Interpolating 't Hooft model between the Instant Form Dynamics and the Light Front Dynamics

Bailing Ma
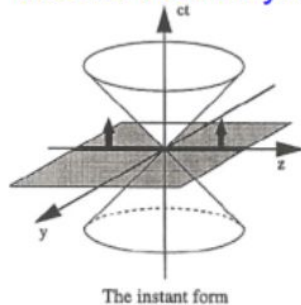
Light Cone 2022 Online: Physics of Hadrons on the Light Front
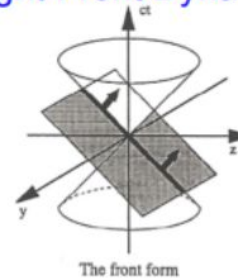
Sep. 21, 2022

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Light Front Dynamics

IFD
Instant Form Dynamics

The instant form

LFD
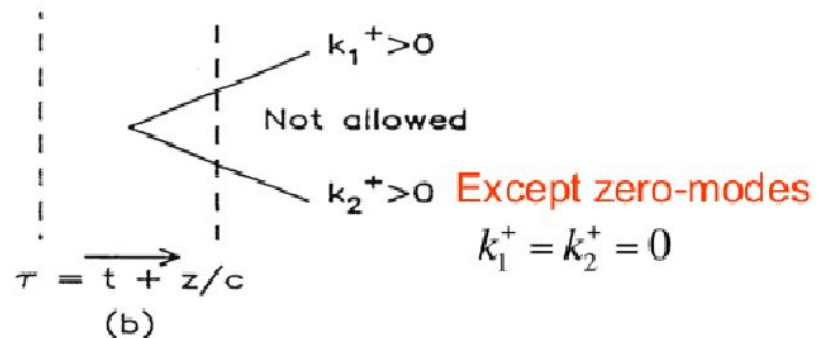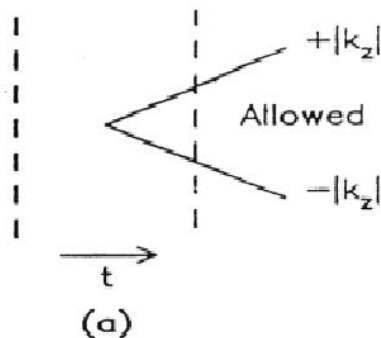**Light-Front Dynamics**

The front form

Energy-Momentum Dispersion Relations

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2}$$

$$p^- = \frac{p_\perp^{\,2} + m^2}{p^+}$$
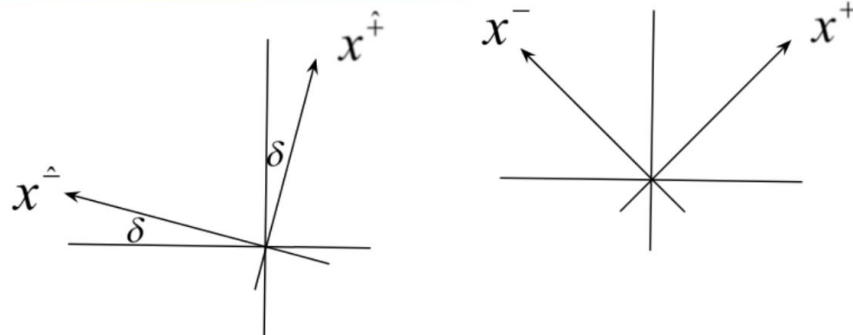
Time-ordered Diagrams: Vacuum fluctuations

$+|k_z|$

Allowed

$-|k_z|$

t

(a)

$k_1^+ > 0$

Not allowed

$k_2^+ > 0$

Except zero-modes

$$k_1^+ = k_2^+ = 0$$

$\tau = t + z/c$

(b)

Interpolation Method

IFD

$$(IFD) \quad 0 \le \delta \le \frac{\pi}{4} \quad (LFD)$$
$$1 \ge \quad C \equiv \cos(2\delta) \quad \ge 0$$

LFD

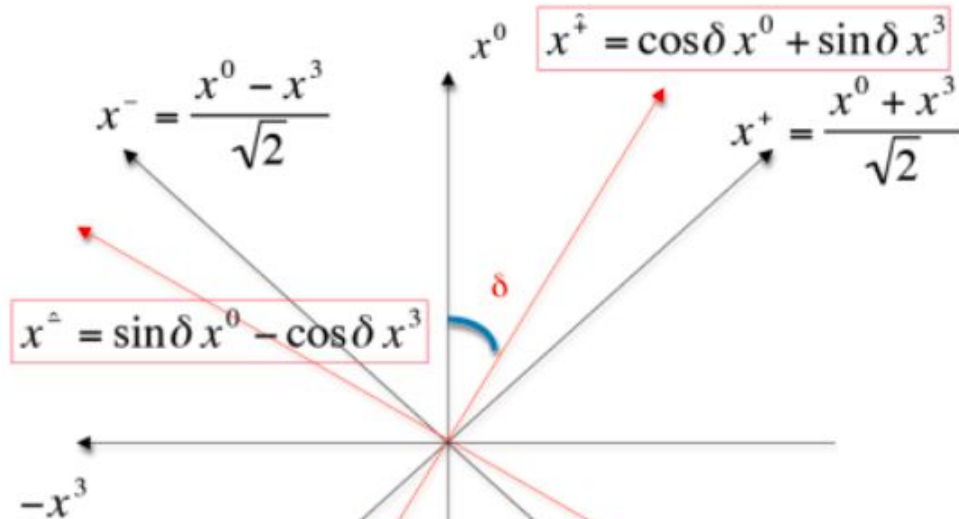x^0

x^3

$x^{\hat{+}}$

$x^{\hat{-}}$

δ

x^-

x^+
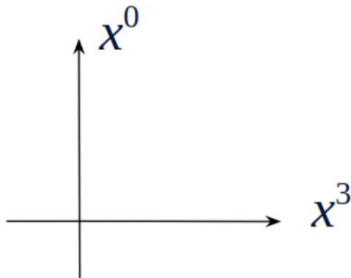
$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

- Relate IFD and LFD, show the whole landscape in between
- Magnify LF zero mode by varying the δ parameter
- Clarify any conceivable confusion between the IMF and LFD

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Works on Interpolation between IFD and LFD

$$x^{\hat{+}} = \cos\delta\, x^0 + \sin\delta\, x^3$$

$$x^- = \frac{x^0 - x^3}{\sqrt{2}}$$

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}$$

$$x^{\hat{-}} = \sin\delta\, x^0 - \cos\delta\, x^3$$

δ

x^0

$-x^3$

K. Hornbostel, PRD45, 3781 (1992) – RQFT
C.Ji and S.Rey, PRD53,5815(1996) – Chiral Anomaly
C.Ji and C. Mitchell, PRD64,085013 (2001) – Poincare Algebra
C.Ji and A. Suzuki, PRD87,065015 (2013) – Scattering Amps
C.Ji, Z. Li and A. Suzuki, PRD91, 065020 (2015) – EM Gauges
Z.Li, M. An and C.Ji, PRD92, 105014 (2015) – Spinors
C.Ji, Z.Li, B.Ma and A.Suzuki, PRD98, 036017(2018) – QED
B.Ma and C.Ji, PRD104, 036004(2021) – QCD$_{1+1}$
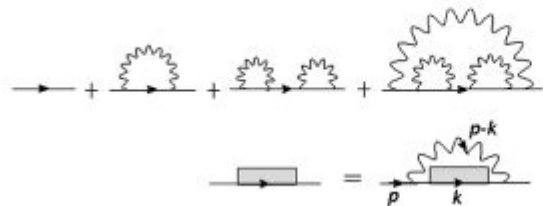
't Hooft model vs IFD QCD$_{3+1}$

$$\mathcal{L} = -\frac{1}{4} F^a_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}a} + \bar{\psi}(i\gamma^{\hat{\mu}} D_{\hat{\mu}} - m)\psi,$$

$$D_{\hat{\mu}} = \partial_{\hat{\mu}} - i g A^a_{\hat{\mu}} t_a$$

$$F^a_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}} A^a_{\hat{\nu}} - \partial_{\hat{\nu}} A^a_{\hat{\mu}} + g f^{abc} A^b_{\hat{\mu}} A^c_{\hat{\nu}}.$$

