1 + i\delta)$, with $m^2, \delta \in \mathbb{R}_+$
- \mathbf{G}_0 becomes:

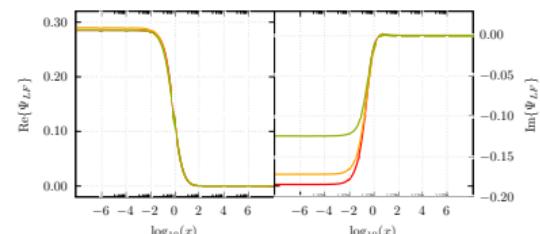
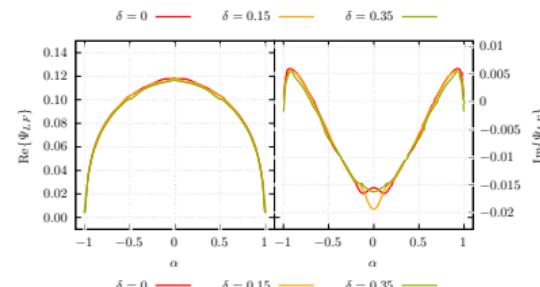
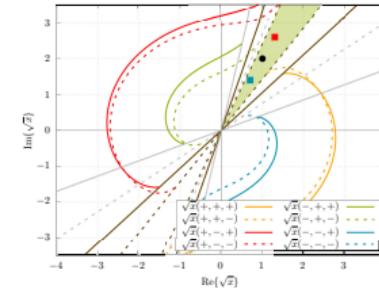
$$\mathbf{G}_0 = D_\phi(q_1, m) D_\phi(q_2, m)$$

- There are now 8 cuts:

$$\sqrt{x}_\pm^{\{\lambda, \nu\}} = \mp(1 \pm \alpha)\sqrt{t}$$

$$\times \left[\omega + i\lambda \sqrt{1 - \omega^2} + \frac{1}{t} \frac{1 + \nu i\delta}{(1 \pm \alpha)^2} \right]$$

- For $\delta < \delta_{crit}$, contour deformation always possible.



Advertisement:

- More details can be found in:

Editors' Suggestion

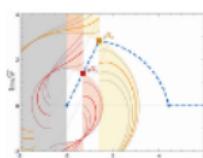
PDF

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Going to the light front with contour deformations

Gernot Eichmann, Eduardo Ferreira, and Alfred Stadler

Phys. Rev. D **105**, 034009 (2022) – Published 10 February 2022



This work introduces a new method for computing correlation functions on the light front. Starting from Bethe-Salpeter equations, the authors present calculations using contour deformation and analytic continuation, finding good agreement with established methods. This approach shows promise as an efficient and generalizable technique with applications to parton distribution functions and related constructions.

Show Abstract 

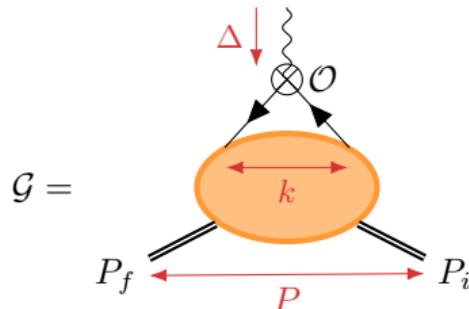
G. Eichmann, EF, A. Stadler; Phys. Rev. D 105, 034009 (2022)

Hadronic Matrix Elements

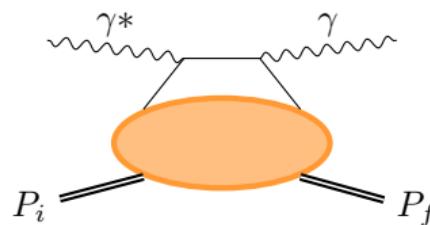
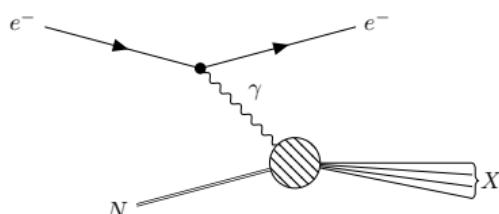
- **Focus:** Understanding hadronic composition and interactions:

$$g_{\alpha\beta\delta\dots}^{\mu\nu\rho\dots} = \langle P_f | T\bar{\psi}(x)\mathcal{O}_{\alpha\beta\delta\dots}^{\mu\nu\rho\dots}\psi(0) | P_i \rangle$$

