Light Cone 2022 conference

Non-perturbative insights into the properties of particles at finite temperature

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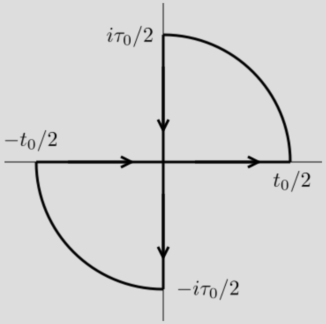
CRC–TR 211 Strong-interaction matter under extreme conditions

Local QFT beyond the vacuum

"Local QFT" → Define QFTs using a core set of physically-motivated assumptions, e.g. causality, Poincaré invariance, positive energy, etc.

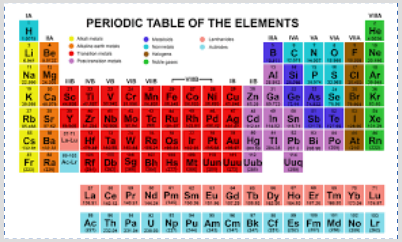
- This approach has led to numerous fundamental ***non-perturbative*** insights:

  → Relationship between Minkowski and Euclidean QFTs



  → *CPT* is a symmetry of *any* QFT

  → Connection between spin & particle statistics

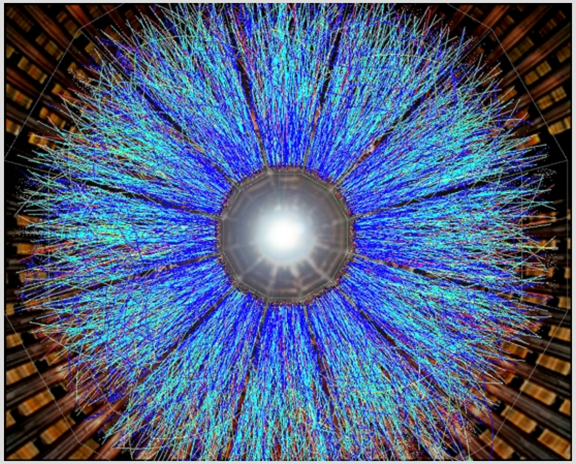


  → Existence of dispersion relations

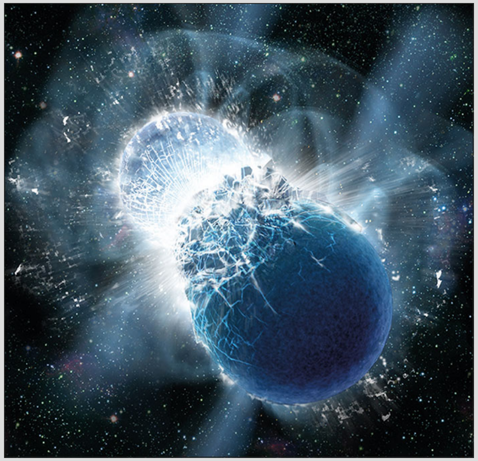
Local QFT beyond the vacuum

- But... local QFT only describes particle dynamics in the *vacuum* state

  $\rightarrow$ What about "extreme environments" where a system is in a state that is either hot, dense, or both?



[Brookhaven National Lab]



[Skyworks Digital Inc.]

- Understanding local QFT in such environments is essential, and yet has received relatively little attention. Some important progress was made by J. Bros and D. Buchholz for temperatures $T > 0$

  $\rightarrow$ See: [*Z. Phys. C* 55 (1992), *Ann. Inst. H.Poincare Phys.Theor.* 64 (1996), *Nucl. Phys. B* 429 (1994), Nucl. Phys. B 627 (2002)]

Non-perturbative implications

- By demanding fields to be **local** ($[\Phi(x),\Phi(y)]=0$ for $(x\text{-}y)^2< 0$) this imposes significant constraints on the structure of correlation functions

$\rightarrow$ For $T=1/\beta > 0$, the thermal commutator has the representation:

$$\boxed{\rho(\omega,\vec{p}) := \mathcal{F}\left[\langle[\phi(x),\phi(y)]\rangle_{\beta}\right] = \int_0^{\infty} ds \int \frac{d^3\vec{u}}{(2\pi)^2}\, \epsilon(\omega)\, \delta\big(\omega^2-(\vec{p}-\vec{u})^2-s\big)\, \widetilde{D}_{\beta}(\vec{u},s)}$$

Note: this is a ***non-perturbative*** representation!

