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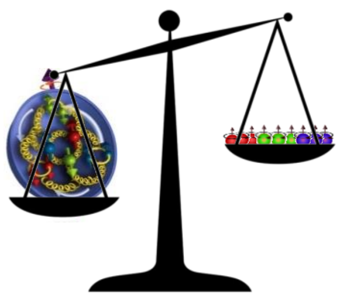
Light Cone 2022 Online :
Physics of Hadrons on the light front



Mass sum rules for the nucleon

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September 23, 2022

Trace decomposition

$$j_D^\mu = T^{\mu\nu} x_\nu \qquad \Rightarrow \qquad \partial_\mu j_D^\mu = T^\mu{}_\mu$$

Conformal symmetry is broken by:
1) constituents' masses
2) quantum corrections

$$T^\mu{}_\mu = \frac{\beta(g)}{2g}\, G^2 + (1+\gamma_m)\, \overline{\psi} m \psi$$

Quark mass matrix

[Crewther (1972)]
[Chanowitz, Ellis (1972)]
[Adler, Collins, Duncan (1977)]
[Collins, Duncan, Joglekar (1977)]
[Nielsen (1977)]

Textbook approach

$$\langle p | T^{\mu\nu}(0) | p \rangle = 2 p^\mu p^\nu$$

$$\Rightarrow \quad \langle p | T^\mu{}_\mu(0) | p \rangle = 2M^2$$

[Shifman, Vainshtein, Zakharov (1978)]
[Donoghue, Golowich, Holstein (1992)]
[Kharzeev (1996)]
[Roberts (2017)]

$\langle \overline{\psi} m \psi \rangle$

Nucleon-meson scattering

$\langle G^2 \rangle$

Near-threshold heavy meson production

$$\boxed{M = \underbrace{\langle \frac{\beta(g)}{2g}\, G^2 \rangle}_{\sim 91\%} + \underbrace{\langle (1+\gamma_m)\, \overline{\psi} m \psi \rangle}_{\sim 9\%}}$$

Nucleon

~ 91% (to be measured)

~ 9% (measurement to be improved)

$\mu = 2\,\text{GeV}$

$$\langle O \rangle \equiv \frac{\langle p_{\text{rest}} | \int \mathrm{d}^3x\, O(x) | p_{\text{rest}} \rangle}{\langle p_{\text{rest}} | p_{\text{rest}} \rangle} = \frac{\langle p_{\text{rest}} | O(0) | p_{\text{rest}} \rangle}{2M}$$

Trace decomposition

Trace anomaly is scale invariant

$$M = \underbrace{\langle \frac{\beta(g)}{2g} G^2 + \gamma_m \overline{\psi} m \psi \rangle}_{\substack{\text{Trace anomaly} \\ \sim 92\%}} + \underbrace{\langle \overline{\psi} m \psi \rangle}_{\substack{\sigma_q \\ \sim 8\%}}$$

Quark and gluon contributions

$$T^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$T_q^{\mu\nu} = \overline{\psi}\gamma^\mu \tfrac{i}{2} \overleftrightarrow{D}^\nu \psi$$
$$T_g^{\mu\nu} = -G^{\mu\lambda} G^\nu{}_\lambda + \tfrac{1}{4} g^{\mu\nu} G^2$$

$$\Rightarrow \boxed{M = g_{\mu\nu} \langle T_q^{\mu\nu} \rangle + g_{\mu\nu} \langle T_g^{\mu\nu} \rangle}$$

[Makino, Suzuki (2014)]
[Hatta, Rajan, Tanaka (2018)]
[Tanaka (2019)]
[Ahmed, Chen, Czakon (2022)]

⚠ **Operator mixing**

$$g_{\mu\nu} T_q^{\mu\nu} = x \, G^2 + (1+y) \, \overline{\psi} m \psi$$
$$g_{\mu\nu} T_g^{\mu\nu} = \left(\frac{\beta(g)}{2g} - x\right) G^2 + (\gamma_m - y) \, \overline{\psi} m \psi$$

x, y are scheme and scale-dependent calculable coefficients

Ji's decomposition

Mass is the rest-frame energy $M = \langle T^{00} \rangle$

Step 1 $\quad T^{\mu\nu} = \underset{(1,1)}{\bar{T}^{\mu\nu}} + \underset{(0,0)}{\hat{T}^{\mu\nu}}$

$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha}$$
$$\hat{T}^{\mu\nu} = \frac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha}$$

[Ji (1995)]

No mixing under Lorentz transformations or renormalization !

$$\Rightarrow \quad M = \underbrace{\langle \bar{T}^{00} \rangle}_{\frac{3}{4}M} + \underbrace{\langle \hat{T}^{00} \rangle}_{\frac{1}{4}M}$$