- 1 space 1 time dimensions, where confinement arises naturally due to the linear potential.
- Large N_c ('t Hooft coupling $\lambda \sim g^2 N_c$ is kept finite while $N_c \rightarrow \infty$ and $g \rightarrow 0$), so that non-planar diagrams are negligible. (Dimensionful quantities are scaled with respect to $\sqrt{2\lambda}$.)
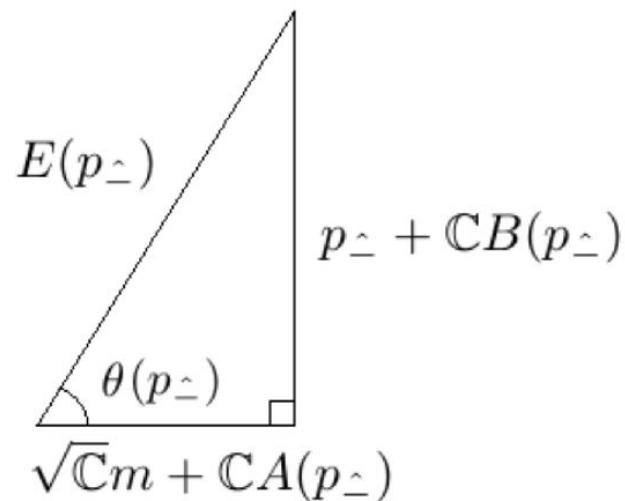- Gluon self-couplings are absent, and there is no ghost, with the use of axial gauge.
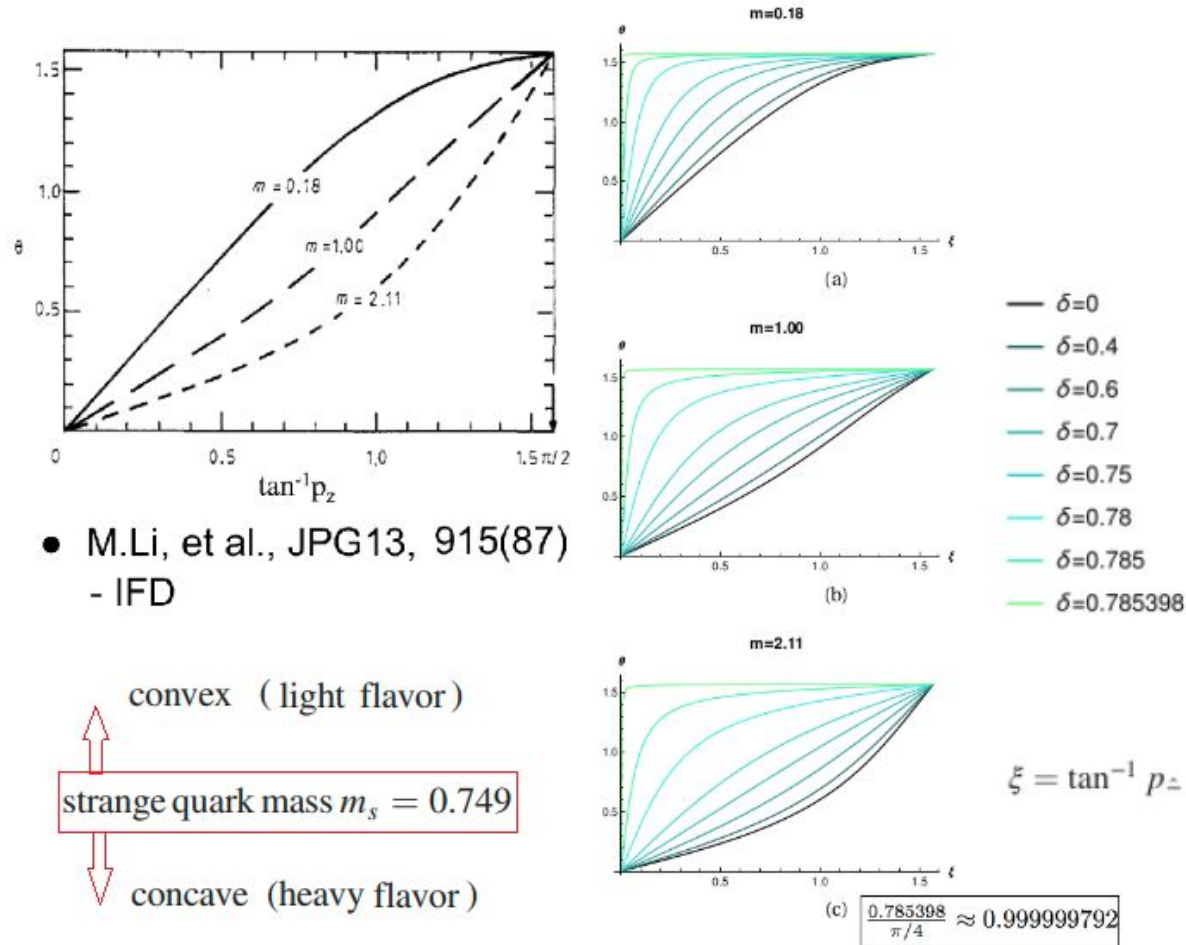
Mass gap equation

$$\Sigma(p_{\hat{-}}) = \sqrt{\mathbb{C}}A(p_{\hat{-}}) + \gamma_{\hat{-}}B(p_{\hat{-}})$$

$$E(p_{\hat{-}})\cos\theta(p_{\hat{-}}) = \sqrt{\mathbb{C}}\,m + \mathbb{C}\cdot\frac{\lambda}{2}\int \frac{d\,k_{\hat{-}}}{\left(p_{\hat{-}}-k_{\hat{-}}\right)^2}\cos\theta(k_{\hat{-}})$$

$$E(p_{\hat{-}})\sin\theta(p_{\hat{-}}) = p_{\hat{-}} + \mathbb{C}\cdot\frac{\lambda}{2}\int \frac{d\,k_{\hat{-}}}{\left(p_{\hat{-}}-k_{\hat{-}}\right)^2}\sin\theta(k_{\hat{-}})$$

Solutions of the mass gap equation : θ

- M.Li, et al., JPG13, 915(87)
  - IFD

convex (light flavor)

strange quark mass $m_s = 0.749$

concave (heavy flavor)

m=0.18

(a)

m=1.00

(b)

m=2.11

(c)

$\frac{0.785398}{\pi/4} \approx 0.999999792$

— $\delta=0$
— $\delta=0.4$
— $\delta=0.6$
— $\delta=0.7$
— $\delta=0.75$
— $\delta=0.78$
— $\delta=0.785$
— $\delta=0.785398$

$\xi = \tan^{-1} p_\perp$

The numerical solutions of $\theta(p_\perp)$ for several interpolation angles, corresponding to the quark mass values in Figure 4 of Ref. Li et al. (1987). All the mass values are in the unit of $\sqrt{2\lambda}$. Note that $\theta(p_\perp)$ is an odd function of $p_\perp$ and only the positive $p_\perp$ range is plotted with the variable $\xi = \tan^{-1} p_\perp$.

