PHYSICS ON AND OFF THE LIGHT CONE

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On the light cone the full symmetry is conformal symmetry not just Lorentz symmetry. Spontaneously breaking conformal symmetry gives masses to particles and takes them off the light cone. Canonical quantization specifies equal time commutators on the light cone. Instant time and light-front commutators are very different, but can be shown to be equivalent by looking at unequal time commutators. We discuss the connection of the light-front approach to the infinite momentum frame approach, and show that vacuum graphs are outside this framework. We show that there is a light-front structure to both AdS/CFT and the eikonal approximation.

1 MINKOWSKI SIGNATURE PREDATES SPECIAL RELATIVITY

Minkowski signature originated in the 19th century and predates 20th century special relativity. Consider the two-dimensional Gauss-Bolyai-Lobachevski geometry with line element

$$ds^2 = \frac{a^2 dr^2}{a^2 + r^2} + r^2 d\theta^2. \tag{1.1}$$

To construct it we introduce a flat three-dimensional space with a line element

$$ds^2 = dx^2 + dy^2 - dt^2 \tag{1.2}$$

as constrained by the hyperbola

$$t^2 - x^2 - y^2 = a^2. \tag{1.3}$$

Eliminating t gives

$$ds^2 = dx^2 + dy^2 - \frac{(xdx + ydy)^2}{a^2 + x^2 + y^2}. \tag{1.4}$$

On introducing polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ we recover (1.1):

$$ds^2 = dr^2 + r^2 d\theta^2 - \frac{r^2 dr^2}{a^2 + r^2} = \frac{a^2 dr^2}{a^2 + r^2} + r^2 d\theta^2. \tag{1.5}$$

The significance of the Gauss-Bolyai-Lobachevski geometry is that it did not obey all of Euclid's axioms, to thus open the door to non-Euclidean Riemannian geometry and eventually to General Relativity. It took 2000 years to find it because it does not embed in a Euclidean geometry with line element $ds^2 = dx^2 + dy^2 + dt^2$ but in a geometry with Minkowski signature instead. Technically, the Gauss-Bolyai-Lobachevski geometry is known as a two-dimensional space of constant negative curvature. Current cosmological studies indicate that we live in a four-dimensional space of constant curvature. We will discuss embedding issues again in AdS/CFT.

The line element

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

is Lorentz invariant. It breaks spacetime up into separate timelike ($ds^2 > 0$), lightlike ($ds^2 = 0$) and spacelike ($ds^2 < 0$) regions. Because ds^2 is not equal to the Euclidean-signatured $-dt^2 - dx^2 - dy^2 - dz^2$, one can have nontrivial solutions to $ds^2 = 0$, with this region being known as the light cone.

Particles that propagate on the light cone are massless. Particles that propagate off the lightcone are massive. To understand the **origin of mass** we thus need to understand how to get off the light cone.

To address the origin of mass we need to identify the **full symmetry** of the light cone.

3 CONFORMAL SYMMETRY - THE FULL SYMMETRY OF THE LIGHT CONE

While the timelike and spacelike regions are Lorentz invariant, the light cone itself has a higher symmetry. Its scale symmetry is immediate since if $\eta_{\mu\nu}dx^\mu dx^\nu = 0$ then on scaling $x^\mu \to \lambda x^\mu$ we see that $\lambda^2\eta_{\mu\nu}dx^\mu dx^\nu$ is zero too.

With 10 constant Poincare parameters ϵ^μ and $\Lambda^\mu{}_\nu$, and 5 constant conformal parameters λ and c^μ the 15 conformal generators transform x^μ and x^2 according to

$$
\begin{aligned}
x^\mu &\to x^\mu + \epsilon^\mu, \qquad x^\mu \to \Lambda^\mu{}_\nu x^\nu, \\
x^\mu &\to \lambda x^\mu, \qquad x^\mu \to \frac{x^\mu + c^\mu x^2}{1 + 2c\cdot x + c^2x^2}, \\
x^2 &\to \lambda^2 x^2, \qquad x^2 \to \frac{x^2}{1 + 2c\cdot x + x^2}.
\end{aligned}
\tag{3.1}
$$

The 10 Poincare generators preserve any x^2, while the 5 conformal generators also preserve $x^2 = 0$.

4 THE CONFORMAL GROUP

The 15 infinitesimal generators act on the coordinates x^μ according to

$$\begin{aligned} &P^\mu = i\partial^\mu, \qquad M^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu), \qquad D = ix^\mu\partial_\mu, \\ &C^\mu = i(x^2\eta^{\mu\nu} - 2x^\mu x^\nu)\partial_\nu, \end{aligned} \tag{4.1}$$

and together they form the 15-parameter $SO(4,2)$ conformal group:

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = i(-\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\rho\nu} + \eta_{\nu\sigma}M_{\rho\mu}), \\ &[M_{\mu\nu}, P_\sigma] = i(\eta_{\nu\sigma}P_\mu - \eta_{\mu\sigma}P_\nu), \quad [P_\mu, P_\nu] = 0, \\ &[M_{\mu\nu}, D] = 0, \quad [D, P_\mu] = -iP_\mu, \\ \\ &[C_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \\ \\ &[M_{\mu\nu}, C_\sigma] = i(\eta_{\nu\sigma}C_\mu - \eta_{\mu\sigma}C_\nu), \\ &[C_\mu, C_\nu] = 0, \quad [D, C_\mu] = iC_\mu. \end{aligned} \tag{4.2}$$

5 SPINORS AND CONFORMAL SYMMETRY

The fundamental representation of the conformal group is a 4-dimensional spinor representation since the 15 Dirac matrices γ_5, γ_μ, $\gamma_\mu\gamma_5$, $[\gamma_\mu, \gamma_\nu]$ also close on the $SO(4,2)$ algebra according to:

$$M_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu], \quad C_\mu + P_\mu = \gamma_\mu, \quad C_\mu - P_\mu = \gamma_\mu\gamma_5, \quad D = \frac{i}{2}\gamma_5. \tag{5.1}$$

The group $SU(2,2)$ is the covering group of $SO(4,2)$ with the 4-dimensional spinor being its fundamental representation.

4-component Dirac spinors are **reducible** under the Lorentz group. They reduce to irreducible left-handed and right-handed Weyl spinors, viz. the Dirac spinor behaves as the $D(1/2,0) \oplus D(0,1/2)$ representation. This is puzzling: why should the fundamental building blocks of matter (viz. fermions) be reducible under the fundamental group (viz. the Lorentz group). Solution: let a bigger group be the fundamental group, one that contains the Lorentz group as a subgroup and under which 4-component Dirac fermions are irreducible.

This is the case for the conformal group, since under it all four Dirac fermion components are **irreducible**, with the complex conformal transformations mixing the left-handed and right-handed spinors, doing so via transformations that are continuous.

Since conformal symmetry has to hold for all spinors no matter what their internal quantum numbers might be, in a conformal invariant theory neutrinos would have to have four components too, with **right-handed neutrinos** being needed to accompany the observed left-handed ones. The weak interaction has to be left-right symmetric: $SU(2)_L \times SU(2)_R \times U(1)$.

Can generalize to the 3-family $SU(6)_L \times SU(6)_R \times U(1)$. This is Quantum Flavordynamics, in which **ALL** of the global chiral symmetry of Quantum Chromodynamics is gauged. Why only gauge its $SU(2)_L \times U(1)$ subgoup – too lopsided.

If the conformal symmetry is exact then all particles are massless. Thus we need to generate mass spontaneously. Thus to get off the light cone we need **spontaneous symmetry breaking**. And since the double well potential $V(\phi) = \lambda\phi^4 - \mu^2\phi^2$ is not conformal invariant, the breaking must be done by radiative loops, to hence be **dynamical**.

We must generate masses dynamically for all massive particles, but especially for right-handed neutrinos since their lack of detection to date means that their masses are much larger than those of the left-handed ones. Thus they must acquire Majorana masses not Dirac masses (i.e., $\langle\Omega|\psi\psi|\Omega\rangle$, not $\langle\Omega|\bar{\psi}\psi|\Omega\rangle$ – see Mannheim, Phys. Rev. D 22, 1729 (1980)). This will break parity and reduce $SU(2)_L \times SU(2)_R \times U(1)$ to $SU(2)_L \times U(1)$ by making right-handed W and Z bosons heavier than the left-handed ones.

So parity must be broken spontaneously. This resolves a puzzle: If time translations and space reflections commute how could the $[H, P]$ commutator not be zero. Answer: it is zero, but parity is broken in the vacuum, i.e., in the states not in the operators.