Step 2 $\quad \bar{T}^{\mu\nu} = \bar{T}^{\mu\nu}_q + \bar{T}^{\mu\nu}_g$

$$\hat{T}^{\mu\nu} = \frac{1}{4}\, g^{\mu\nu} \left[x\, G^2 + (1+y)\, \overline{\psi} m \psi \right] + \frac{1}{4}\, g^{\mu\nu} \left[\left(\frac{\beta(g)}{2g} - x \right) G^2 + (\gamma_m - y)\, \overline{\psi} m \psi \right]$$

[Ji (1995)]	**[Yang *et al.* (2015)]**	**[Metz, Pasquini, Rodini (2020)]**
$x = 0, \quad y = \gamma_m$	$x = 0, \quad y = 0$	Arbitrary $x,\, y$

Ji's decomposition

Step 3 Reshuffling of quark mass contributions

$$\boxed{M = \langle \underbrace{\bar{T}_q^{00} - \tfrac{3}{4}\,\overline{\psi} m \psi}_{\frac{3}{4}\left[\langle x\rangle_q M - \sigma_q\right]} \rangle + \langle \underbrace{\overline{\psi} m \psi}_{\sigma_q} \rangle + \langle \underbrace{\bar{T}_g^{00}}_{\frac{3}{4}\langle x\rangle_g M} \rangle + \langle \underbrace{\tfrac{1}{4}\left[\frac{\beta(g)}{2g}\,G^2 + \gamma_m\,\overline{\psi} m \psi\right]}_{\frac{1}{4}\left[M - \sigma_q\right]} \rangle}$$

$\mu = 2\,\mathrm{GeV}$ **~ 38%** **~ 8%** **~ 31%** **~ 23%**

LF momentum fractions

$$\langle x\rangle_{q,g} \equiv \frac{1}{p^+}\,\frac{\langle p|\int \mathrm{d}^2x_\perp\,\mathrm{d}x^-\,T^{++}_{q,g}(x)|p\rangle}{\langle p|p\rangle} = \frac{\langle p|T^{++}_{q,g}(0)|p\rangle}{2(p^+)^2} \underset{g^{++}=0}{=} \frac{\langle p|\bar{T}^{++}_{q,g}(0)|p\rangle}{2(p^+)^2}$$

Deep inelastic scattering

Momentum sum rule

$$\langle x\rangle_q + \langle x\rangle_g = 1$$

Energy-momentum tensor (EMT)

What is the physical meaning of the individual terms in trace and Ji's decompositions ?

Strategy: look at the whole EMT [C.L. (2018)]

$$\langle p|T^{\mu\nu}_a(0)|p\rangle = 2p^\mu p^\nu A_a(0) + 2M^2 g^{\mu\nu}\bar{C}_a(0) \qquad a = q, g$$

$$\langle p|T^{\mu\nu}(0)|p\rangle = 2p^\mu p^\nu$$

Four-momentum sum rules [Ji (1998)]

$$\boxed{\begin{aligned}\sum_a A_a(0) &= 1\\ \sum_a \bar{C}_a(0) &= 0\end{aligned}}$$

NB: $\langle x\rangle_a = A_a(0)$

Why two sum rules ?

What is the meaning of $\bar{C}_a(0)$?

Energy-momentum tensor (EMT)

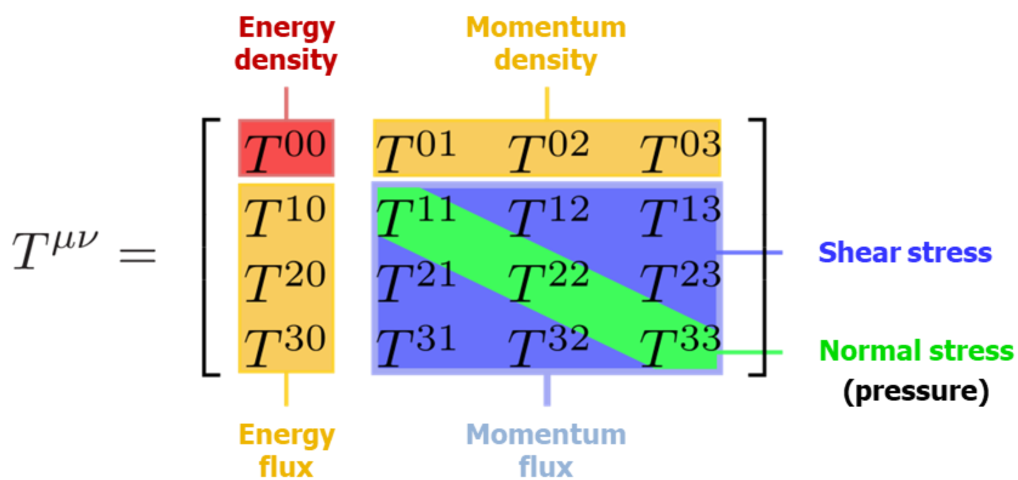
Classical perfect fluid

$$\Theta^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p\, g^{\mu\nu}$$

$$a = q, g \qquad \langle p|T_a^{\mu\nu}(0)|p\rangle = 2p^\mu p^\nu A_a(0) + 2M^2 g^{\mu\nu}\bar{C}_a(0)$$