Solutions of the mass gap equation : E

- M.Li, et al., JPG13, 915(87)
  - IFD

$$\xi = \tan^{-1} p_\perp$$

$$\frac{0.785398}{\pi/4} \approx 0.999999792$$
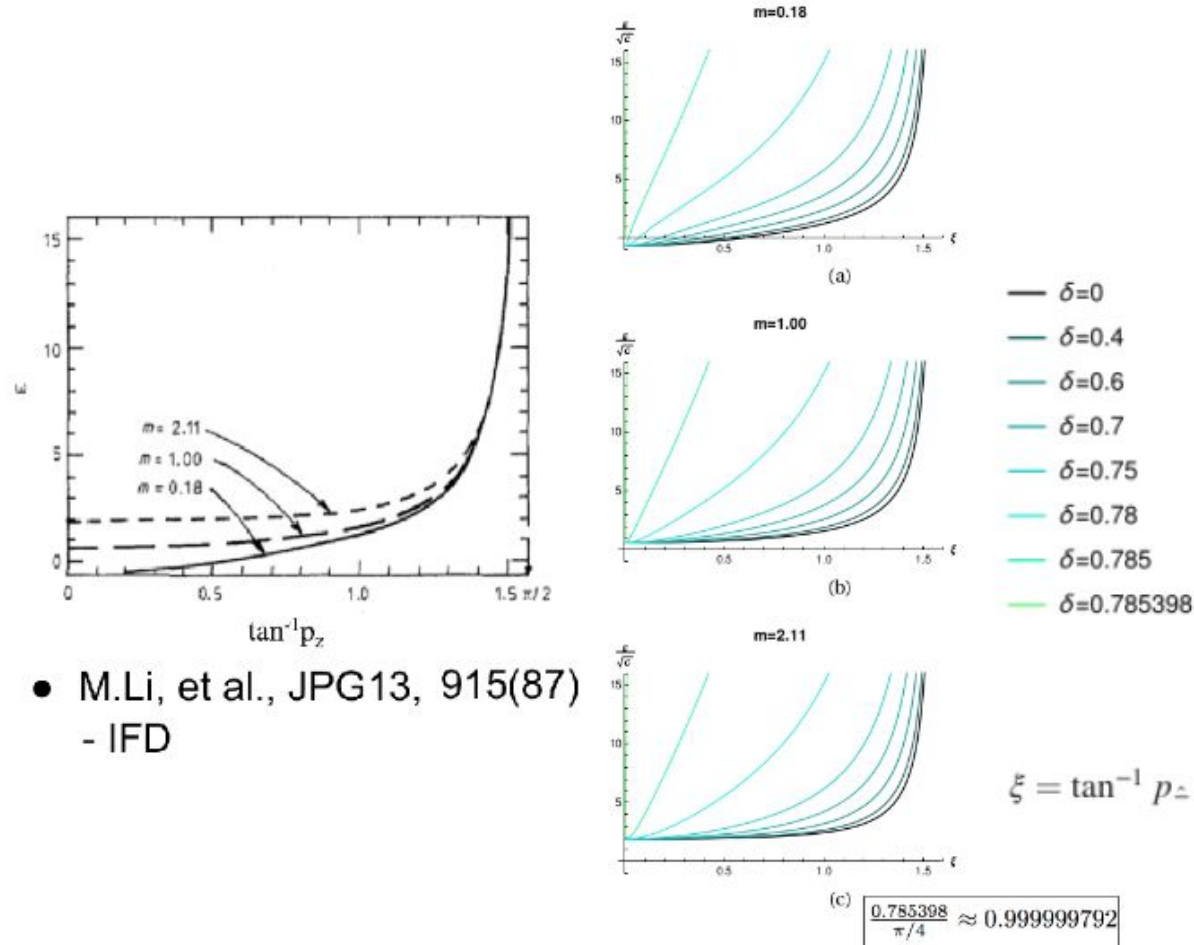
The solutions of $E(p_\perp)/\sqrt{C}$ for several interpolation angles for different choices of quark mass corresponding to the quark mass values in Figure 5 of Ref. Li et al. (1987). All quantities are in proper units of $\sqrt{2\lambda}$. $E(p_\perp)$ is an even function of $p_\perp$. We plot only for positive $p_\perp$ with the variable $\xi = \tan^{-1} p_\perp$.

Dressed quark propagator

Free Propagator

$$S_f(p) = \frac{1}{\not{p} - m + i\varepsilon}$$

$$\boxed{\Sigma(p) = \Sigma_s(p) + \Sigma_v(p)\not{p}}$$

Interacting Propagator

$$S(p) = \frac{1}{\not{p} - m - \Sigma(p) + i\varepsilon}$$

$$= \frac{F(p)}{\not{p} - M(p) + i\varepsilon}$$

$$F(p) = \left(1 - \Sigma_v(p)\right)^{-1}$$ **"Wave function renormalization factor"**

$$M(p) = \frac{m + \Sigma_s(p)}{1 - \Sigma_v(p)}$$ **"Renormalized fermion mass function"**

Ji, C.-R. (2012). Pion loops in chiral perturbation theory and light-front dynamics. Few Body Syst., 52:421–426.

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Energy-momentum dispersion relation for a dressed quark

Free particle in IFD

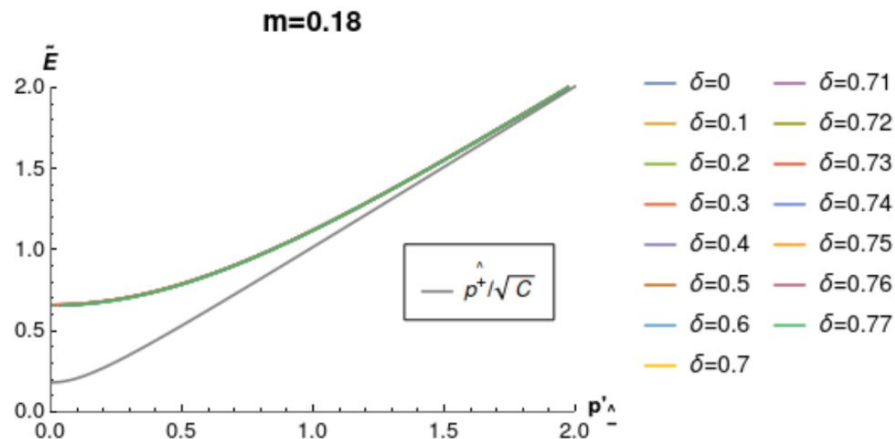Free particle in interpolation

Dressed particle in interpolation

$$\frac{F(p'_\perp)E(p'_\perp)}{\sqrt{C}} = \sqrt{p'^2_\perp + M(p'_\perp)^2} \equiv \tilde{E}(p'_\perp)$$

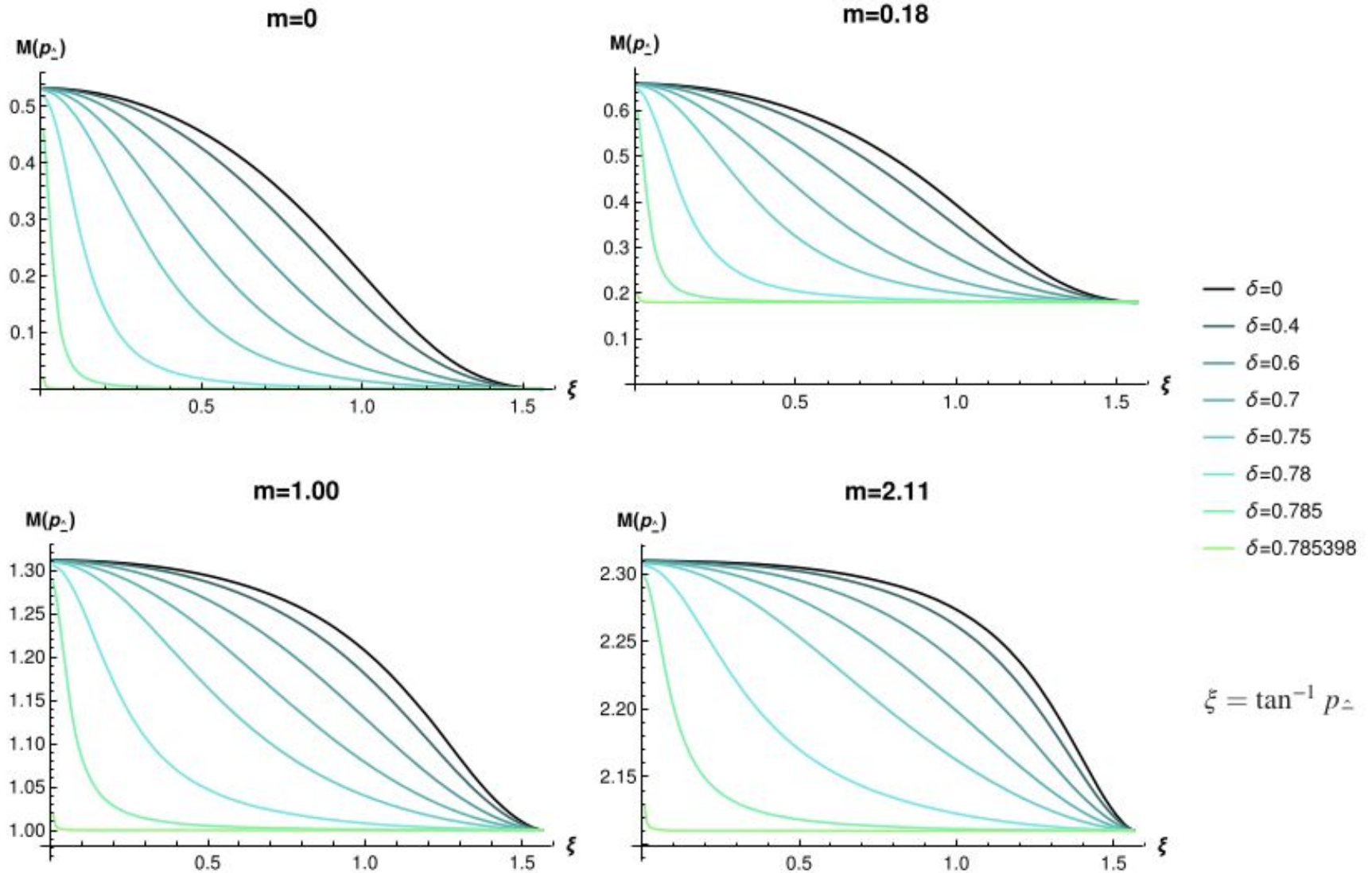$$E = \sqrt{m^2 + p_z{}^2}$$

$$p'^{\hat{+}} = \sqrt{m^2 + p'^2_\perp}$$

$$\tilde{E}(p'_\perp) = \sqrt{M(p'_\perp)^2 + p'^2_\perp}$$

where the interpolating rescaled variables $p'^{\hat{+}} = \dfrac{p^{\hat{+}}}{\sqrt{\mathbb{C}}};\ p'_\perp = \dfrac{p_\perp}{\sqrt{\mathbb{C}}}.$

m=0.18

δ=0	δ=0.71
δ=0.1	δ=0.72
δ=0.2	δ=0.73
δ=0.3	δ=0.74
δ=0.4	δ=0.75
δ=0.5	δ=0.76
δ=0.6	δ=0.77
δ=0.7	