"Thermal spectral density"

- In the limit of vanishing temperature one recovers the well-known *Källén-Lehmann* spectral representation:

$$\boxed{\rho(\omega,\vec{p}) \xrightarrow{\beta\to\infty} 2\pi\epsilon(\omega)\int_0^{\infty} ds\ \delta(p^2-s)\,\rho(s)}$$

e.g. $\rho(s)=\delta(s\text{-}m^2)$ for a massive free theory

Important question: *what does the thermal spectral density* $\widetilde{D}_{\beta}(\boldsymbol{u},s)$ *look like?*

Non-perturbative implications

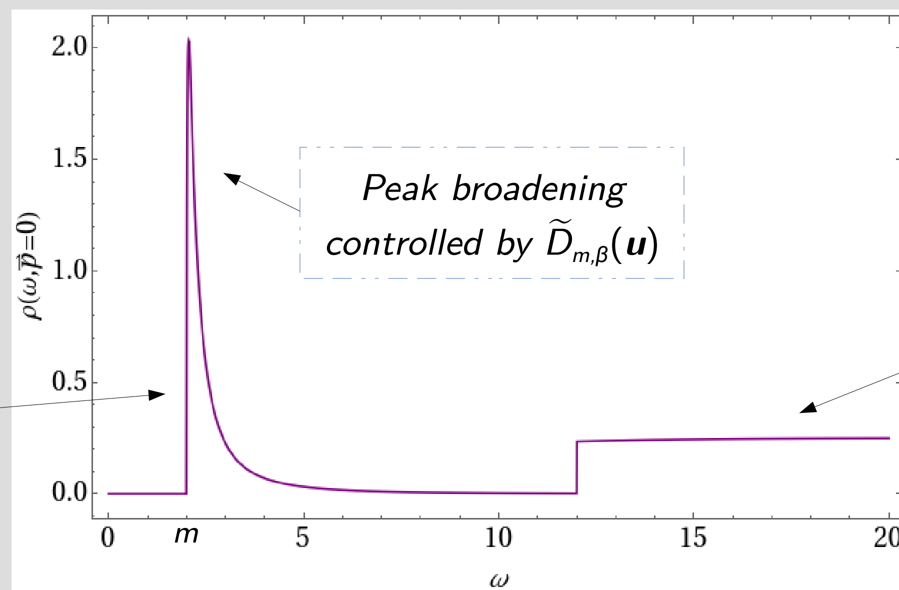
- A natural decomposition [Bros, Buchholz, *NPB* 627 (2002)] is:

$$\boxed{\widetilde{D}_{\beta}(\vec{u},s) = \widetilde{D}_{m,\beta}(\vec{u})\,\delta(s-m^2) + \widetilde{D}_{c,\beta}(\vec{u},s)}$$

"Damping factor"

Continuous component

Causes $T=0$ mass pole m to be <u>screened</u> by thermal effects



Peak broadening controlled by $\widetilde{D}_{m,\beta}(\boldsymbol{u})$

Fixes T-dependence of continuous spectral contributions

$\rightarrow$ **Damping factors hold the key to understanding in-medium effects!**

In-medium observables from Euclidean data

- In many instances, $T > 0$ *Euclidean* data is used to calculate observables, e.g. spectral functions $\rho_\Gamma(\omega,\boldsymbol{p})$ from $C_\Gamma(\tau,\vec{x}) = \langle O_\Gamma(\tau,\vec{x})\, O_\Gamma(0,\vec{0})\rangle_T$ where O_Γ is some particle-creating operator

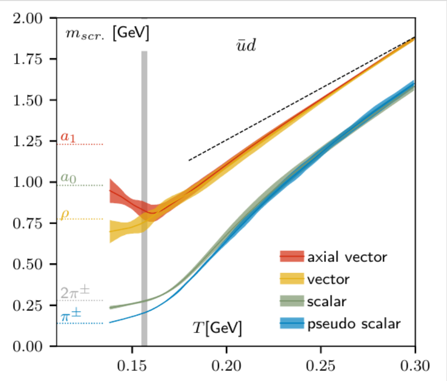
$$\widetilde{C}_\Gamma(\tau,\vec{p}) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\cosh\left[\left(\frac{\beta}{2}-|\tau|\right)\omega\right]}{\sinh\left(\frac{\beta}{2}\omega\right)}\, \rho_\Gamma(\omega,\vec{p})$$

$\rightarrow$ Determine $\rho_\Gamma(\omega,\boldsymbol{p})$ given $C_\Gamma(\tau,\boldsymbol{p})$: *problem is ill-conditioned, need more information!*

- Another quantity of interest in lattice studies is the spatial correlator

$$C_\Gamma(x_3) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau\, C_\Gamma(\tau,\vec{x}) = \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} e^{ip_3x_3} \int_0^\infty \frac{d\omega}{\pi\omega}\, \rho_\Gamma(\omega, p_1 = p_2 = 0, p_3)$$

- Large-x_3 behaviour $C_\Gamma(x_3) \sim \exp(-m_{scr}|x_3|)$ used to extract "screening masses" $m_{scr}(T)$



[HotQCD collaboration, *Phys. Rev. D* 100 (2019)]

Spectral functions from lattice data

Goal: Use the additional constraints imposed by *locality* to improve the extraction of spectral functions from Euclidean data

- Locality implies a general connection between the spatial correlator and thermal spectral density [P.L., PRD 106 (2022); P.L, O. Philipsen, 2207.14718]

$$\boxed{C(x_3) = \frac{1}{2}\int_0^\infty ds \int_{|x_3|}^\infty dR\; e^{-R\sqrt{s}} D_\beta(R,s)}$$

$\rightarrow$ The lightest $T{=}0$ states dominate: $\boxed{C(x_3) \approx \frac{1}{2}\sum_{i=1}^{n}\int_{|x_3|}^\infty dR\; e^{-m_i R} D_{m_i,\beta}(R)}$

n particle states with masses m_i

- Once the damping factors of the dominant states are known, one can then use these to compute their contribution to $\rho(\omega, \boldsymbol{p})$