If we now make the conformal symmetry local we are led to conformal gravity with action

$$I_W = -\alpha_g \int d^x (-g)^{1/2} C_{\lambda\mu\nu\tau} C^{\lambda\mu\nu\tau},$$

where $C_{\lambda\mu\nu\tau}$ is the Weyl tensor, and not led to Einstein gravity with action

$$I_{EH} = (1/16\pi G) \int d^4x (-g)^{1/2} R^{\alpha}{}_{\alpha}.$$

With conformal gravity (like Einstein gravity a pure metric theory of gravity that also contains the Schwarzschild solution needed for the solar system) we solve [Mannheim, Prog. Part. Nucl. Phys. 94, 125 (2017)] the dark matter, dark energy/cosmological constant and quantum gravity problems, all in one go. Extrapolating Einstein gravity beyond the solar system is where the all problems come from.
Continuing Einstein gravity to galaxies gives the dark matter problem.

Continuing Einstein gravity to cosmology gives the dark energy/cosmological constant problem.

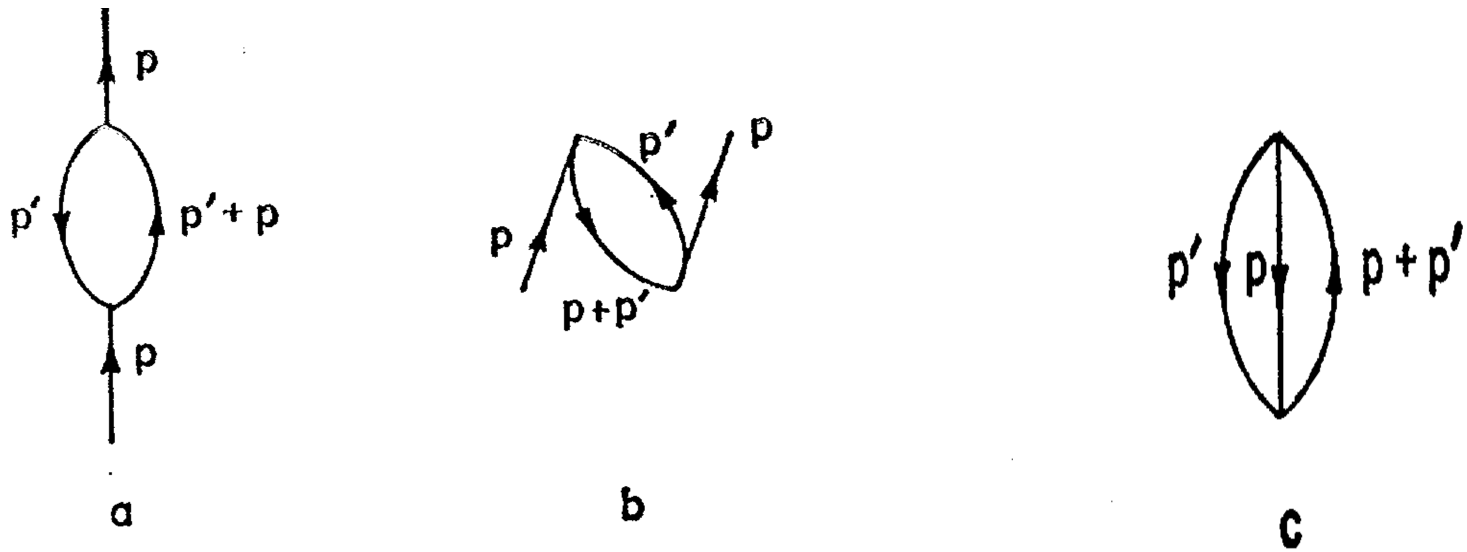
Quantizing Einstein gravity and continuing the theory far off the mass shell gives the renormalization problem.

Conclusion: With Einstein gravity we are extrapolating the wrong theory.

The cosmological constant problem is related to mass generation and thus addressed and solved by the dynamical symmetry breaking mechanism that gets us off the light cone.

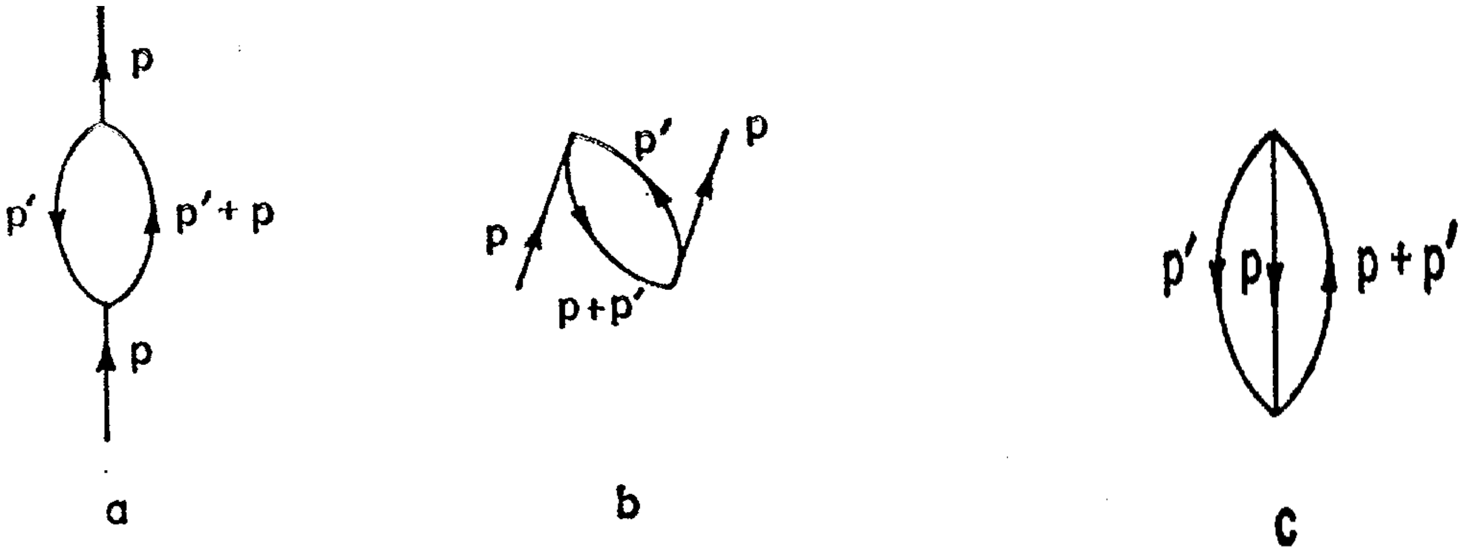
7 INFINITE MOMENTUM FRAME

In 1966 Weinberg (Phys. Rev. 150, 1313 (1966)) showed that instant-time quantization perturbation theory would be simplified in the frame in which an observer moved with an infinite three-momentum with respect to the center of mass system of a scattering process, i.e., $p^3 = \alpha P$ where P is large and α is a constant and $p^0 = [(p^3)^2 + (p^1)^2 + (p^2)^2 + m^2]^{1/2} \to \alpha P + [(p^1)^2 + (p^2)^2 + m^2]/2\alpha P$.



Specifically, Graph (b) would be suppressed with respect to Graph (a). Graph (c) was not discussed. In Weinberg's case the x^0 time axis runs up the diagram and the analysis was made using old-fashioned perturbation theory. Old-fashioned (i.e. pre-Feynman) perturbation theory is off the energy shell but on the mass shell. (The Feynman approach is off the mass shell).

8 LIGHT-FRONT VARIABLES



In 1969 Chang and Ma (Phys. Rev. 180, 1506 (1969)) recovered Weinberg's infinite momentum frame result by working with the light-front variables $p^+ = p^0 + p^3$, $p^- = p^0 - p^3$. Under a Lorentz boost in the z direction with velocity u these variables transform as

$$p^0 + p^3 \rightarrow (p^0 + p^3)\left(\frac{1+u}{1-u}\right)^{1/2}, \quad p^0 - p^3 \rightarrow (p^0 - p^3)\left(\frac{1-u}{1+u}\right)^{1/2}. \tag{8.1}$$

Setting $(1+u)^{1/2}/(1-u)^{1/2} = 1/2P$, $p^3 = \alpha P$, for large P and $(p^0)^2 - (p^3)^2 = p^+p^- = m^2 + (p^1)^2 + (p^2)^2$ we obtain

$$p^0 + p^3 \rightarrow \frac{2\alpha P}{2P} = \alpha, \quad p^0 - p^3 \rightarrow \frac{[m^2 + (p^1)^2 + (p^2)^2]}{2\alpha P}2P = \frac{[m^2 + (p^1)^2 + (p^2)^2]}{2\alpha}, \tag{8.2}$$

i.e., we recover the momenta used by Weinberg. With this choice a Green's function as evaluated with a complex plane p_+ contour becomes equal to Graph (a) when Graph (a) is evaluated with a complex plane p_0 contour an inifinite momentum frame with large p^3.