$$\Rightarrow \quad \langle T_a^{\mu\nu}\rangle = M\left(\begin{array}{c|ccc} A_a(0)+\bar{C}_a(0) & 0 & 0 & 0 \\ \hline 0 & -\bar{C}_a(0) & 0 & 0 \\ 0 & 0 & -\bar{C}_a(0) & 0 \\ 0 & 0 & 0 & -\bar{C}_a(0) \end{array}\right)$$

[C.L. (2018)]



Rest energy decomposition

Mass is the rest-frame energy $M = \langle T^{00} \rangle$ **[C.L. (2018)]**

$$\boxed{M = \underbrace{\langle T_q^{00} \rangle}_{\text{Quark total energy}} + \underbrace{\langle T_g^{00} \rangle}_{\text{Gluon total energy}}}$$

$$\langle T_a^{00} \rangle = [A_a(0) + \bar{C}_a(0)]M$$

Refinements

$$\langle T_q^{00} \rangle = \underbrace{\langle T_q^{00} - \overline{\psi} m \psi \rangle}_{\text{Quark kinetic + potential energy}} + \underbrace{\langle \overline{\psi} m \psi \rangle}_{\text{Quark rest energy}}$$

Seems legit

$$\langle T_g^{00} \rangle = \underbrace{\langle T_g^{00} - \tfrac{1}{4}\left[\frac{\beta(g)}{2g}\, G^2 + \gamma_m\, \overline{\psi} m \psi\right] \rangle}_{\text{No obvious meaning}} + \underbrace{\langle \tfrac{1}{4}\left[\frac{\beta(g)}{2g}\, G^2 + \gamma_m\, \overline{\psi} m \psi\right] \rangle}_{\text{No obvious meaning}}$$

Mechanical equilibrium

[von Laue (1911)]
[Landau, Lifshitz (1951)]
[Deser (1976)]
[Dudas, Pirjol (1991)]
[C.L. (2018)]
[C.L., Metz, Pasquini, Rodini (2021)]

Virial theorem $\langle T^{ii}\rangle = 0$

$$\boxed{\underbrace{\langle T^{ii}_q\rangle}_{\text{Quark pressure-volume work}} + \underbrace{\langle T^{ii}_g\rangle}_{\text{Gluon pressure-volume work}} = 0}$$

$$\langle T^{ii}_a\rangle = -\bar{C}_a(0)M$$

Trace decomposition

$$\langle T^\mu{}_\mu\rangle = \underbrace{\langle T^{00}\rangle}_{=M} - \sum_i \underbrace{\langle T^{ii}\rangle}_{=0}$$

Ji's decomposition

$$\begin{aligned}\langle T^{00}\rangle &= \langle T^{00} - \tfrac{1}{4}\,T^\mu{}_\mu\rangle + \langle\tfrac{1}{4}\,T^\mu{}_\mu\rangle \\ &= \Big[\tfrac{3}{4}\,\underbrace{\langle T^{00}\rangle}_{=M} + \tfrac{1}{4}\sum_i \underbrace{\langle T^{ii}\rangle}_{=0}\Big] + \Big[\tfrac{1}{4}\,\underbrace{\langle T^{00}\rangle}_{=M} - \tfrac{1}{4}\sum_i \underbrace{\langle T^{ii}\rangle}_{=0}\Big]\end{aligned}$$

- Both mix rest energy with mechanical equilibrium !
- Separate quark and gluon contributions have no clear relation to mass since $\langle T^{ii}_a\rangle \neq 0$

Operator structure

Ji's decomposition

$$M = \langle \bar{T}_q^{00} - \tfrac{3}{4}\,\overline{\psi} m \psi\rangle + \langle \overline{\psi} m \psi\rangle + \langle \bar{T}_g^{00}\rangle + \langle \tfrac{1}{4}\Big[\frac{\beta(g)}{2g}\,G^2 + \gamma_m\,\overline{\psi} m \psi\Big]\rangle$$

1) Traceless and trace parts do not mix under renormalization
2) Trace part contains an anomaly