- In QCD, perhaps the simplest spatial correlator example is that of the light quark pseudo-scalar meson operator $\boxed{\mathcal{O}^a_{\mathrm{PS}} = \overline{\psi}\gamma_5 \frac{\tau^a}{2}\psi}$

Spectral functions from lattice data

- In [P.L, O. Philipsen, 2207.14718] the pseudo-scalar correlator lattice data from [Rohrhofer et al. *PRD* **100** (2019)] was used to compute the spectral function $\rho_{PS}(\omega, \boldsymbol{p})$

  - **Step 1**: Perform fits to the spatial correlator data $C_{PS}(x_3)$ to obtain the functional dependence at different temperatures (T=220–960 MeV)

    $\rightarrow$ The ansatz $A \exp(-Bx_3)+C \exp(-Dx_3)$ describes the data well

    Contribution of 2 lowest-lying states, π and π^*

  - **Step 2**: Calculate the π and π^* damping factors from $C_{PS}(x_3)$

    $\rightarrow$ Exponential contribution to $C_{PS}(x_3)$ implies: $\boxed{D_{m_i,\beta}(\vec{x}) = \alpha_i \, e^{-\gamma_i |\vec{x}|}}$

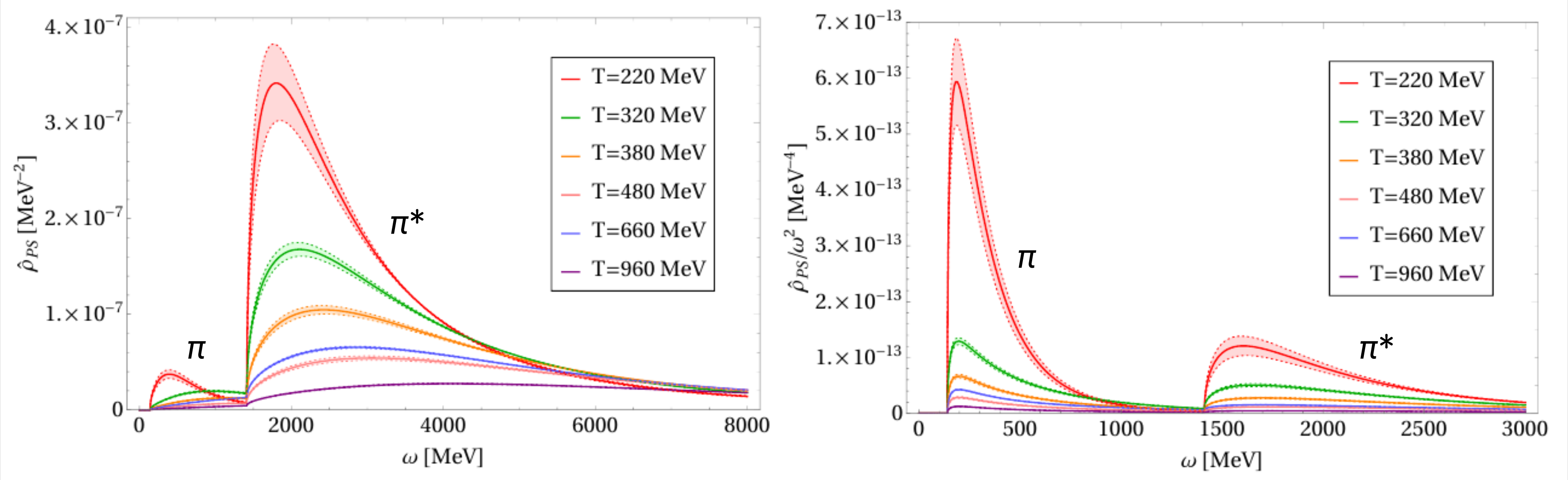
  - **Step 3**: Use $D_{m,\beta}(\boldsymbol{x})$ and the spectral representation to compute $\rho_{PS}(\omega, \boldsymbol{p})$

    $\rightarrow$ Implies form:

$$\rho_{\rm PS}(\omega,\vec{p}) = \epsilon(\omega) \Bigg[\theta(\omega^2 - m_\pi^2) \frac{4\,\alpha_\pi \gamma_\pi \sqrt{\omega^2 - m_\pi^2}}{(|\vec{p}|^2 + m_\pi^2 - \omega^2)^2 + 2(|\vec{p}|^2 - m_\pi^2 + \omega^2)\gamma_\pi^2 + \gamma_\pi^4} + \theta(\omega^2 - m_{\pi^*}^2) \frac{4\,\alpha_{\pi^*} \gamma_{\pi^*} \sqrt{\omega^2 - m_{\pi^*}^2}}{(|\vec{p}|^2 + m_{\pi^*}^2 - \omega^2)^2 + 2(|\vec{p}|^2 - m_{\pi^*}^2 + \omega^2)\gamma_{\pi^*}^2 + \gamma_{\pi^*}^4} \Bigg]$$

Spectral functions from lattice data

- Using the T-dependence of the lattice fit parameters one obtains:



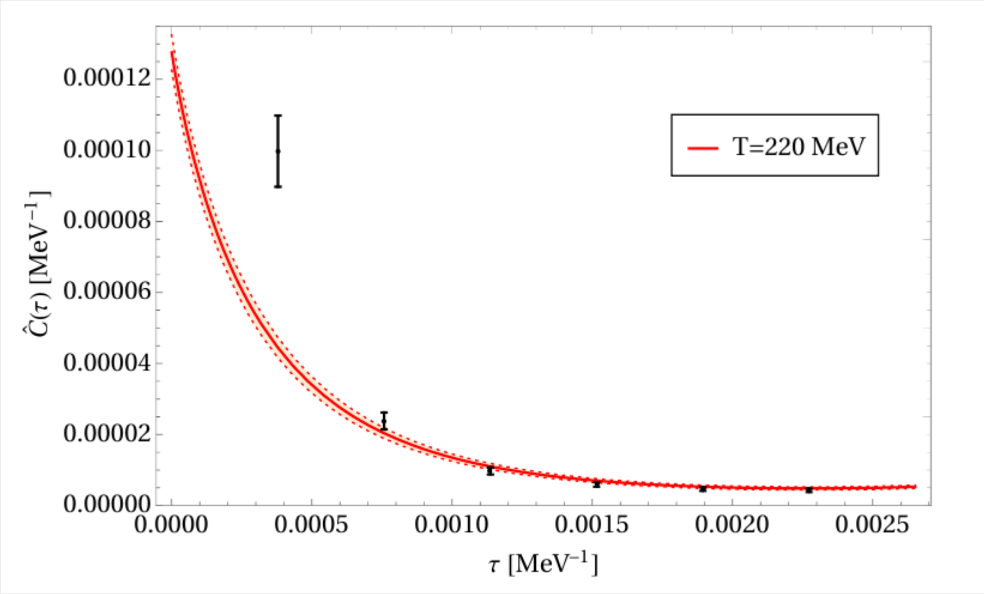
→ π and π^* dominate the spectral function at these temperatures

→ The π has a pronounced peak at T=220 MeV, *non-perturbative effects are still important!*

- The π and π^* states gradually "melt away" as T increases due to the more frequent interactions with the thermal medium

A non-perturbative test

- Given the full analytic structure of $\rho_{PS}(\omega,\boldsymbol{p})$, one can use this to *predict* the form of the corresponding temporal correlator $\widetilde{C}_{PS}(\tau,\boldsymbol{p}{=}0)$
- $\widetilde{C}_{PS}(\tau,\boldsymbol{p}{=}0)$ has a very different $\rho_{PS}(\omega,\boldsymbol{p})$ dependence $\rightarrow$ *non-trivial test!*
- Using the $T{=}220$ MeV temporal data from [Rohrhofer et al. *PLB* **802** (2020)] one obtains the following result:



- The prediction matches the data well for large τ, and then begins to undershoot $\rightarrow$ missing contributions from higher excited states (π^{**} etc.)

Summary & outlook

- Local QFT can be extended to systems with $T > 0$, and this has important implications, including:

  → Spectral representations for thermal correlation functions

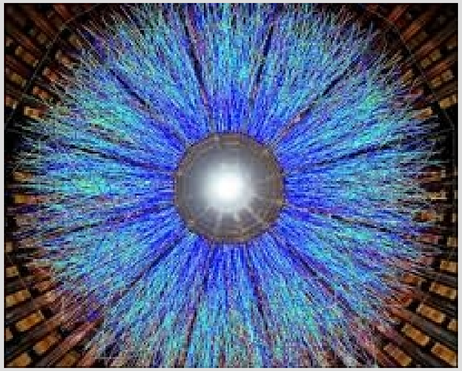
  → Ability to extract real-time observables from Euclidean data

  → Dynamical interpretation of in-medium effects

- Applying this framework to lattice QCD data enables one to calculate spectral functions in a new way → *evidence for distinct pion states, even at high T*

- So far only real scalar fields $\Phi(x)$ with $T > 0$ considered, but this approach can be generalised (higher spin, non-vanishing density). *Work in progress!*

→ This framework provides a way of obtaining ***non-perturbative*** insights into the phase structure of QFTs, and the resulting in-medium phenomena



[Brookhaven National Lab]

Backup: *Local QFT*

- In the 1960s, A. Wightman and R. Haag pioneered an approach which set out to answer the fundamental question "*what is a QFT?*"
- The resulting approach, *Local QFT*, defines a QFT using a core set of physically motivated axioms

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a, \alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^\uparrow}$.*

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a,1) = e^{iP^\mu a_\mu}$.*

Axiom 3 (Uniqueness of the vacuum). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

Axiom 4 (Field operators). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi_l^{(\kappa)}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

Axiom 5 (Relativistic covariance). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^\uparrow}$:*

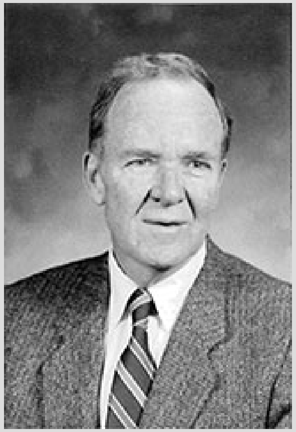
$$U(a,\alpha)\varphi_i^{(\kappa)}(x)U(a,\alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1})\varphi_j^{(\kappa)}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^\uparrow}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^\uparrow}$.