$— \hat{p}^+/\sqrt{C}$

m=0

δ=0	δ=0.71
δ=0.1	δ=0.72
δ=0.2	δ=0.73
δ=0.3	δ=0.74
δ=0.4	δ=0.75
δ=0.5	δ=0.76
δ=0.6	δ=0.77
δ=0.7	
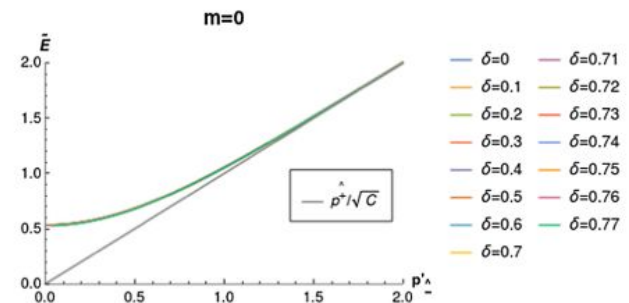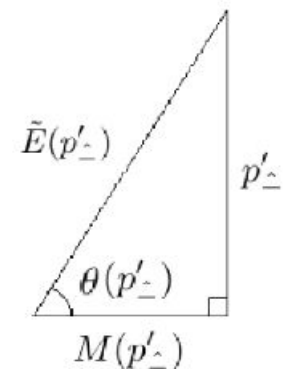
Confinement ⟷ Asymptotic freedom

Running quark mass function

Wavefunction renormalization factor

$$m \lesssim 0.56$$

$$\xi = \tan^{-1} p_{\hat{\ominus}}$$
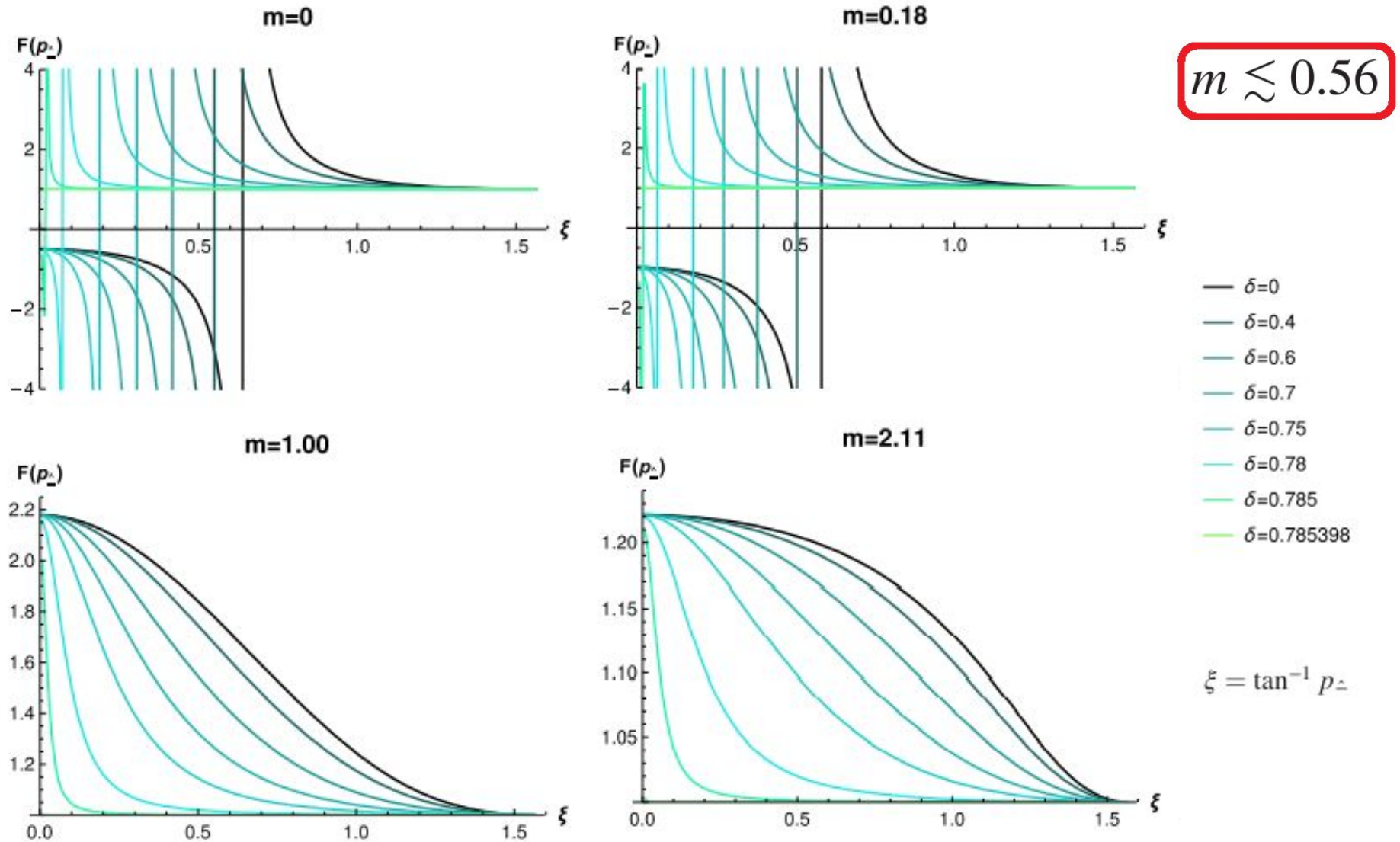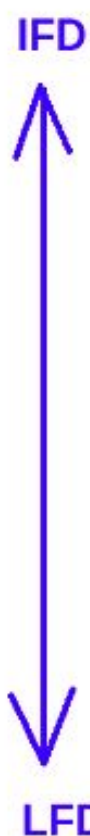
Chiral condensate: zero mass case

$$< \bar{\psi} \psi > |_{\text{ren}} = -\frac{N_c}{2\pi} \int_{-\infty}^{+\infty} d p'_\perp \left[\cos \theta(p'_\perp) - \cos \theta_f(p'_\perp) \right].$$

δ	number of grid points	$< \bar{\psi} \psi > \mid_{m \to 0} / N_c$
0	200	−0.285209
	600	−0.287508
0.4	200	−0.285164
	600	−0.287496
0.6	200	−0.284792
	600	−0.287375
0.7	200	−0.283836
	600	−0.287059
0.75	200	−0.281837
	600	−0.286396
0.78	200	−0.296575
	600	−0.291334
0.785	600	−0.298104
	1000	−0.294377
	3000	−0.290590
0.78535	1000	−0.304659
	3000	−0.294134
	5000	−0.291964

IFD

LFD

Analytic value is

$$-1/\sqrt{12} \approx -0.29$$

Burkardt, M. (1996).
Phys. Rev. D, 53:933–938.

Zhitnitsky, A. R. (1985).
Phys. Lett. B, 165:405–409.

Zhitnitsky, A. R. (1996).
Phys. Rev. D, 53:5821–5833.