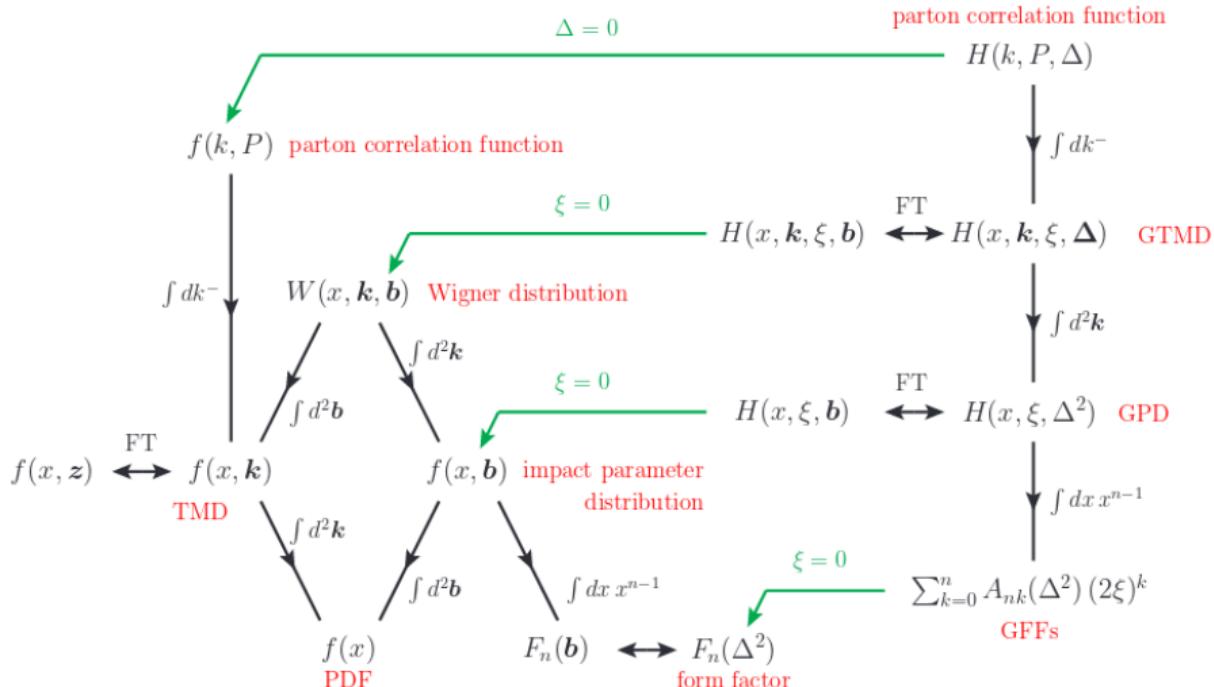
$$g_{\alpha\beta\delta\dots}^{\mu\nu\rho\dots} = \sum_j H_j(P, k, \Delta) (\tau_j)_{\alpha\beta\delta\dots}^{\mu\nu\rho\dots} (P, k, \Delta)$$



- These interactions probe the partonic distributions and interactions of Hadrons
- **Main idea:** Processes = Hard scattering \times Hadronic structure functions
(Review: Section 2 of Belitsky, Radyushkin, 2005), ...



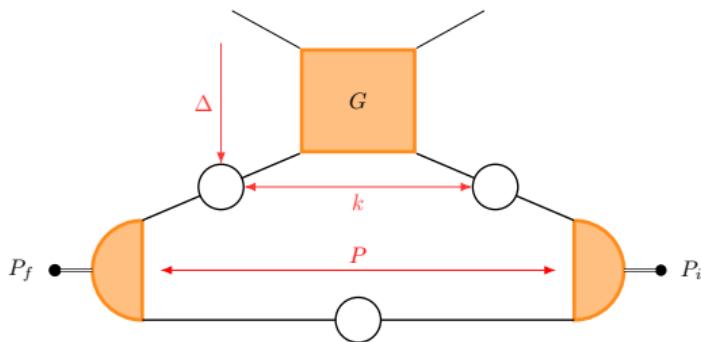
Moving further



(Picture: Diehl, 2016); (Diehl, 2003); (Meissner, Goeke, Metz, Schlegel; 2008); (Meissner, Metz, Schlegel; 2009), other talks this week, ...