There is a caveat. In the infinite momentum frame case the flow of time is forward in x^0, while the flow of time in the light-front case is forward in $x^+ = x^0 + x^3$. But for timelike or lightlike events $(x^0)^2 - (x^3)^2 = x^+x^- \geq (x^1)^2 + (x^2)^2$ is positive, where $x^- = x^0 - x^3$. Thus x^+x^- is positive. Consequently, x^+ and x^- have the same sign. And thus for $x^0 = (x^+ + x^-)/2 > 0$ (a Lorentz invariant for timelike or lightlike events) it follows that x^+ is positive too.

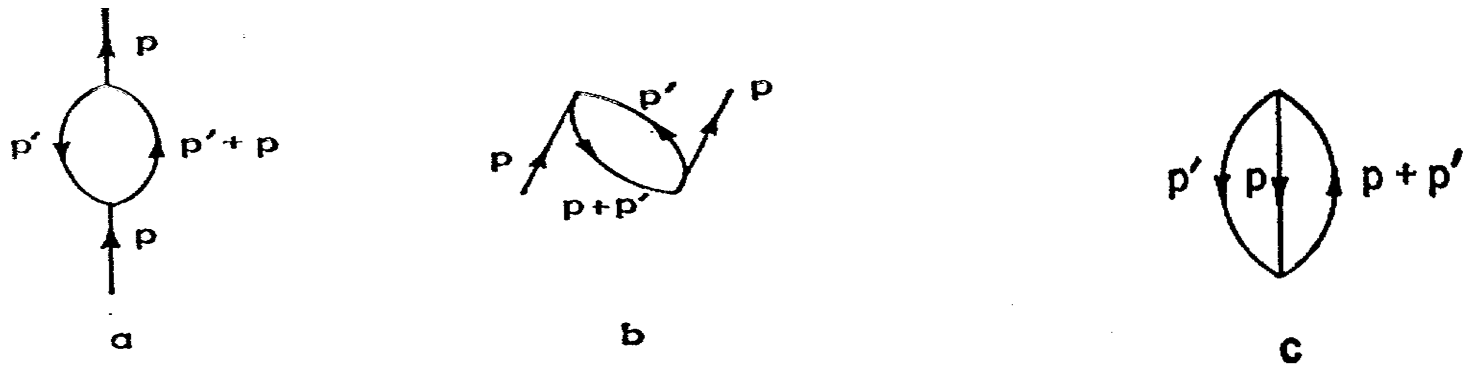
Thus for timelike or lightlike events, forward in x^+ is the same as forward in x^0.

8.1 Relativistic Eikonalization and the Light-Front Approach

In eikonalization of a scalar field ϕ one introduces a phase $\phi = Ae^{iT}$ and sets $\partial_\mu T = dx^\mu/dq$ where q is an affine parameter that measures distance along the eikonal ray. It is immediately suggested to set $T = \int^x k_\mu dx^\mu$. However, if we do set $T = \int^x k_\mu dx^\mu$ we would obtain $T = \int^x (dx_\mu/dq)dx^\mu = \int^x (dx_\mu/dq)(dx^\mu/dq)dq = \int^x k_\mu k^\mu dq$, and with $k_\mu k^\mu = 0$, $(dx_\mu/dq)(dx^\mu/dq) = 0$, such a T would vanish identically.

Thus instead we set T equal to the non-vanishing $T = \int^x k_- dx^-$. Then with $k_+ = 0$, $k_1 = 0$, $k_2 = 0$ one still has $k_\mu k^\mu = 0$ even as T is then nonzero (the vanishing of k_+, k_1 and k_2 does not restrict k_- while still keeping $k_\mu k^\mu = 4k_+k_- - k_1^2 - k_2^2$ zero). Thus while non-relativistic eikonalization occurs with the normal to the wavefront being in the x^3 direction so that a non-vanishing eikonal phase T is given by $T = \int^x k_3 dx^3$, $\partial_3 T = k_3 = dx_3/dq$, in relativistic eikonalization the normal is in the longitudinal x^- direction, with a non-vanishing eikonal phase T being given by $T = \int^x k_- dx^-$, $\partial_+ T = 0$, $\partial_- T = k_- = dx_-/dq$.

9 THE TAKEAWAY



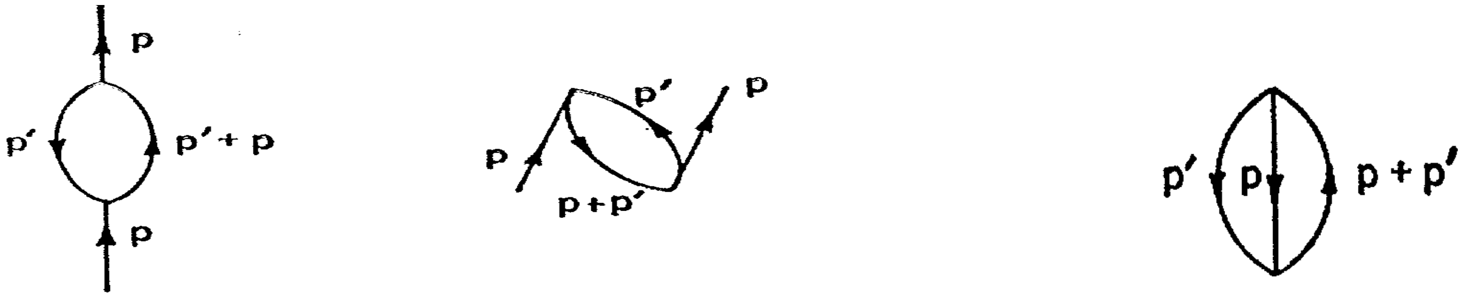
In their work Chang and Ma showed that

for Graph (a) x^+ is positive and all the p^- poles have both p^- and p^+ positive,

for Graph (b) x^+ is negative and all the p^- poles have both p^- and p^+ negative,

for Graph (c) x^+ is zero and so is p^+. But if p^+ is zero then p^- is infinite. Thus $p_+ = p^-/2$ is infinite too, just as it should be since it is the conjugate of x^+. $(\Delta x^+ \Delta p_+ > \hbar)$.

However, and this is the key point, **all of these statements are true without going to the infinite momentum frame.** They thus can define a strategy for evaluating diagrams as diagrams are segregated by the sign of the time variable x^+. And since x^+ is positive for scattering processes they only involve positive p^- and p^+, with the p^- pole contributions then corresponding to old-fashioned perturbation theory diagrams. Only needing positive p^- and p^+ provides enormous computational benefits.



a b c

But what about the instant-time graphs that are not at infinite momentum. Are they different from or the same as the light-front graphs. And if they are different, then which ones describe the real world. In Mannheim, Lowdon and Brodsky, Phys. Rep. 891, 1 (2021) they were shown to be the same. Thus Graph (a) in light front is equivalent to Graphs (a) and (b) in instant time quantization **at any momenta**. Essentially in c-number Feynman diagrams the transformation from instant-time coordinates to light-front coordinates is just a change of variables.

The vacuum Graph (c) is expressly non-zero, something known as early as 1969. However, it involves $p^+ = 0$ zero modes, whose evaluation is tricky. Resolved in Mannheim, Lowdon and Brodsky, Phys. Lett. B 797, 134916 (2019). The procedure is to construct the vacuum graph as the $x^+ = 0$ limit of the time-ordered Feynman diagram $\langle\Omega|[\theta(x^+)\phi(x)\phi(0) + \theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle$. The $x^+ = 0$ limit is the limit of two time orderings (forward and backward), even though Graphs (a) and (b) only involve the forward $x^+ > 0$. Thus the vacuum graph cannot be evaluated using old-fashioned three dimensional on mass shell perturbation theory (though non vacuum graphs can be). The vacuum graph must be evaluated as a four-dimensional off shell Feynman diagram, to thus contain information that is not accessible using the three-dimensional approach. It is then explicitly nonzero.

10 LIGHT-FRONT QUANTIZATION – THE TIP OF THE LIGHT CONE

Instead of replacing instant-time momenta by light-front momenta in Feynman diagrams, we can obtain a fully-fledged light-front quantum field theory by constructing equal x^+ commutators rather than equal x^0 commutators. For a scalar field [Neville and Rohrlich, Nuovo Cimento A 1, 625 (1971)]

Scalar field light-front commutators at equal x^+

$$\begin{aligned}
[\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)] &= -\frac{i}{4}\epsilon(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2),\\
[\phi(x^+,x^1,x^2,x^-),2\partial_-\phi(x^+,y^1,y^2,y^-)] &= i\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^- - y^-).
\end{aligned} \tag{10.1}$$

Scalar field instant-time commutators at equal x^0

$$\begin{aligned}
[\phi(x^0,x^1,x^2,x^3),\partial_0\phi(x^0,y^1,y^2,y^3)] &= i\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3),\\
[\phi(x^0,x^1,x^2,x^3),\phi(x^0,y^1,y^2,y^3)] &= 0.
\end{aligned} \tag{10.2}$$

Gauge field instant-time commutators at equal x^0

$$\begin{aligned}
[A_\nu(x^0,x^1,x^2,x^3),\partial_0 A_\mu(x^0,y^1,y^2,y^3)] &= -ig_{\mu\nu}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3),\\
[A_\nu(x^0,x^1,x^2,x^3),A_\mu(x^0,y^1,y^2,y^3)] &= 0.
\end{aligned} \tag{10.3}$$

Using gauge fixing, for light-front gauge fields we obtain (Mannheim, Lowdon and Brodsky 2021)

Gauge field light-front commutators at equal x^+

$$\begin{aligned}
[A_\nu(x^+,x^1,x^2,x^-),2\partial_- A_\mu(x^+,y^1,y^2,y^-)] &= -ig_{\mu\nu}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^- - y^-),\\
[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)] &= \frac{i}{4}g_{\mu\nu}\epsilon(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2).
\end{aligned} \tag{10.4}$$

Analogous results in the non-Abelian case.