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Chiral condensate: varying mass case
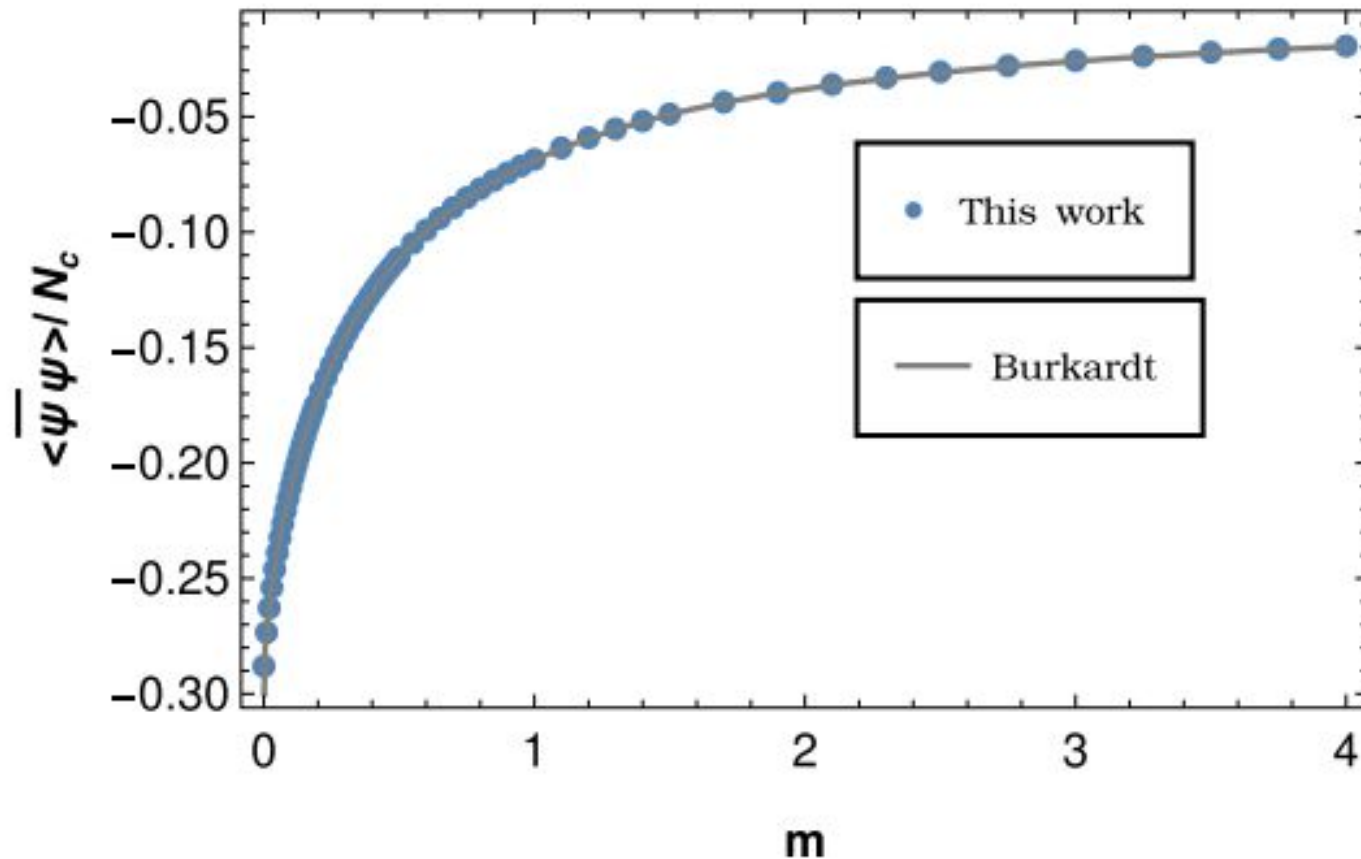
Numerical results of the condensation $\langle \bar{\psi}\psi \rangle|_{ren}$ as a function of m in comparison with the analytic result in M. Burkardt, Phys. Rev. D **53**, 933 (1996). All quantities are in proper units of $\sqrt{2\lambda}$.

Bound state equation

$$\Gamma(r,p) = \frac{i\lambda}{2\pi} \int \frac{dk_{\hat{-}} dk_{\hat{+}}}{(p_{\hat{-}} - k_{\hat{-}})^2} S(p)\gamma^{\hat{+}}\Gamma(r,k)\gamma^{\hat{+}} S(p-r)$$
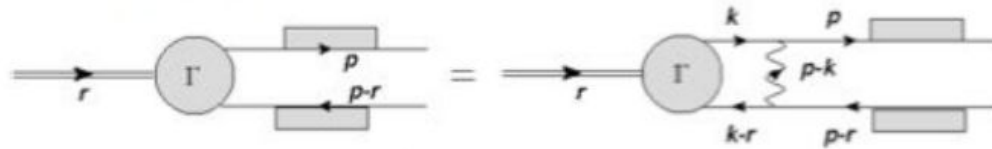
$$\left[-r_{\hat{+}} + \frac{-\mathbb{S}p_{\hat{-}} + E(p_{\hat{-}})}{\mathbb{C}} + \frac{\mathbb{S}(p_{\hat{-}} - r_{\hat{-}}) + E(p_{\hat{-}} - r_{\hat{-}})}{\mathbb{C}} \right] \hat{\phi}_+(r_{\hat{-}}, p_{\hat{-}})$$

$$= \lambda \int \frac{dk_{\hat{-}}}{(p_{\hat{-}} - k_{\hat{-}})^2} \left[C(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_+(r_{\hat{-}}, k_{\hat{-}}) - S(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_-(r_{\hat{-}}, k_{\hat{-}}) \right],$$

$$\left[r_{\hat{+}} + \frac{-\mathbb{S}(p_{\hat{-}} - r_{\hat{-}}) + E(p_{\hat{-}} - r_{\hat{-}})}{\mathbb{C}} + \frac{\mathbb{S}p_{\hat{-}} + E(p_{\hat{-}})}{\mathbb{C}} \right] \hat{\phi}_-(r_{\hat{-}}, p_{\hat{-}})$$

$$= \lambda \int \frac{dk_{\hat{-}}}{(p_{\hat{-}} - k_{\hat{-}})^2} \left[C(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_-(r_{\hat{-}}, k_{\hat{-}}) - S(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_+(r_{\hat{-}}, k_{\hat{-}}) \right].$$

LFD $\left[\mathcal{M}^2 - \frac{m^2 - 2\lambda}{x} - \frac{m^2 - 2\lambda}{1-x} \right] \phi(x) = -2\lambda \int_0^1 \frac{dy}{(x-y)^2} \phi(y)$

0

Meson spectroscopy

LFD
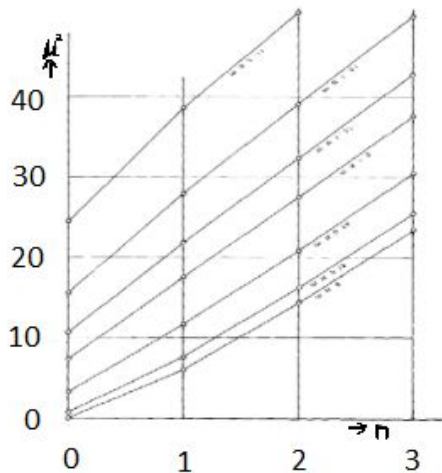
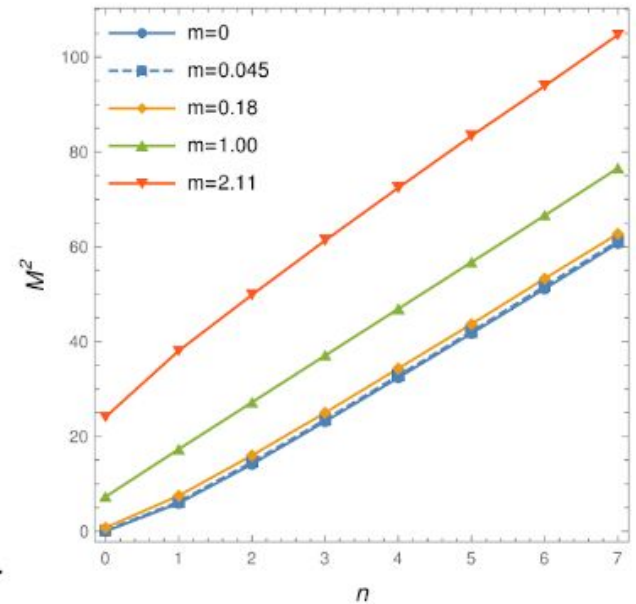't Hooft, G. (1974).

Nucl. Phys. B, 75:461–470.