Axiom 6 (Local (anti-)commutativity). *If the support of the test functions f, g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

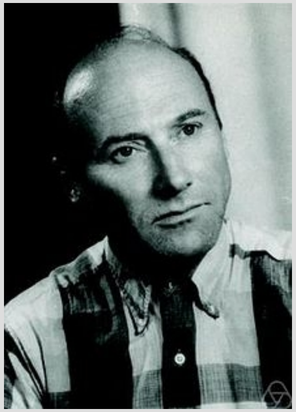
$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f)\varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g)\varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.



A. Wightman

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).]



R. Haag

[R. Haag, *Local Quantum Physics*, Springer-Verlag (1992).]

Backup: *Local QFT beyond the vacuum*

- **Idea**: Look for a generalisation of the standard axioms that is compatible with $T > 0$, and approaches the vacuum case for $T \to 0$

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a,\alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^\uparrow}$.*

→ H_β is defined for fixed $\beta = 1/T$

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a,1) = e^{iP^\mu a_\mu}$.*

→ Replaced by the KMS condition

$$\langle \Omega_\beta | \phi(x_1)\cdots\phi(x_k)\phi(x_{k+1})\cdots\phi(x_n)|\Omega_\beta\rangle$$
$$= \langle \Omega_\beta | \phi(x_{k+1})\cdots\phi(x_n)\phi(x_1 + i(\beta,\vec{0}))\cdots\phi(x_k + i(\beta,\vec{0}))|\Omega_\beta\rangle$$

Axiom 3 (Uniqueness of the vacuum). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

→ Instead, thermal background state $|\Omega_\beta>$

Axiom 4 (Field operators). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi_l^{(\kappa)}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

✓ Fields are still distributions

Axiom 5 (Relativistic covariance). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^\uparrow}$:*

$$U(a,\alpha)\varphi_i^{(\kappa)}(x)U(a,\alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1})\varphi_j^{(\kappa)}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^\uparrow}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^\uparrow}$.

→ The fields no longer transform under general unitary Lorentz transformations

Axiom 6 (Local (anti-)commutativity). *If the support of the test functions f,g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f)\varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g)\varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.

✓ Locality is unaffected by the properties of the background state. *This is important!*

Backup: *Damping factors from Euclidean FRG data*

- Locality constraints imply that particle damping factors $D_{m,\beta}(\boldsymbol{x})$ can also be calculated from Euclidean momentum space data [P.L., PRD 106 (2022)]

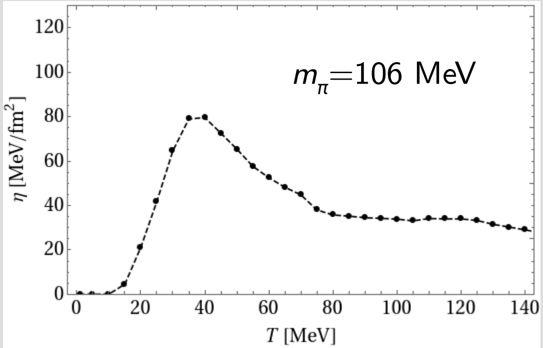
$$\boxed{D_{m,\beta}(\vec{x}) \sim e^{|\vec{x}|m} \int_0^\infty \frac{d|\vec{p}|}{2\pi}\, 4|\vec{p}| \sin(|\vec{p}||\vec{x}|)\, \widetilde{G}_\beta(0,|\vec{p}|).}$$

p-space Euclidean propagator

Holds for large separation $|\boldsymbol{x}|$

- In [P.L., R.-A. Tripolt, PRD 106 (2022)] pion propagator data from the quark-meson model (FRG calculation) was used to compute the damping factor at different values of T via the analytic relation above
- Fits to the resulting data were consistent with the form: $\boxed{D_{m_\pi,\beta}(\vec{x}) = \alpha_\pi\, e^{-\gamma_\pi|\vec{x}|}}$
- $D_{m,\beta}(\boldsymbol{x})$ can then be used as input for calculations, e.g. shear viscosity