The instant-time and light-front commutators are completely different.

11 INSTANT-TIME AND LIGHT-FRONT ANTICOMMUTATORS

Fermion instant-time anticommutators at equal x^0

$$\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(x^0,y^1,y^2,y^3)\right\} = \delta_{\alpha\beta}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3). \tag{11.1}$$

Fermion light-front anticommutators at equal x^+

$$\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\} = \Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{11.2}$$

[Chang, Root and Yan, Phys. Rev. D 7, 1133 (1973).]

Non-Invertible Projectors

$$\Lambda^\pm = \tfrac{1}{2}(1\pm\gamma^0\gamma^3),\quad \Lambda^+ + \Lambda^- = I,\quad (\Lambda^+)^2=\Lambda^+,\quad (\Lambda^-)^2=\Lambda^-,\quad \Lambda^+\Lambda^-=0,\quad \gamma^\pm=\gamma^0\pm\gamma^3,\quad (\gamma^\pm)^2=0,$$

$\psi_{(\pm)} = \Lambda_\pm\psi$, $\quad\psi_{(-)}$ **is a constrained variable:** (11.3)

$$\psi_{(-)}(x^+,x^1,x^2,x^-) = -\frac{i}{4}\int du^-\epsilon(x^--u^-)[-i\gamma^0(\gamma^1\partial_1+\gamma^2\partial_2)+m\gamma^0]\psi_{(+)}(x^+,x^1,x^2,u^-). \tag{11.4}$$

$$\left\{[\psi_{(+)}]_\nu(x),[\psi^\dagger_{(-)}]_\sigma(y)\right\} = \tfrac{i}{8}\epsilon(x^--y^-)[i(\gamma^-\gamma^1\partial^x_1+\gamma^-\gamma^2\partial^x_2)-m\gamma^-]_{\nu\sigma}\delta(x^1-y^1)\delta(x^2-y^2), \tag{11.5}$$

$$\begin{aligned}&\left\{\psi^{(-)}_\mu(x^+,x^1,x^2,x^-),[\psi^\dagger_{(-)}]_\nu(x^+,y^1,y^2,y^-)\right\}\\ &= \frac{1}{16}\Lambda^-_{\mu\nu}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\int du^-\epsilon(x^--u^-)\epsilon(y^--u^-)\delta(x^1-y^1)\delta(x^2-y^2).\end{aligned} \tag{11.6}$$

The instant-time and light-front anticommutators are completely different and even not invertible.

In instant time the light cone is $x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0$. Thus when $x^0 = 0$ it follows that $x^1 = x^2 = x^3 = 0$, to thus put us at the tip of the light cone.

In light front the light cone is $x^+x^- - x_1^2 - x_2^2 = 0$. Thus when $x^+ = 0$ it follows only that $x^1 = x^2 = 0$. However x^- is not constrained, to thus allow for an $\epsilon(x^-)$ term in equal x^+ commutators.

But does this mean that equal x^0 quantization and equal x^+ quantization correspond to different physical theories? So are they different not just in principle but in practice also?

No. Since

13 UNEQUAL TIME COMMUTATORS AND ANTICOMMUTATORS

Following Mannheim, Phys. Rev. D 102, 025020 (2020):
UNEQUAL TIME Scalar instant-time commutator

$$\begin{aligned}
i\Delta(x-y) &= [\phi(x^0,x^1,x^2,x^3),\phi(y^0,y^1,y^2,y^3)] \\
&= \int \frac{d^3p d^3q}{(2\pi)^3(2p)^{1/2}(2q)^{1/2}}\Big([a(\vec{p}),a^\dagger(\vec{q})]e^{-ip\cdot x+iq\cdot y}+[a^\dagger(\vec{p}),a(\vec{q})]e^{ip\cdot x-iq\cdot y}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3 2p}\big(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\big) \\
&= -\frac{i}{2\pi}\frac{\delta(x^0-y^0-|\vec{x}-\vec{y}|)-\delta(x^0-y^0+|\vec{x}-\vec{y}|)}{2|\vec{x}-\vec{y}|} \\
&= -\frac{i}{2\pi}\epsilon(x^0-y^0)\delta[(x^0-y^0)^2-(x^1-y^1)^2-(x^2-y^2)^2-(x^3-y^3)^2]. \qquad (13.1)
\end{aligned}$$

Since it holds at ALL times, it also holds at EQUAL light front time.

Substitute $x^0=(x^++x^-)/2$**,** $x^3=(x^+-x^-)/2$**,** $y^0=(y^++y^-)/2$**,** $y^3=(y^+-y^-)/2$**:**

$$i\Delta(x-y)=-\frac{i}{2\pi}\epsilon[\tfrac{1}{2}(x^++x^--y^+-y^-)]\delta[(x^+-y^+)(x^--y^-)-(x^1-y^1)^2-(x^2-y^2)^2]. \qquad (13.2)$$

$$i\Delta(x-y)\big|_{x^+=y^+} = [\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)]=-\frac{i}{4}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \qquad (13.3)$$

At $x^+=y^+$ **UNEQUAL instant-time commutator is EQUAL light-front time commutator**

Light-front quantization is instant-time quantization, and does not need to be independently postulated.

UNEQUAL TIME Abelian gauge field instant-time commutator

$$[A_\nu(x^0,x^1,x^2,x^3),A_\mu(y^0,y^1,y^2,y^3)] = ig_{\mu\nu}\Delta(x-y)$$
$$= -\frac{i}{2\pi}g_{\mu\nu}\epsilon(x^0-y^0)\delta[(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2]. \tag{13.4}$$

Leads to

$$[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)] = \frac{i}{4}g_{\mu\nu}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{13.5}$$

At $x^+=y^+$ UNEQUAL instant-time commutator is EQUAL light-front time commutator Similar result holds for non-Abelian gauge field.

14 FERMION UNEQUAL INSTANT-TIME ANTICOMMUTATOR

$$\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(y^0,y^1,y^2,y^3)\right\} = \left[(i\gamma^\mu\gamma^0\partial_\mu\right]_{\alpha\beta} i\Delta(x-y). \tag{14.1}$$

Apply projector and set $x^+=y^+$

$$\Lambda^+_{\alpha\gamma}\left\{\psi_\gamma(x^+,x^1,x^2,x^-),\psi_\delta(x^+,y^1,y^2,y^-)\right\}\Lambda^+_{\delta\beta}$$
$$= \left\{[\psi_{(+)}(x^+,x^1,x^2,x^-)]_\alpha,[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\} = \Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{14.2}$$

At $x^+=y^+$ UNEQUAL instant-time anticommutator is EQUAL light-front time anticommutator. Can also derive anticommutators involving bad fermions in the same way.

All cases discussed in Mannheim, Phys. Rev. D 102, 025020 (2020).

Light-front quantization is instant-time quantization, and does not need to be independently postulated. The seemingly different structure between EQUAL instant-time and EQUAL light-front time commutators is actually a consequence of the structure of UNEQUAL instant-time time commutators and anticommutators as restricted to equal x^0 or equal x^+.

Now the transformation $x^+ = x^0 + x^3$, $x^- = x^0 - x^3$ is not a Lorentz transformation but a translation, i.e., a general coordinate transformation. But for theories that are Poincare invariant this is a symmetry. Thus:

GENERAL RULE: ANY TWO DIRECTIONS OF QUANTIZATION THAT CAN BE CONNECTED BY A GENERAL COORDINATE TRANSFORMATION DESCRIBE THE SAME THEORY.

BUT IN THE QUANTUM THEORY TRANSLATIONS ARE UNITARY TRANSFORMATIONS. THUS INSTANT-TIME AND LIGHT-FRONT THEORIES ARE UNITARILY EQUIVALENT, AND ARE THUS ONE AND THE SAME THEORY.