Writing the hadronic correlation

- We write the hadronic correlation using elements calculated via functional methods:



- G is the four-point quark correlation function, calculated with scattering equation.
- The quark propagator is calculated via quark DSE.
- The BSWF is calculated via the meson BSE.

(Mezrag, arXiv:1507.05824); (Diehl, Gousset, 1998); (Tiburzi, Miller, 2003);
(Mezrag, Chang, Moutarde, Roberts, Rodríguez-Quintero, Sabatié, Schmidt, 2015);
(Cloët, Roberts, 2018), many many others, ...

$$G^{[\Gamma]}(P, k, \Delta) = \frac{1}{2} \text{Tr} \left[\int dk^- \int \frac{d^4 z}{2\pi^4} e^{ik \cdot z} \langle P_f | \bar{\psi}(z) \mathcal{W} \Gamma \psi(0) | P_i \rangle \right]$$

- Partonic distributions are calculated by integrating the correlator in k^- and taking appropriate traces.

Conclusions

- ▶ We explored a new way to calculate the LFWFs and PDAs.
- ▶ Very good agreement with the established Nakanishi method.
- ▶ We can also tackle extensions to the scalar toy model: unequal masses and complex conjugate mass poles in the propagators — features of many QCD calculations.
- ▶ We can calculate beyond the $M^2 = 4m^2$ threshold.
- ▶ The contour deformation method can also be applied to other correlation functions — just need a region free of singularities where the path can be deformed.
- ▶ We have also explored a way forward for the calculation of partonic structure functions.

Thank you!

BACKUP

Nakanishi Method

- ▶ BSWF defined from a smooth weight function $g(x, \alpha)$.

$$\Psi(q, P) = \frac{1}{m^4} \int_0^\infty dx' \int_{-1}^1 d\alpha' \frac{g(x', \alpha')}{[\kappa + 1 + x' + (1 - \alpha'^2)t]^3}, \quad \kappa = \frac{1}{m^2} \left(q - \frac{\alpha'}{2} P \right)^2.$$

- ▶ Light front quantities obtained from the weight function g , for example LFWF:

$$\Psi_{LF} = \frac{\mathcal{N}}{m^2} \int_0^\infty dx' \frac{g(x', \alpha)}{[x' + 1 + x + (1 - \alpha^2)t]^2}$$

- ▶ The BSE can be rewritten for g :

$$\int_0^\infty dx' \frac{g(x', \alpha)}{[x' + 1 + x + (1 - \alpha^2)t]^2} = c \int_0^\infty dx' \int_{-1}^1 d\alpha' V(x, x', \alpha, \alpha') g(x', \alpha')$$

$$V(x, x', \alpha, \alpha') = \frac{K(x, x', \alpha, \alpha') + K(x, x', -\alpha, -\alpha')}{2[x + 1 + (1 - \alpha^2)t]}$$

$$K(x, x', \alpha, \alpha') = \int_0^1 dv \frac{\theta(\alpha - \alpha')(1 - \alpha)^2}{[v(1 - \alpha)(x' + 1 + (1 - \alpha'^2)t) + (1 - v)C]^2}$$

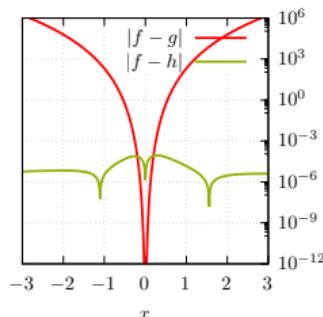
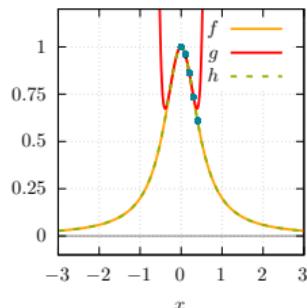
$$C = (1 - \alpha')(1 + x + (1 - \alpha^2)t) + (1 - \alpha) \left(\frac{\beta}{v} + x' \right)$$