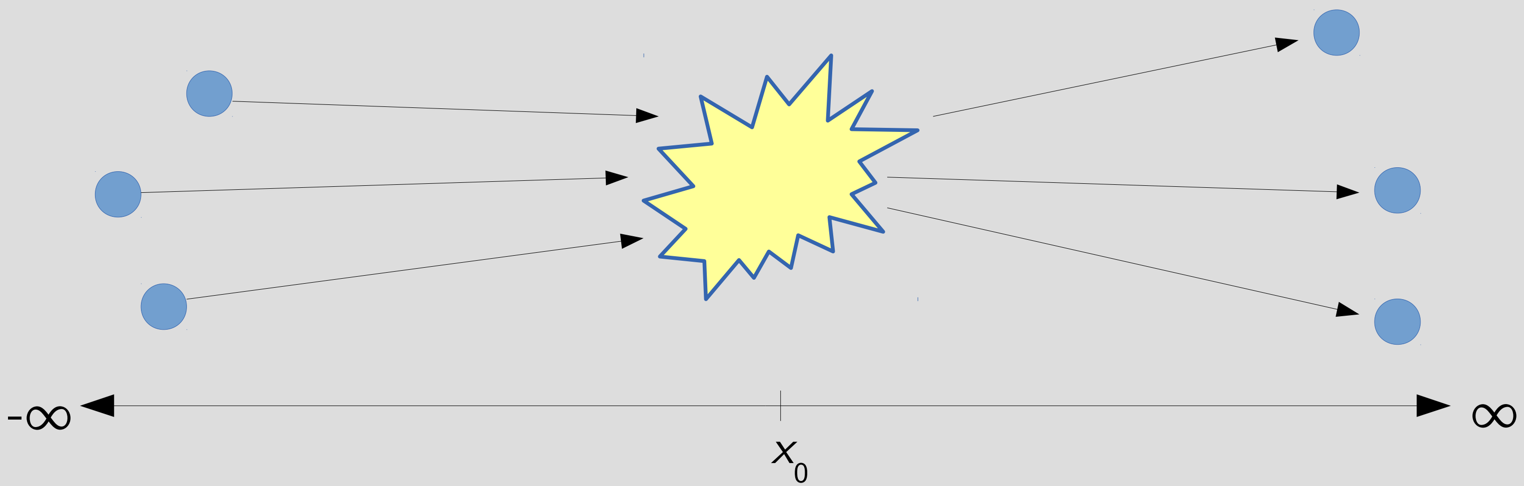
Similar qualitative features to results from chiral perturbation theory



Backup: *Damping factors from asymptotic dynamics*

- Since all observable quantities are computed using correlation functions, which are characterised by damping factors, one can use these to gain new insights into the properties of QFTs when $T>0$

- It has been proposed [Bros, Buchholz, NPB 627 (2002)] that these quantities are controlled by the large-time x_0 dynamics of the theory



Important: Interactions with the thermal background persist, even for large x_0

$\rightarrow$ ***Need to take this into account in definition of scattering states!***

Backup: *Damping factors from asymptotic dynamics*

- Idea: thermal scattering states are defined by imposing an asymptotic field condition [Bros, Buchholz, NPB 627 (2002)]:

Asymptotic fields Φ_0 are assumed to satisfy dynamical equations, but only at large x_0

In Φ^4 theory

"Asymptotic coupling"

$$(\partial^2 + m^2)\phi_0(x) + \frac{\lambda}{3!}\phi_0^3(x) \xrightarrow{|x_0|\to\infty} 0$$

"Asymptotic mass"

- Given that the thermal spectral density has the decomposition

$$\boxed{\widetilde{D}_\beta(\vec{u}, s) = \widetilde{D}_{m,\beta}(\vec{u})\,\delta(s - m^2) + \widetilde{D}_{c,\beta}(\vec{u}, s)}$$

it follows that:

1. The continuous contribution to $\langle\Omega_\beta|\phi(x)\phi(y)|\Omega_\beta\rangle$ is **suppressed** for large x_0
2. The particle damping factor $\widetilde{D}_{m,\beta}(\boldsymbol{u})$ is **uniquely fixed** by the asymptotic field equation

- This means that the non-perturbative thermal effects experienced by particle states are entirely controlled by the asymptotic dynamics!

Backup: Φ^4 theory for $T > 0$

- Applying the asymptotic field condition for Φ^4 theory, the resulting damping factors have the form [Bros, Buchholz, NPB 627 (2002)]:

$\rightarrow$ **For $\boldsymbol{\lambda} < 0$:** $$\boxed{D_{m,\beta}(\vec{x}) = \frac{\sin(\kappa|\vec{x}|)}{\kappa|\vec{x}|}}$$ $\rightarrow$ **For $\boldsymbol{\lambda} > 0$:** $$\boxed{D_{m,\beta}(\vec{x}) = \frac{e^{-\kappa|\vec{x}|}}{\kappa_0|\vec{x}|}}$$

where κ is defined with $r = m/T$: $$\kappa = T\sqrt{|\lambda|}K(r), \qquad K(r) = \sqrt{\int \frac{d^3\hat{\vec{q}}}{(2\pi)^3 2\sqrt{|\hat{\vec{q}}|^2 + r^2}} \frac{1}{e^{\sqrt{|\hat{\vec{q}}|^2+r^2}} - 1}}$$

$\rightarrow$ *The parameter κ has the interpretation of a thermal width: $\kappa \rightarrow 0$ for $T \rightarrow 0$*, or equivalently κ^{-1} *is mean-free path*

- Now that one has the exact dependence of $D_{m,\beta}(\boldsymbol{x})$ on the external physical parameters, in this case T, m and λ, one can use this to calculate observables ***analytically***