16 UNITARY EQUIVALENCE VIA TRANSLATION INVARIANCE

So far the discussion has only dealt with free theory commutators, and they just happen to be c-numbers. However, for interacting theories we can only discuss matrix elements. With

$$[\hat{P}_\mu, \phi] = -i\partial_\mu \phi, \quad [\hat{P}_\mu, \hat{P}_\nu] = 0 \tag{16.1}$$

to all orders in perturbation theory because of Poincare invariance, we introduce

$$U(\hat{P}_0, \hat{P}_3) = \exp(ix^3 \hat{P}_0)\exp(ix^0 \hat{P}_3). \tag{16.2}$$

It effects

$$U\phi(IT; x^0, x^1, x^2, -x^3)U^{-1} = \phi(IT; x^0 + x^3, x^1, x^2, x^0 - x^3) = \phi(LF; x^+, x^1, x^2, x^-) \tag{16.3}$$

Then with a light-front vacuum of the form $|\Omega_F\rangle = U|\Omega_I\rangle$ we obtain

$$\begin{aligned} -i\langle\Omega_I|[\phi(IT; x^0, x^1, x^2, -x^3), \phi(0)]|\Omega_I\rangle &= -i\langle\Omega_I|U^\dagger U[\phi(IT; x^0, x^1, x^2, -x^3), \phi(0)]U^\dagger U|\Omega_I\rangle \\ &= -i\langle\Omega_F|[\phi(LF; x^+, x^1, x^2, x^-), \phi(0)]|\Omega_F\rangle, \end{aligned} \tag{16.4}$$

to all orders in perturbation theory. We thus establish the unitary equivalence of matrix elements of instant-time and light-front commutators to all orders.

The same equivalence holds for the all-order Lehmann representations. For the instant-time case we have

$$\langle\Omega|[\phi(IT;x),\phi(IT;y)]|\Omega\rangle = \frac{1}{(2\pi)^3}\int_0^\infty d\sigma^2\rho(\sigma^2,IT)\int d^4q\epsilon(q_0)\delta(q^2-\sigma^2)e^{-iq\cdot(x-y)}$$
$$= \int_0^\infty d\sigma^2\rho(\sigma^2,IT)i\Delta(IT,FREE;x-y,\sigma^2), \tag{16.5}$$

where

$$\rho(q^2,IT)\theta(q_0) = (2\pi)^3\sum_n \delta^4(p_\mu^n-q_\mu)|\langle\Omega|\phi(0)|p_\mu^n\rangle|^2, \quad \hat{P}_\mu|p_\mu^n\rangle = p_\mu^n|p_\mu^n\rangle, \tag{16.6}$$

as written in instant-time momentum eigenstates.
For the light-front case we have

$$\langle\Omega|[\phi(LF;x),\phi(LF;y)]|\Omega\rangle = \frac{2}{(2\pi)^3}\int_0^\infty d\sigma^2\rho(\sigma^2,LF)\int d^4q\epsilon(q_+)\delta(q^2-\sigma^2)e^{-iq\cdot(x-y)}.$$
$$= \int_0^\infty d\sigma^2\rho(\sigma^2,LF)i\Delta(LF,FREE;x-y,\sigma^2), \tag{16.7}$$

where

$$\rho(q_\mu,LF) = \frac{(2\pi)^3}{2}\sum_n \delta^4(p_\mu^n-q_\mu)|\langle\Omega|\phi(0)|p_\mu^n\rangle|^2 = \rho(q^2,LF)\theta(q_+), \tag{16.8}$$

as written in light-front momentum eigenstates. Then with

$$U|p_0^n\rangle = |p_+^n\rangle, \quad U|p_3^n\rangle = |p_-^n\rangle, \quad U|p_1^n\rangle = |p_1^n\rangle, \quad U|p_2^n\rangle = |p_2^n\rangle \tag{16.9}$$

we obtain the all-order

$$\langle\Omega|[\phi(IT;x),\phi(IT;y)]|\Omega\rangle = \langle\Omega|[\phi(LF;x),\phi(LF;y)]|\Omega\rangle. \tag{16.10}$$

With the all-order momentum operators having real and complete eigenspectra we have the all-order

$$\hat{P}_\mu(IT) = \sum |p^n(IT)\rangle p^n_\mu(IT)\langle p^n(IT)|, \quad \hat{P}_\mu(LF) = \sum |p^n(LF)\rangle p^n_\mu(LF)\langle p^n(LF)|. \tag{16.11}$$

With eigenvalues not changing under a unitary transformation, we obtain

$$\begin{aligned}\hat{P}_0(IT) = U\hat{P}_0(IT)U^{-1} = U\sum |p^n(IT)\rangle p^n_0\langle p^n(IT)|U^\dagger \\ = \sum |p^n(LF)\rangle(p^n_+ + p^n_-)\langle p^n(LF)| = \hat{P}_+(LF) + \hat{P}_-(LF).\end{aligned} \tag{16.12}$$

Given (16.11) and (16.12), there initially appears to be a mismatch between the eigenstates of $\hat{P}_0(IT)$ and $\hat{P}_+(LF)$. However, for any timelike set of instant-time momentum eigenvalues we can Lorentz boost p_1, p_2 and p_3 to zero, to yield

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 0, \quad p_0 = m. \tag{16.13}$$

If we impose this same $p_1 = 0$, $p_2 = 0$, $p_3 = 0$ condition on the light-front momentum eigenvalues we would set $p_+ = p_-$, $p^2 = 4p_+^2 = m^2$, and thus obtain

$$p_1 = 0, \quad p_2 = 0, \quad p_+ = p_-, \quad p_0 = 2p_+ = m \tag{16.14}$$

When written in terms of contravariant vectors with $p^\mu = g^{\mu\nu}p_\nu$ this condition takes the form

$$p^0 = p^- = m. \tag{16.15}$$

Thus in **the instant-time rest frame** the eigenvalues of the contra variant $\hat{P}^0(IT)$ and $\hat{P}^-(LF)$ coincide. In this sense then instant-time and light-front Hamiltonians are equivalent. And non-relativistic in the light-front case still means $p_3 = 0$, i.e., $p_+ = p_-$, and not $p_- = p^+/2 = 0$.

17 AdS/CFT

Work done by Guth, Kaiser, Mannheim and Nayeri, as reported in Mannheim: Brane-Localized Gravity (World Scientific 2005).

An AdS_5 geometry can be described by the flat $M(4,2)$ metric

$$ds^2 = dW^2 + dX^2 + dY^2 + dZ^2 - dU^2 - dV^2, \tag{17.1}$$

as subject to the constraint

$$U^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = \ell^2. \tag{17.2}$$

We introduce

$$X = R\,\sin\theta\,\cos\phi \;\;,\;\; Y = R\,\sin\theta\,\sin\phi \;\;,\;\; Z = R\,\cos\theta \;\;, \tag{17.3}$$

$$\eta_1 = t \;\;,\;\; \eta_+ = \ell \;\;,\;\; \eta_- = \ell + \frac{r^2 - t^2}{\ell} \;\;,\;\; \eta_4 = r \tag{17.4}$$

$$\chi_1 \quad = -t \;\;,\;\; \chi_+ = -\ell \;,\;\; \chi_- = \ell + \frac{t^2 - r^2}{\ell} \;\;,\;\; \chi_4 = -r \;\;. \tag{17.5}$$

Then on setting

$$\begin{aligned} U \quad &= \eta_1 \cosh\left(\frac{w}{\ell}\right) + \chi_1 \sinh\left(\frac{w}{\ell}\right) \;,\; V + W = \eta_+ \cosh\left(\frac{w}{\ell}\right) + \chi_+ \sinh\left(\frac{w}{\ell}\right) \;, \\ V - W \quad &= \eta_- \cosh\left(\frac{w}{\ell}\right) + \chi_- \sinh\left(\frac{w}{\ell}\right) \;,\; R = \eta_4 \cosh\left(\frac{w}{\ell}\right) + \chi_4 \sinh\left(\frac{w}{\ell}\right) \end{aligned} \tag{17.6}$$

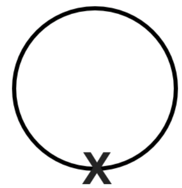
we obtain

$$ds^2 = \quad dw^2 + e^{-2w/\ell}\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - dt^2\right] \;\;. \tag{17.7}$$

to thus embed a 4-dimensional Minkowski surface in a 5-dimensional AdS_5 bulk. We note that the embedding has to done with light-front variables.

18 THE NAMBU-JONA-LASINIO CHIRAL FOUR-FERMION MODEL AS A MEAN-FIELD THEORY

Nambu and Jona-Lasinio Phys. Rev. 122, 345 (1961).