Schlessinger Point Method

- Numerical analytic continuation method:

$$R(\omega) = \frac{f(\omega_1)}{1 + \frac{a_1(\omega - \omega_1)}{1 + \frac{a_2(\omega - \omega_2)}{1 + \frac{a_3(\omega - \omega_3)}{\dots}}}}$$

- $\{a_i\}$ obtained by imposing $R(\omega_i) = f(\omega_i)$



- Recurrence relations:

$$R(\omega) = \frac{f(\omega_1)}{1 + \mathcal{Z}_1} = \frac{f(\omega_1)}{1 + \frac{a_1(\omega - \omega_1)}{1 + \frac{a_2(\omega - \omega_2)}{1 + \frac{a_3(\omega - \omega_3)}{\dots}}}} = \dots$$

$$\mathcal{Z}_k = \frac{a_k(\omega - \omega_k)}{1 + \mathcal{Z}_{k+1}} \Leftrightarrow \mathcal{Z}_{k+1} = \frac{a_k(\omega - \omega_k)}{\mathcal{Z}_k} - 1,$$
$$\omega = \omega_k \implies \mathcal{Z}_k = 0$$

$$f(\omega_2) = \frac{f(\omega_1)}{1 + a_1(\omega_2 - \omega_1)},$$

$$\mathcal{Z}_1 = \frac{f(\omega_1)}{f(\omega_2)} - 1 \Leftrightarrow a_1 = \frac{\mathcal{Z}_1}{\omega_2 - \omega_1},$$

Why not do one more iteration?

- ▶ Do *one more iteration* for a value of $\omega = W \in \mathbb{C}$, with the obtained Ψ

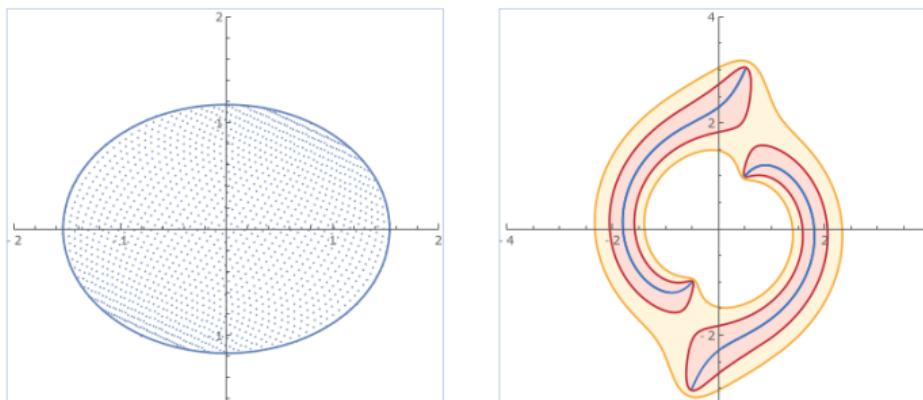
$$\Psi(x, W, t, \alpha) = \mathcal{N} \int_0^\infty dx' \int_{-1}^1 d\omega' \mathcal{K}(x, x', W, \omega') \Psi(x', \omega', t, \alpha)$$

- ▶ **Problem:** Kernel cuts will change

- ▶ For $\omega \in \mathbb{C}$, and $y, \omega' \in [-1, 1]$, Ω turns into a region bounded by the $r(\theta)$ ellipse, with $\omega = a + ib$ and $\sqrt{1 - \omega^2} = c + id$:

$$r(\theta) = \sqrt{a^2 + c^2} \sqrt{\cos^2 \theta + E^2 \sin^2 \theta} \quad E = \begin{cases} \frac{d^2}{a^2} & \alpha \neq 0 \\ \frac{b^2}{1+b^2} & \alpha = 0 \end{cases}$$

- ▶ Kernel cuts will eventually overlap



Cuts for complex conjugate mass poles

$$\text{Im}\{\sqrt{\tau}\} \text{Re}\{i\sqrt{1+i\delta}\} < \text{Im}\{i\sqrt{1+i\delta}\} \text{Re}\{\sqrt{\tau}\}.$$