IFD
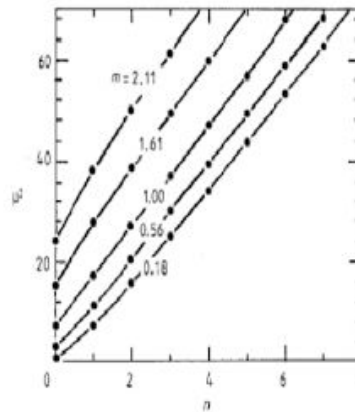
Bars, I. and Green, M. B. (1978).
Phys. Rev. D, 17:537–545.

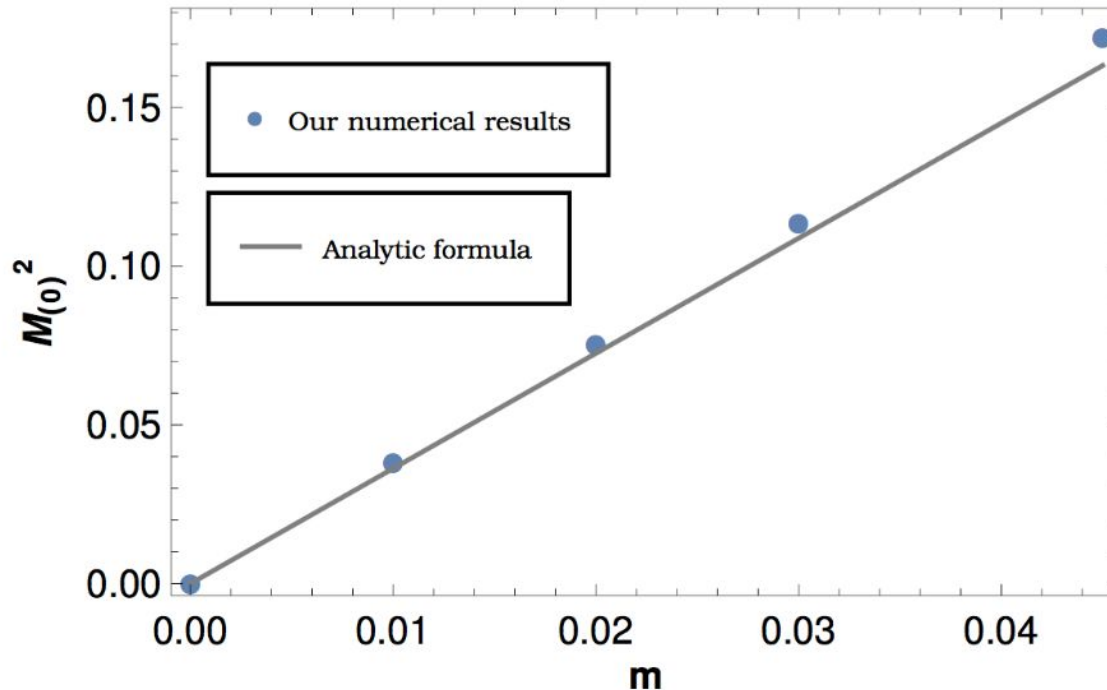Li, M., Wilets, L., and Birse, M. C. (1987).
J. Phys. G, 13:915–923.

Jia, Y., Liang, S., Li, L., and Xiong, X. (2017).
JHEP, 11:151.

Interpolation form

Ma, B. and Ji, C.-R. (2021).
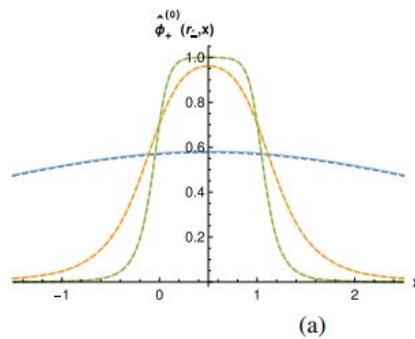Phys. Rev. D, 104:036004.

Gell-Mann - Oakes - Renner relation

$$\mathcal{M}_\pi^2 = -\frac{4m < \bar{\psi}\psi >}{f_\pi^2} = \sqrt{\frac{8\pi^2 m^2 \lambda}{3}} \qquad f_\pi = \sqrt{N_c/\pi}$$

Brower, R. C., Spence, W. L., and Weis, J. H. (1979). Bound states and asymptotic limits for quantum chromodynamics in two dimensions. Phys. Rev. D, 19:3024–3049.

Wavefunction solutions

Ground state wave functions $\hat{\phi}_+^{(0)}(r_\perp, x)$ and $\hat{\phi}_-^{(0)}(r_\perp, x)$ for $m = 0$. All quantities are in proper units of $\sqrt{2\lambda}$.

IFD $(\delta=0)$
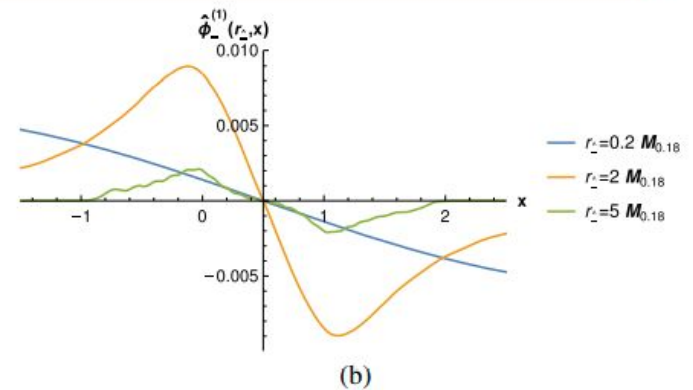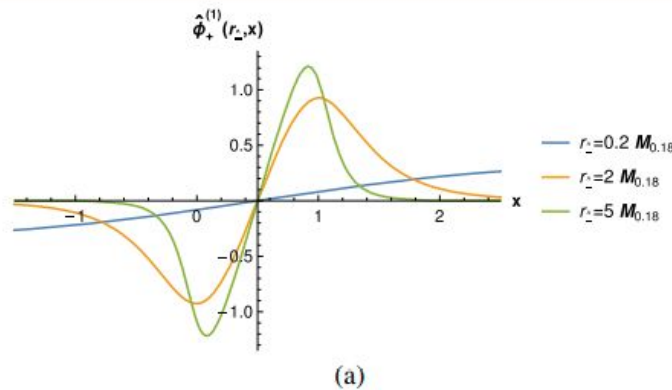
(a)

(b)

Interpolation

$(\delta=0.6)$

(c)

(d)
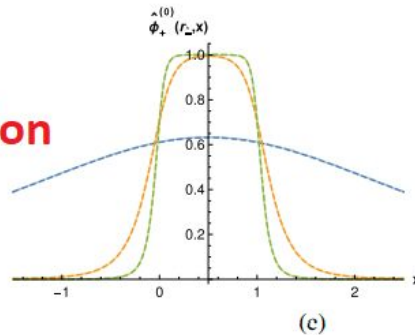
the analytic solution in the interpolating formulation is given by

$$\hat{\phi}_\pm^{(0)}(r_\perp, p_\perp) = \frac{1}{2}\left(\cos\frac{\theta(r_\perp - p_\perp) - \theta(p_\perp)}{2} \pm \sin\frac{\theta(r_\perp - p_\perp) + \theta(p_\perp)}{2}\right)$$
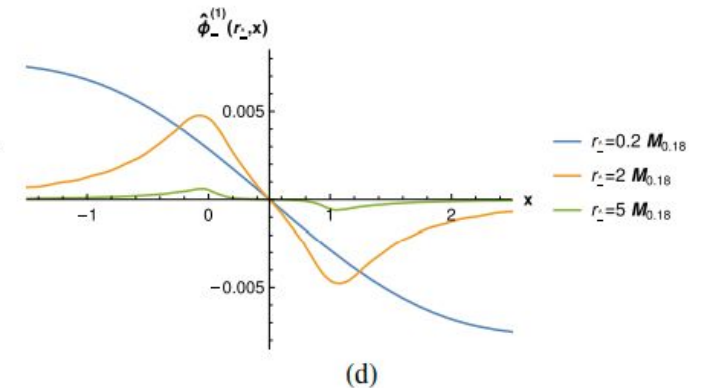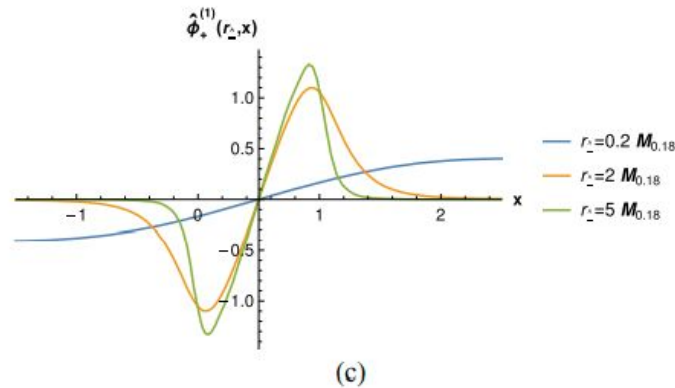
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Wavefunction solutions

First excited state wave functions $\hat{\phi}_+^{(1)}(r_-, x)$ and $\hat{\phi}_-^{(1)}(r_-, x)$ for $m = 0$.