Introduce mass term with m as a trial parameter and note $m^2/2g$ term

$$\begin{aligned} I_{\rm NJL} &= \int d^4x \left[i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{g}{2}[\bar{\psi}\psi]^2 - \frac{g}{2}[\bar{\psi}i\gamma_5\psi]^2 \right] \\ &= \int d^4x \left[i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + \frac{m^2}{2g} \right] + \int d^4x \left[-\frac{g}{2}\left(\bar{\psi}\psi - \frac{m}{g}\right)^2 - \frac{g}{2}\left(\bar{\psi}i\gamma_5\psi\right)^2 \right] \\ I_{\rm NJL} &= I_{\rm MF} + I_{\rm RI}, \quad \text{mean field plus residual interaction} \end{aligned} \tag{18.1}$$

Hartree-Fock approximation

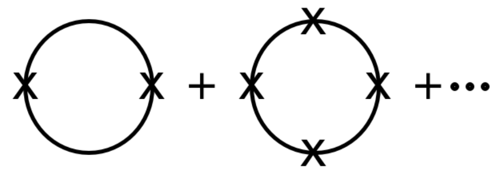
$$\langle\Omega_{\rm m}| \left[\bar{\psi}\psi - \frac{m}{g}\right]^2 |\Omega_{\rm m}\rangle = \langle\Omega_{\rm m}| \left[\bar{\psi}\psi - \frac{m}{g}\right] |\Omega_{\rm m}\rangle^2 = 0, \tag{18.2}$$

$$\langle\Omega_{\rm m}|\bar{\psi}\psi|\Omega_{\rm m}\rangle = -i\int \frac{d^4k}{(2\pi)^4}{\rm Tr}\left[\frac{1}{\not{p} - m + i\epsilon}\right] = \frac{m}{g}, \tag{18.3}$$

Satisfied by self-consistent M, and defines g^{-1}

$$-\frac{M\Lambda^2}{4\pi^2} + \frac{M^3}{4\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right) = \frac{M}{g}. \tag{18.4}$$

18.1 Vacuum Energy



$$\begin{aligned}\epsilon(m) &= i\int\frac{d^4p}{(2\pi)^4}\text{Tr}\ \ln\left[\not{p}-m+i\epsilon\right]-i\int\frac{d^4p}{(2\pi)^4}\text{Tr}\ \ln\left[\not{p}+i\epsilon\right]\\ &= -\frac{m^2\Lambda^2}{8\pi^2}+\frac{m^4}{16\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right)+\frac{m^4}{32\pi^2} \qquad (18.5)\end{aligned}$$

is quadratically divergent.

$$\begin{aligned}\tilde{\epsilon}(m) &= \epsilon(m)-\frac{m^2}{g}\\ &= \frac{m^4}{16\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right)-\frac{m^2M^2}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)+\frac{m^4}{32\pi^2}. \qquad (18.6)\end{aligned}$$

is only log divergent, with double-well potential emerging, but still cutoff dependent.

18.2 Higgs-Like Lagrangian

Vacuum to vacuum functional due to $m(x)\bar{\psi}\psi$: $\langle\Omega(t=-\infty)|\Omega(t=+\infty)\rangle = e^{iW(m(x))}$

$$W(m(x)) = \sum\frac{1}{n!}\int d^4x_1...d^4x_n G_0^n(x_1,...,x_n)m(x_1)...m(x_n),$$

$$W(m(x)) = \int d^4x\left[-\epsilon(m(x)) + \frac{1}{2}Z(m(x))\partial_\mu m(x)\partial^\mu m(x) + ...\right].$$

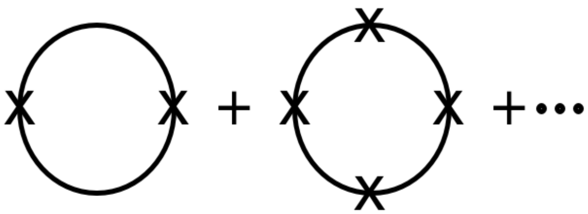


Figure 1: Vacuum energy density $\epsilon(m)$ via an infinite summation of massless graphs with zero-momentum point $m\bar{\psi}\psi$ insertions.

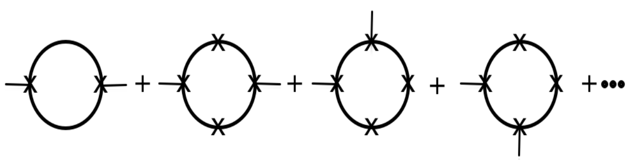


Figure 2: $\Pi_{\rm S}(q^2, m(x))$ developed as an infinite summation of massless graphs, each with two point $m\bar{\psi}\psi$ insertions carrying momentum q_μ (shown as external lines), with all other point $m\bar{\psi}\psi$ insertions carrying zero momentum.

Eguchi and Sugawara (Phys. Rev. D 10, 4257 (1974)), Mannheim (Phys. Rev. D 14, 2072 (1976)):

$$I_{\rm EFF} = \int\frac{d^4x}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)\left[\frac{1}{2}\partial_\mu m(x)\partial^\mu m(x) + m^2(x)M^2 - \frac{1}{2}m^4(x)\right]. \tag{18.7}$$

Set $\phi = \langle\Omega_m|\bar{\psi}(1+\gamma^5)\psi|\Omega_m\rangle$. Couple to an axial gauge field via $\bar{\psi}g_A\gamma^\mu\gamma^5 A_{\mu 5}\psi$. Get effective Higgs:

$$I_{\rm EFF} = \int\frac{d^4x}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)\left[\frac{1}{2}|(\partial_\mu - 2ig_{\rm A}A_{\mu 5})\phi(x)|^2 + |\phi(x)|^2M^2 - \frac{1}{2}|\phi(x)|^4 - \frac{g_{\rm A}^2}{6}F_{\mu\nu 5}F^{\mu\nu 5}\right]. \tag{18.8}$$

18.3 The Collective Goldstone and Higgs Modes when the Fermion is Massive

$$\begin{aligned}
\Pi_{\rm S}(q^2,M) &= -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[\frac{1}{\not p - M + i\epsilon}\frac{1}{\not p + \not q - M + i\epsilon}\right] \\
&= -\frac{\Lambda^2}{4\pi^2}+\frac{M^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)+\frac{(4M^2-q^2)}{8\pi^2}+\frac{(4M^2-q^2)}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right) \\
&-\frac{1}{8\pi^2}\frac{(4M^2-q^2)^{3/2}}{(-q^2)^{1/2}}\ln\left(\frac{(4M^2-q^2)^{1/2}+(-q^2)^{1/2}}{(4M^2-q^2)^{1/2}-(-q^2)^{1/2}}\right).
\end{aligned} \tag{18.9}$$

$$\begin{aligned}
\Pi_{\rm P}(q^2,M) &= -i\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\left[i\gamma_5\frac{1}{\not p - M + i\epsilon}i\gamma_5\frac{1}{\not p + \not q - M + i\epsilon}\right] \\
&= -\frac{\Lambda^2}{4\pi^2}+\frac{M^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)-\frac{q^2}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)+\frac{(4M^2-q^2)}{8\pi^2} \\
&+\frac{(8M^4-8M^2q^2+q^4)}{8\pi^2(-q^2)^{1/2}(4M^2-q^2)^{1/2}}\ln\left(\frac{(4M^2-q^2)^{1/2}+(-q^2)^{1/2}}{(4M^2-q^2)^{1/2}-(-q^2)^{1/2}}\right).
\end{aligned} \tag{18.10}$$

$$T_{\rm S}(q^2) \;=\; \frac{R_{\rm S}^{-1}}{(q^2-4M^2)}, \qquad T_{\rm P}(q^2)=\frac{R_{\rm P}^{-1}}{q^2}, \tag{18.11}$$

$$R_{\rm S} \;=\; R_{\rm P}=\frac{1}{8\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right). \tag{18.12}$$

The scalar Higgs mass is finite and of order the dynamical fermion mass, and residue is determined.

19 ADD IN SCALE INVARIANCE

In QED at a fixed point (Johnson, Baker and Willey Phys. Rev 136, B1111 (1961); Phys. Rev. 163, 1699 (1964))) compatibility of the short-distance Wilson expansion and the propagator $S^{-1}(p) \sim \not{p} - (-p^2)^{\gamma_\theta(\alpha)/2}$ gives $\gamma_\theta(\alpha) = -1$ (Mannheim Phys. Rev. D 12, 1772 (1975)). Specifically, in a scale invariant theory the Wilson expansion is of the form

$$T(\psi(x)\bar{\psi}(0)) \;=\; \langle\Omega_0|T(\psi(x)\bar{\psi}(0))|\Omega_0\rangle + (\mu^2 x^2)^{\gamma_\theta(\alpha)/2} : \psi(0)\bar{\psi}(0) : \tag{19.1}$$

where the normal ordering is done with respect to the unbroken massless vacuum $|\Omega_0\rangle$. Now take matrix element in the spontaneously broken vacuum $|\Omega_m\rangle$, to obtain

$$\tilde{S}(p) = \frac{1}{\not{p}} + (-p^2)^{(-\gamma_\theta(\alpha)/2-2)}, \quad \tilde{S}^{-1}(p) = \not{p} - (-p^2)^{(-\gamma_\theta(\alpha)-2)/2}. \tag{19.2}$$

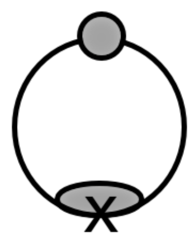
Compatibility with $S^{-1}(p) \sim \not{p} - (-p^2)^{\gamma_\theta(\alpha)/2}$ then gives $\gamma_\theta(\alpha) = -\gamma_\theta(\alpha) - 2$, i. e. $\gamma_\theta(\alpha) = -1$.