Backup: Φ^4 theory for $T > 0$

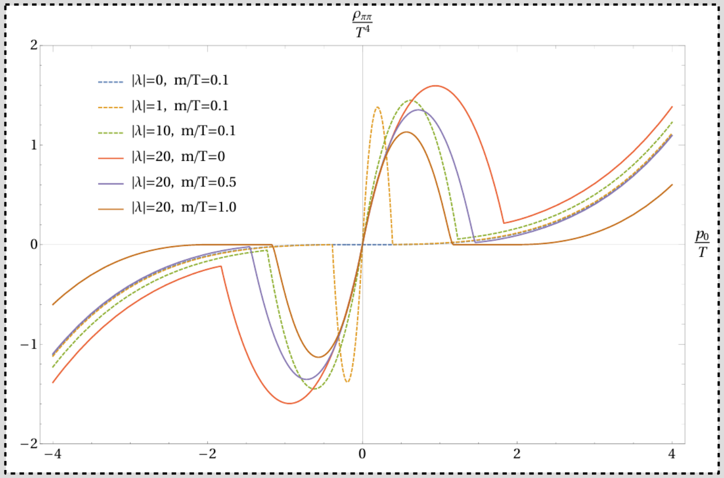
- Of particular interest is the *shear viscosity* η, which measures the resistance of a medium to sheared flow

  $\rightarrow$ This quantity can be determined from the spectral function of the spatial traceless energy-momentum tensor

$$\boxed{\rho_{\pi\pi}(p_0) = \lim_{\vec{p}\to 0} \mathcal{F}\left[\langle\Omega_\beta|\left[\pi^{ij}(x), \pi_{ij}(y)\right]|\Omega_\beta\rangle\right](p)}$$

  ... and η is recovered via the *Kubo relation* $\boxed{\eta = \frac{1}{20}\lim_{p_0\to 0}\frac{d\rho_{\pi\pi}}{dp_0}}$

- Using $D_{m,\beta}(\boldsymbol{x})$ for $\lambda < 0$, the EMT spectral function $\rho_{\pi\pi}$ has the form:

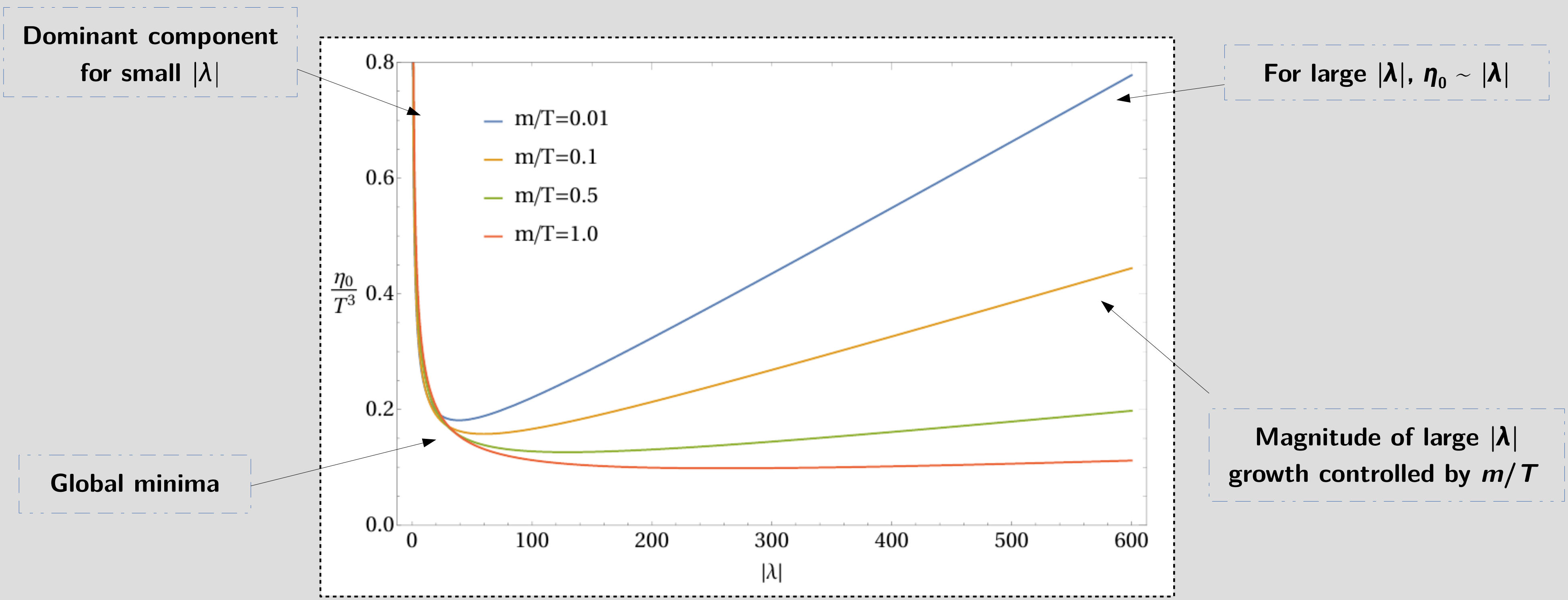


- The presence of interactions causes resonant peaks to appear $\rightarrow$ peaked when $p_0 \sim \kappa = 1/\ell$
- For $\lambda \to 0$ the free-field result is recovered, as expected
- The dimensionless ratio m/T controls the magnitude of the peaks