Quasi-PDF

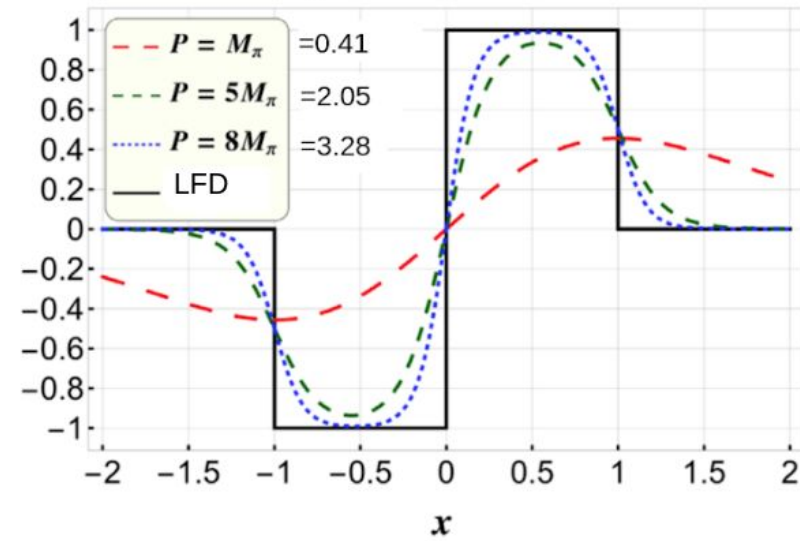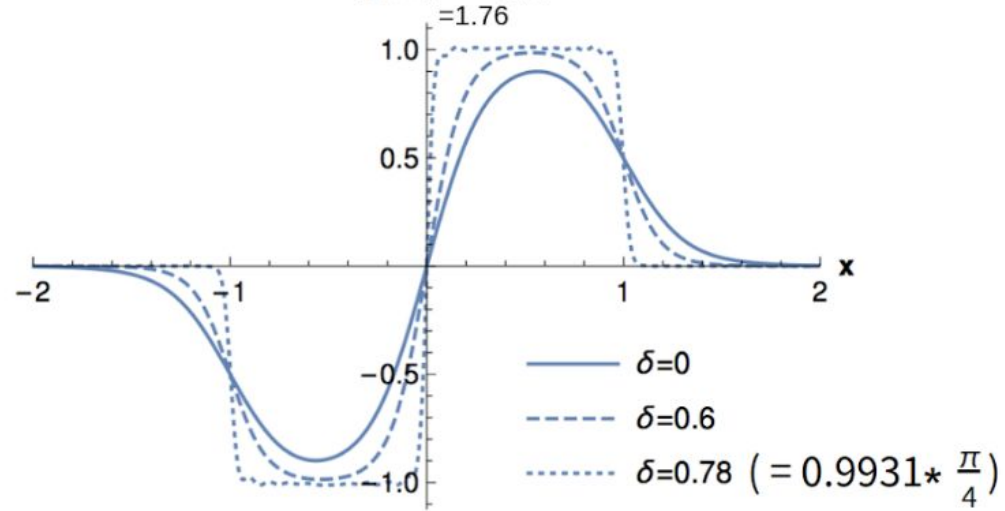Quark quasi-PDFs and light-front PDF for the chiral pion.

Interpolating "quasi-PDFs" for the chiral pion.

All quantities are in proper units of $\sqrt{2\lambda}$.

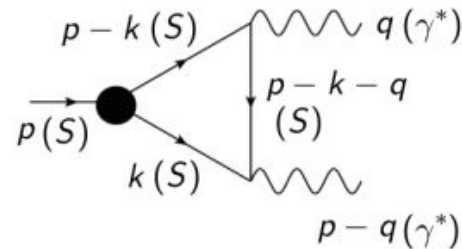Jia, Y., Liang, S., Xiong, X., and Yu, R. (2018). Phys. Rev. D, 98:054011.

Ma, B. and Ji, C.-R. (2021). Phys. Rev. D, 104:036004.

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Outlook

- Transition form factors in 1+1 dimensional models
  - scalar loop
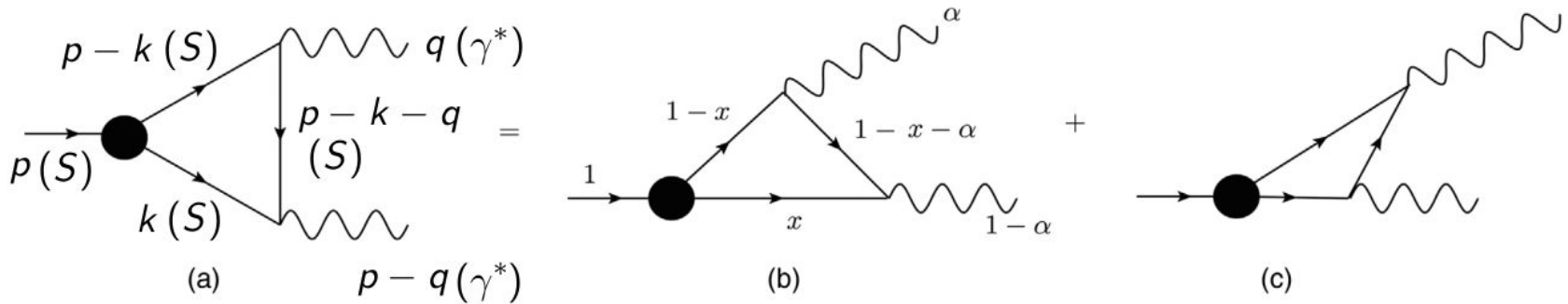 - fermion loop
  - 't Hooft model

- Drell-Yan-West frame ($q^+ = 0$) can no longer be used, as it results in $q^2 = 0$.
- Choosing a $q^+ \neq 0$ frame, one can also directly access not only the space-like ($q^2 < 0$), but also the time-like ($q^2 > 0$) momentum regions.
- Define a parameter such that