20 SCALE INVARIANT QED COUPLED TO FOUR FERMI THEORY AT $\gamma_\theta(\alpha) = -1$

$$
\begin{aligned}
\mathcal{L}_{\rm QED-FF} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi - \frac{g}{2}[\bar{\psi}\psi]^2 - \frac{g}{2}[\bar{\psi}i\gamma_5\psi]^2 \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\bar{\psi}\psi + \frac{m^2}{2g} - \frac{g}{2}\left(\bar{\psi}\psi - \frac{m}{g}\right)^2 - \frac{g}{2}\left(\bar{\psi}i\gamma_5\psi\right)^2 \\
&= \mathcal{L}_{\rm QED-MF} + \mathcal{L}_{\rm QED-RI}.
\end{aligned}
\tag{20.1}
$$

$$
\tilde{S}^{-1}(p) = \not{p} - m\left(\frac{-p^2 - i\epsilon}{\mu^2}\right)^{-1/2} + i\epsilon, \quad \tilde{\Gamma}_{\rm S}(p,p,0) = \left(\frac{-p^2 - i\epsilon}{\mu^2}\right)^{-1/2}
\tag{20.2}
$$

as renormalized at μ^2. With dimension of $(\bar{\psi}\psi)^2$ dropping from 6 to 4 when $\gamma_\theta = -1$, quadratic divergences become logarithmic, and four-fermion interaction becomes renormalizable to all orders in g (Mannheim, Phys. Lett. B 773, 604 (2017)).

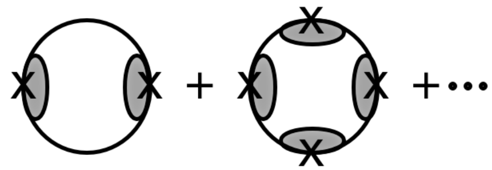


$$
\langle \Omega_m|\bar{\psi}\psi|\Omega_m\rangle = -\frac{m\mu^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{m\mu}\right) = \frac{m}{g}.
\tag{20.3}
$$

$$
-\frac{\mu^2}{4\pi^2}\ln\left(\frac{\Lambda^2}{M\mu}\right) = \frac{1}{g}, \qquad M = \frac{\Lambda^2}{\mu}\exp\left(\frac{4\pi^2}{\mu^2 g}\right).
\tag{20.4}
$$

Gap equation gives $-g \sim 1/\ln\Lambda^2$. Thus g is negative, i.e. attractive, and becomes very small as $\Lambda \to \infty$, with BCS-type essential singularity in gap equation at $g = 0$. Hence dynamical symmetry breaking with weak coupling.

20.1 Vacuum Energy

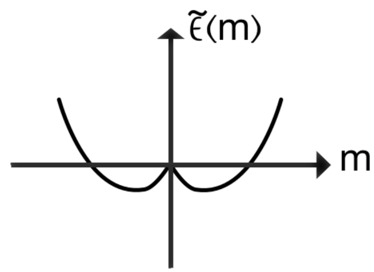


$$\epsilon(m) = \frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\ \ln\left[1 - \frac{m^2}{p^2+i\epsilon}\left(\frac{-p^2-i\epsilon}{\mu^2}\right)^{-1}\right] = -\frac{m^2\mu^2}{8\pi^2}\left[\ln\left(\frac{\Lambda^2}{m\mu}\right)+\frac{1}{2}\right] \tag{20.5}$$

and is only log divergent. Due to presence of $m^2/2g$ term we obtain the completely finite

$$\tilde{\epsilon}(m) = \epsilon(m) - \frac{m^2}{2g} = \frac{m^2\mu^2}{16\pi^2}\left[\ln\left(\frac{m^2}{M^2}\right) - 1\right], \tag{20.6}$$

which we recognize as a double-well potential, dynamically induced.



We thus see the power of dynamical symmetry breaking. It reduces divergences. Moreover, since $m^2/2g$ is a cosmological term, dynamical symmetry breaking has a control over the cosmological constant problem that an elementary Higgs field potential does not. When coupled to conformal gravity (as needed for quartic divergence in the vacuum energy that we ignored), the cosmological constant problem is completely solved.

20.2 Higgs-Like Lagrangian

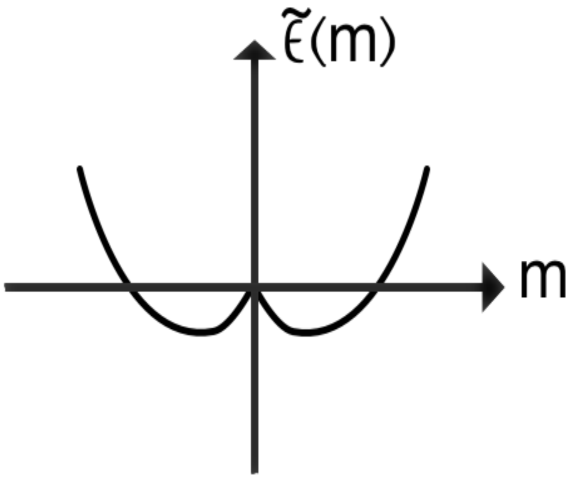


Figure 3: Dynamically generated double-well potential for the renormalized vacuum energy density when $\gamma_\theta(\alpha) = -1$.

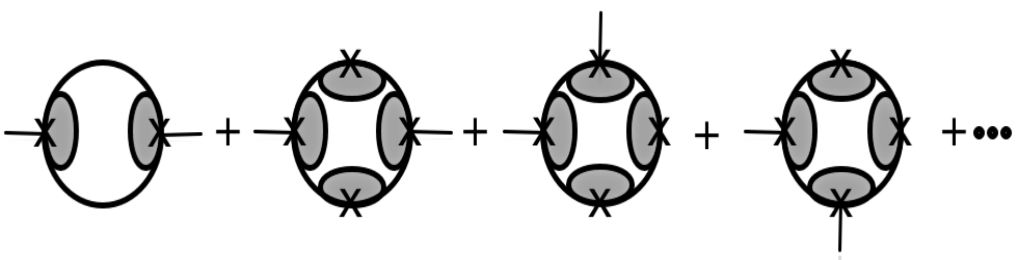
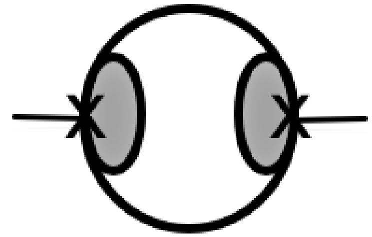


Figure 4: $\Pi_{\rm S}(q^2, m(x))$ developed as an infinite summation of massless graphs, each with two dressed $m\bar{\psi}\psi$ insertions carrying momentum q_μ (shown as external lines), with all other dressed $m\bar{\psi}\psi$ insertions carrying zero momentum.