Backup: Φ^4 theory for $T > 0$

- Applying Kubo's relation, the shear viscosity η_0 arising from the asymptotic states can be written [P.L., R.-A. Tripolt, J. M. Pawlowski, D. H. Rischke, PRD 104 (2021)]

$$\boxed{\eta_0 = \frac{T^3}{15\pi}\left[\frac{\mathcal{K}_3\left(\frac{m}{T},0,\infty\right)}{\sqrt{|\lambda|}} + \sqrt{|\lambda|}\,\mathcal{K}_1\left(\frac{m}{T},0,\infty\right) + \frac{\mathcal{K}_4\left(\frac{m}{T},\sqrt{|\lambda|}K\left(\frac{m}{T}\right),\sqrt{|\lambda|}K\left(\frac{m}{T}\right)\right)}{4|\lambda|}\right]}$$



$\rightarrow$ For fixed coupling, η_0/T^3 is entirely controlled by functions of m/T

Backup: Φ^4 theory for $T > 0$

- What about the case $\lambda > 0$? $\rightarrow$ η_0 *diverges!*

  Why? – *The particle damping factor $D_{m,\beta}(\boldsymbol{u})$ does not decay rapidly enough at large momenta*

- This characteristic is related to the "bad" UV behaviour of the quartic interaction, i.e. the triviality of Φ^4 appears to have an impact beyond $T=0$!

- In [PRD 104, 065010 (2021)] it was shown more generally that the finiteness of η_0 is related to the existence of thermal equilibrium

  If the KMS condition holds $\Longrightarrow$ η_0 is finite

- This procedure demonstrates that asymptotic dynamics can be used to explore the non-perturbative properties of QFTs when T>0

  $\rightarrow$ *Can also calculate other observables, e.g. transport coefficients, entropy density, pressure, etc.*

Backup: *spectral representations*

- For thermal asymptotic states, the spectral function $\rho_{\pi\pi}$ has the form

$$\rho_{\pi\pi}(p_0) = \sinh\left(\frac{\beta}{2}p_0\right) \int \frac{d^3\vec{q}}{(2\pi)^4}\frac{2}{3}|\vec{q}|^4 \int_{-\infty}^{\infty} dq_0 \, \frac{\widetilde{C}_\beta(q_0,\vec{q})\,\widetilde{C}_\beta(p_0-q_0,\vec{q})}{\sinh\left(\frac{\beta}{2}q_0\right)\sinh\left(\frac{\beta}{2}(p_0-q_0)\right)}$$

… which after applying the generalised KL representation, together with the Kubo relation, implies

$$\begin{aligned}\eta_0 = \frac{T^5}{240\pi^5}\int_0^\infty ds \int_0^\infty dt \int_0^\infty d|\vec{u}| \int_0^\infty d|\vec{v}| \, |\vec{u}||\vec{v}| \, \widetilde{D}_\beta(\vec{u},s)\,\widetilde{D}_\beta(\vec{v},t) \\ \times \Bigg[4\left[1+\epsilon(|\vec{u}|-|\vec{v}|)\right]\left\{\frac{|\vec{v}|}{T}\mathcal{I}_3\left(\frac{\sqrt{t}}{T},0,\infty\right) + \frac{|\vec{v}|^3}{T^3}\mathcal{I}_1\left(\frac{\sqrt{t}}{T},0,\infty\right)\right\} \\ + \left\{\mathcal{I}_4\left(\frac{\sqrt{t}}{T},\frac{|\vec{v}|}{T},\frac{s-t+(|\vec{u}|+|\vec{v}|)^2}{2(|\vec{u}|+|\vec{v}|)T}\right) + \epsilon(|\vec{u}|-|\vec{v}|)\,\mathcal{I}_4\left(\frac{\sqrt{t}}{T},\frac{|\vec{v}|}{T},\frac{s-t+(|\vec{v}|-|\vec{u}|)^2}{2(|\vec{v}|-|\vec{u}|)T}\right)\right\}\Bigg]\end{aligned}$$

- The model dependence of η_0 factorises, and is controlled by the thermal spectral density $D_\beta(\mathbf{u},s)$

Backup: *Euclidean spectral relations*

- One can use the assumptions of local QFT at finite T to put constraints on the the structure of Euclidean correlation functions

  → From the KMS condition and locality: $$\mathcal{W}_E(\tau,\vec{x}) = \frac{1}{\beta}\sum_{N=-\infty}^{\infty} w_N(\vec{x})e^{\frac{2\pi i N}{\beta}\tau}$$

- The Fourier coefficients of the Euclidean two-point function are then related to the thermal damping factors as follows [P.L., PRD 106 (2022)]:

$$w_N(\vec{x}) = \frac{1}{4\pi|\vec{x}|}\left[D_m(\vec{x})e^{-|\vec{x}|\sqrt{m^2+\omega_N^2}} + \int_0^\infty ds\, e^{-|\vec{x}|\sqrt{s+\omega_N^2}}D_c(\vec{x},s)\right]$$

→ *The continuous component $D_c(\boldsymbol{x},s)$ is exponentially suppressed!*

- $\omega_N = 2\pi N T$ are the Matsubara frequencies. For $N=0$ this leads to:

$$\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau\, \mathcal{W}_E(\tau,\vec{x}) \sim \frac{1}{4\pi|\vec{x}|} D_{m,\beta}(\vec{x})\, e^{-|\vec{x}|m}$$