$$p = (p^+, p^-) = \left(p^+, \frac{M^2}{2p^+} \right)$$

$$q = (q^+, q^-) = \left(\alpha p^+, \frac{M^2}{2p^+} - \frac{q'^2}{2(1-\alpha)p^+} \right)$$
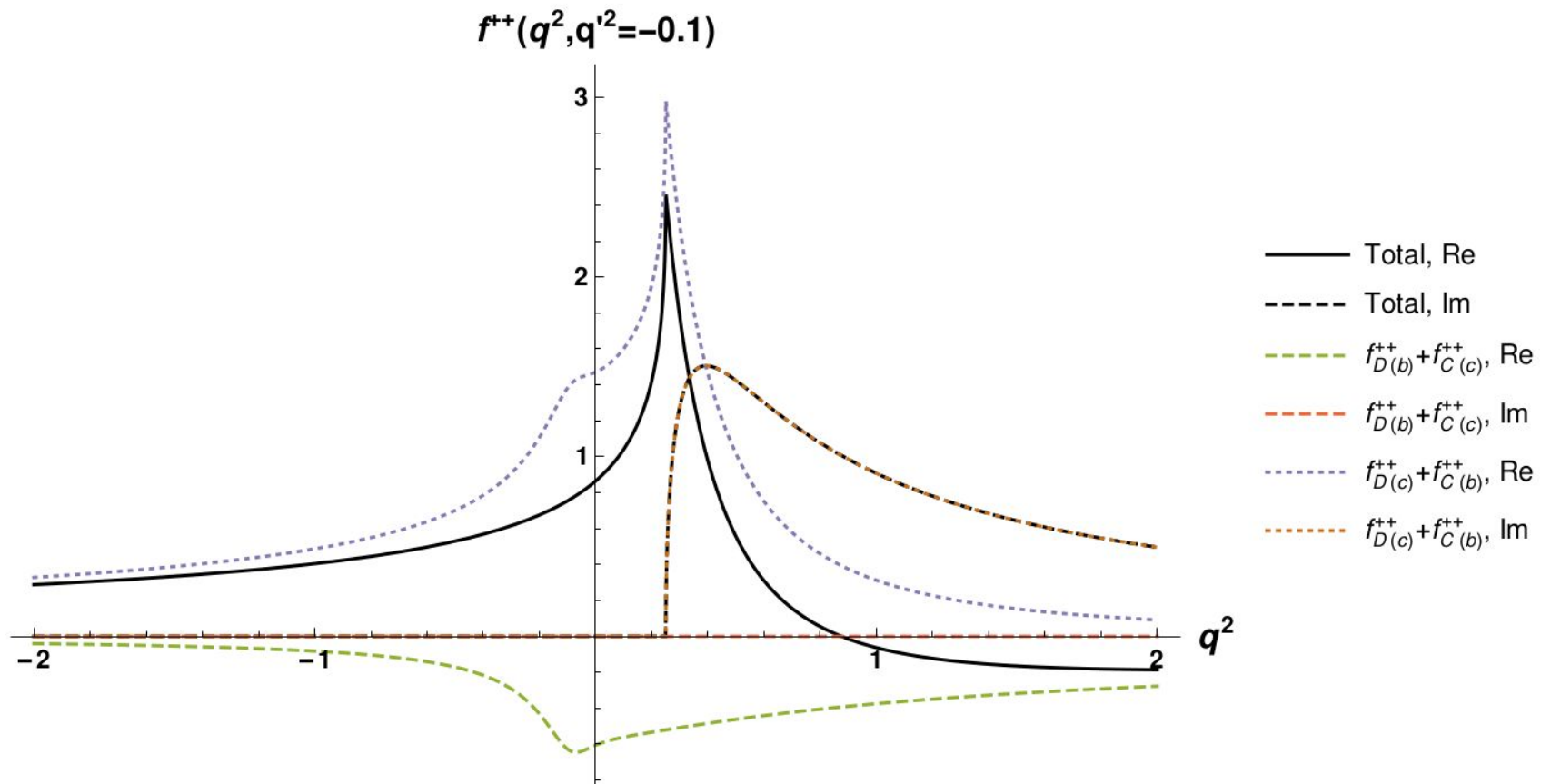
$$q' = p - q = (q'^+, q'^-) = \left((1-\alpha)p^+, \frac{q'^2}{2(1-\alpha)p^+} \right)$$

Outlook

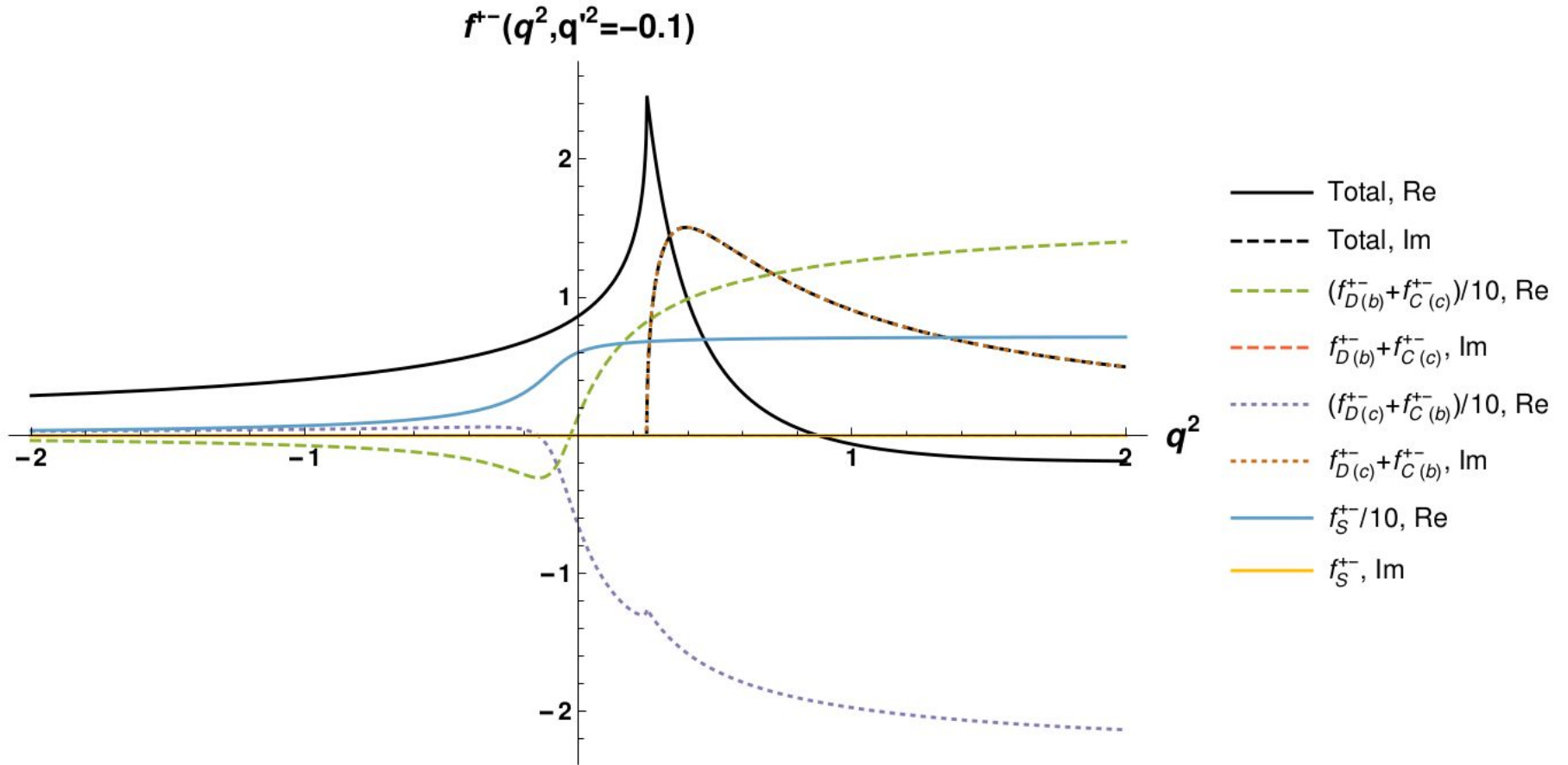- Assuming $\Gamma_i^{\mu\nu} = f_i(q^2, q'^2)\left(g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu}\right)$, where the index *i* means each individual Light-Front Time-Ordered (LFTO) diagram, we calculated the contributions from each LFTO diagrams to the form factor.

Outlook

$f^{++}(q^2, q'^2 = -0.1)$

Legend:
- Total, Re (solid black)
- Total, Im (dashed black)
- $f^{++}_{D(b)} + f^{++}_{C(c)}$, Re
- $f^{++}_{D(b)} + f^{++}_{C(c)}$, Im
- $f^{++}_{D(c)} + f^{++}_{C(b)}$, Re
- $f^{++}_{D(c)} + f^{++}_{C(b)}$, Im

The numerical results of individual x^+-ordered contributions to the form factor for the case of $m = 0.25\ GeV$, $M = 0.14\ GeV$, and $q'^2 = -0.1\ GeV^2$, normalized to $F(q^2 = 0, q'^2 = 0) = 1$, by picking the $++$ component of the current. Note that $f^{++}_{D(b)} = f^{++}_{C(c)}$ and $f^{++}_{D(c)} = f^{++}_{C(b)}$.

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Outlook

The numerical results of individual x^+-ordered contributions to the form factor for the case of $m = 0.25\ GeV$, $M = 0.14\ GeV$, and $q'^2 = -0.1\ GeV^2$, normalized to $F(q^2 = 0, q'^2 = 0) = 1$, by picking the $+-$ component of the current. Note that $f_{D(b)}^{+-} = f_{C(c)}^{+-}$ and $f_{D(c)}^{+-} = f_{C(b)}^{+-}$.

Outlook

- This shows that each individual LFTO contribution to the form factor depends on which component one takes to make the calculation.
- Each individual contribution in our case is Valence(V) versus Non-Valence (NV), but in other scenarios, e.g. proton mass decomposition calculation, it may be like, quark contribution, gluon contribution, angular momentum contribution, etc. contributions to the $T^{\mu\nu}$ tensor.
- Our theoretical simulation shows that for each individual contribution, one cannot assume it is gauge invariant, and there may be more than one form factor in this case. Only when one adds all the contributions together, the total result is gauge invariant with only one form factor.

Thank you!