Mannheim Nucl. Phys. B 143, 285 (1978):

$$\begin{aligned}
\mathcal{L}_{\rm EFF} &= -\tilde{\epsilon}(m(x)) - \frac{1}{2}m(x)[\Pi_{\rm S}(-\partial_\mu\partial^\mu, m(x)) - \Pi_{\rm S}(0, m(x))]m(x) + ... \\
&= -\frac{m^2(x)\mu^2}{16\pi^2}\left[\ln\left(\frac{m^2(x)}{M^2}\right) - 1\right] + \frac{3\mu}{256\pi m(x)}\partial_\mu m(x)\partial^\mu m(x) + ... \qquad (20.7)
\end{aligned}$$

20.3 The Collective Tachyon Modes when the Fermion is Massless



$$\Pi_{\rm S}(q^2, m=0) = \Pi_{\rm P}(q^2, m=0) = -\frac{\mu^2}{4\pi^2}\left[\ln\left(\frac{\Lambda^2}{(-q^2)}\right) - 3 + 4\ \ln 2\right]. \qquad (20.8)$$

$$\begin{aligned} T_{\rm S}(q^2) &= \frac{g}{1 - g\Pi_{\rm S}(q^2)} = \frac{1}{g^{-1} - \Pi_{\rm S}(q^2)}, \\ T_{\rm P}(q^2) &= \frac{g}{1 - g\Pi_{\rm P}(q^2)} = \frac{1}{g^{-1} - \Pi_{\rm P}(q^2)}, \end{aligned} \qquad (20.9)$$

$$q^2 = -M\mu e^{4\ln 2 - 3} = -0.797M\mu, \qquad (20.10)$$

$$T_{\rm S}(q^2) = T_{\rm P}(q^2) = \frac{31.448M\mu}{(q^2 + 0.797M\mu)} \qquad (20.11)$$

20.4 The Collective Goldstone Mode when the Fermion is Massive

$$\begin{aligned}\Pi_{\rm P}(q^2=0,m) &= -4i\mu^2\int\frac{d^4p}{(2\pi)^4}\frac{(p^2)(-p^2)-m^2\mu^2}{((p^2+i\epsilon)^2+m^2\mu^2)^2}.\\ &= 4i\mu^2\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+i\epsilon)^2+m^2\mu^2}=\frac{1}{g}.\end{aligned} \tag{20.12}$$

$$T_{\rm P}(q^2)=\frac{128\pi M}{7\mu q^2}=\frac{57.446M}{\mu q^2}. \tag{20.13}$$

20.5 The Collective Higgs Mode when the Fermion is Massive

$$q_0({\rm Higgs}) = (1.480-0.017i)(M\mu)^{1/2}, \qquad q^2({\rm Higgs})=(2.189-0.051{\rm i}){\rm M}\mu. \tag{20.14}$$

$$q_0({\rm Higgs}) = (1.480-0.017i)M, \qquad q^2({\rm Higgs})=(2.189-0.051{\rm i}){\rm M}^2, \tag{20.15}$$

Higgs mass is close to dynamical fermion mass, but above threshold, and thus has a width. In a double well elementary Higgs field theory Higgs mass is real. Width can be used to distinguish an elementary Higgs from a dynamical one.

One thing more: We can cancel the quartic divergence in the matter sector vacuum energy by a a quartic divergence in the gravity sector, provided the gravity sector is conformal gravity. Together with dynamical mass generation this then controls the cosmological constant (Mannheim, Prog. Part. Nucl. Phys. 94, 125 (2017)).

21 FITTING THE ACCELERATING UNIVERSE DATA

To summarize: The vacuum energy momentum tensor of a massive free fermion has a quartic divergence, a quadratic divergence, a logarithmic divergence, and a finite part. Conformal gravity takes care of the quartic divergence and the finite part. Critical scaling in the matter sector and the reduction in the dimension of $d_\theta(\alpha)$ from three to two reduces the quadratic divergence to logarithmic, and the mean field induced $-m^2/2g$ term when mass is generated dynamically takes care of the resulting logarithmic divergence, with an associated dynamically generated Higgs boson having no hierarchy problem.

With S now denoting the occupied positive energy state contribution $\langle\Omega_M|b\bar{\psi}\psi b^\dagger|\Omega_M\rangle - \langle\Omega_M|\bar{\psi}\psi|\Omega_M\rangle$, the relevant effective conformal invariant Lagrangian is given by

$$\mathcal{L}_{\rm EFF} = \frac{S^4}{16\pi^2} - \frac{S^2}{512\pi} R^\alpha{}_\alpha, \tag{21.1}$$

It acts as an effective repulsive gravity theory with a negative cosmological $G_{\rm EFF} = -32/S^2$. (Mannheim, Ap. J. **391**, 429 (1992)). However local gravity is still attractive.

$T^{\mu\nu}_{\rm M}$ is given by

$$T^{\mu\nu}_{\rm M} = (\rho_{\rm M} + p_{\rm M})U^\mu U^\nu + p_{\rm M} - \frac{S^2}{256\pi}\left(R^{\mu\nu} - \frac{g^{\mu\nu}}{2}R^\alpha{}_\alpha\right) - g^{\mu\nu}\frac{S^4}{16\pi^2}. \tag{21.2}$$

In a conformal Robertson-Walker cosmology we can show that $-1 \leq q_0 \leq 0$ no matter what the value of the parameters. The associated luminosity function is given by

$$d_L = -\frac{c}{H_0}\frac{(1+z)^2}{q_0}\left(1 - \left[1 + q_0 - \frac{q_0}{(1+z)^2}\right]^{1/2}\right), \tag{21.3}$$

with a best fit (Mannheim Prog. Part. Nucl. Phys. **56**, 340 (2006)) with $q_0 = -0.37$, a value that non-trivially is right in the $-1 \leq q_0 \leq 0$ range.

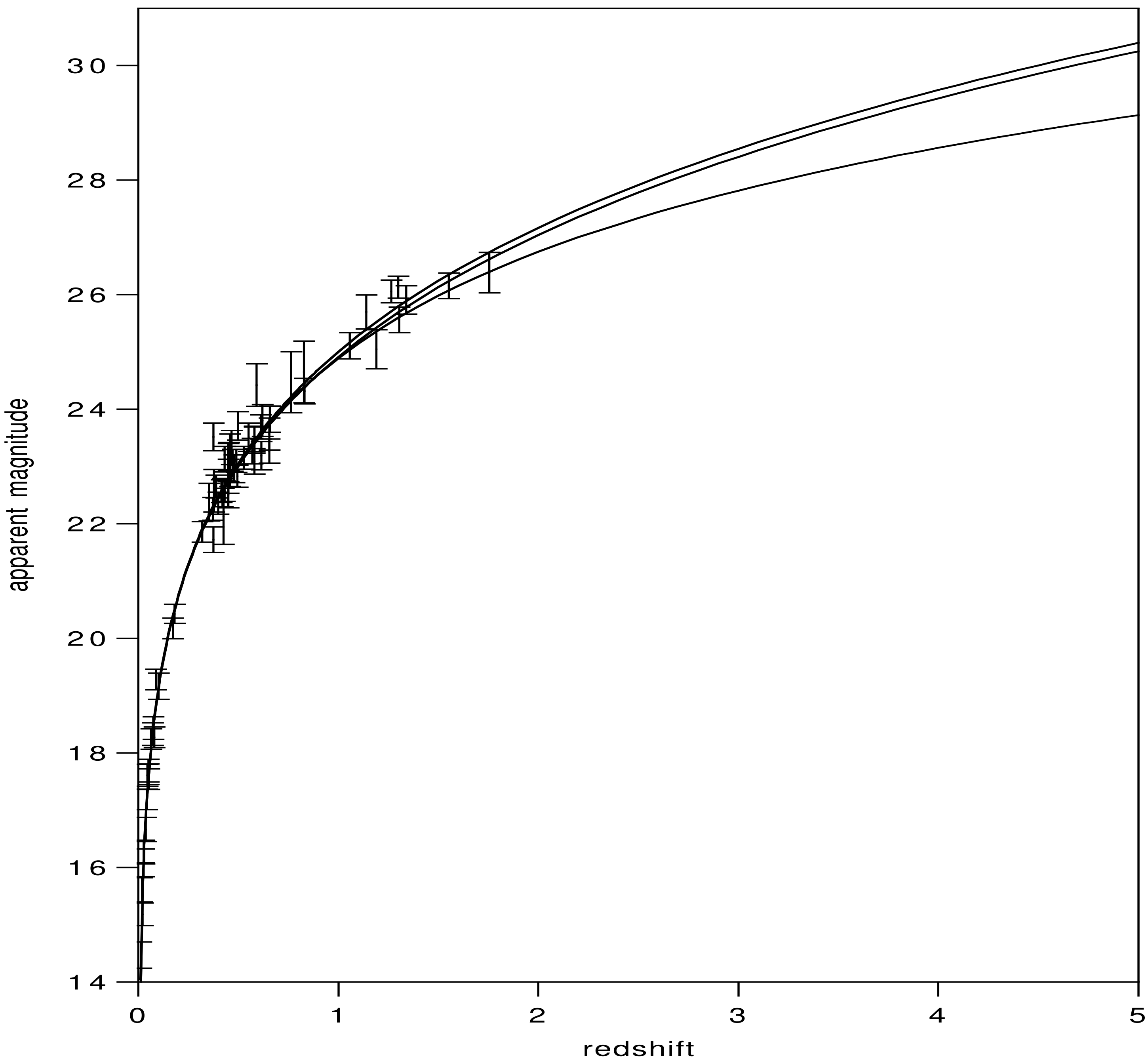


Figure 5: Hubble plot expectations for $q_0 = -0.37$ (highest curve) and $q_0 = 0$ (middle curve) conformal gravity and for $\Omega_M(t_0) = 0.3$, $\Omega_\Lambda(t_0) = 0.7$ standard gravity (lowest curve).

22 THE MORAL OF THE STORY

There is a lot of interesting physics on the light cone and even more interesting physics off it.