Reasoning 1

Renormalized traceless operators have the classical form

[Voloshin, Zakharov (1980)]
[Luke, Manohar, Savage (1992)]
[Ji (1995)]
[Kharzeev (1996)]
[Ji (2021)]

$$\bar{T}_q^{00} = \overline{\psi}\vec{\gamma}\cdot i\vec{D}\psi + \tfrac{3}{4}\,\overline{\psi} m \psi$$

$$\bar{T}_g^{00} = \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

$$\Rightarrow \quad T^{00} = T^{00}_{\text{bare}} + \underbrace{\tfrac{1}{4}\Big[\frac{\beta(g)}{2g}\,G^2 + \gamma_m\,\overline{\psi} m \psi\Big]}_{\text{« Quantum anomalous energy »}}$$

Reasoning 2

Dimensional regularization and minimal subtraction

$T^{\mu\nu}$ is a conserved current and hence is not renormalized

[Collins (1984)]
[Makino, Suzuki (2014)]
[Hatta, Rajan, Tanaka (2018)]
[Metz, Pasquini, Rodini (2020)]
[C.L., Metz, Pasquini, Rodini (2021)]

$$T^{00} = T^{00}_{\text{bare}}$$

$$\Rightarrow \quad \bar{T}^{00} = \overline{\psi}\vec{\gamma}\cdot i\vec{D}\psi + \tfrac{3}{4}\,\overline{\psi} m \psi + \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2) - \tfrac{1}{4}\Big[\frac{\beta(g)}{2g}\,G^2 + \gamma_m\,\overline{\psi} m \psi\Big]$$

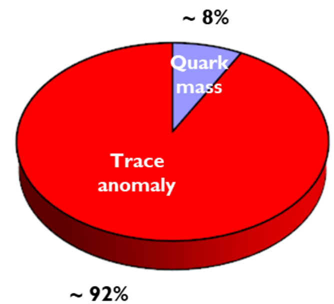
Separation into quark and gluon contributions is scheme and scale dependent

Mass sum rules (in D2 scheme and μ = 2 GeV)

[Metz, Pasquini, Rodini (2020)]
[C.L., Metz, Pasquini, Rodini (2021)]

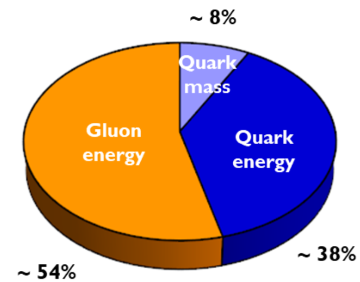
Trace decomposition

$$M = \langle \frac{\beta(g)}{2g} G^2 + \gamma_m \overline{\psi} m \psi \rangle + \langle \overline{\psi} m \psi \rangle$$



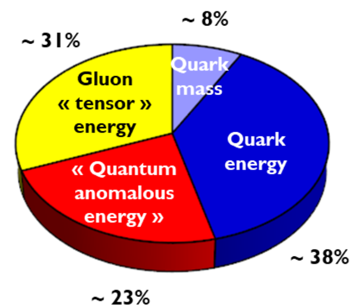
Rest energy decomposition

$$M = \langle \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2) \rangle + \langle \overline{\psi} \vec{\gamma} \cdot i\vec{D} \psi \rangle + \langle \overline{\psi} m \psi \rangle$$



Ji's decomposition

$$M = \underbrace{\langle \bar{T}_g^{00} \rangle + \langle \tfrac{1}{4}\left[\frac{\beta(g)}{2g} G^2 + \gamma_m \overline{\psi} m \psi \right] \rangle}_{\langle \frac{1}{2}(\vec{E}^2 + \vec{B}^2) - \frac{1}{4}\left[\frac{\beta(g)}{2g} G^2 + \gamma_m \overline{\psi} m \psi \right] \rangle} + \underbrace{\langle \bar{T}_q^{00} - \tfrac{3}{4} \overline{\psi} m \psi \rangle}_{\langle \overline{\psi} \vec{\gamma} \cdot i\vec{D} \psi \rangle} + \langle \overline{\psi} m \psi \rangle$$



D2 scheme: the whole anomaly is attributed to the gluon sector

$$\Longrightarrow \quad g_{\mu\nu} \langle T_q^{\mu\nu} \rangle = [A_q(0) + 4\bar{C}_q(0)] M = \sigma_q$$

Conclusions and last comments

- Mass can be decomposed in several ways,
  but it is crucial that each contribution receives a clear physical meaning

- Scheme and scale dependences enter as soon as
  one separates quark and gluon contributions

- Renormalization requires great care in presence of trace anomaly

- All decompositions require at most two independent inputs : $\langle x \rangle_q,\ \sigma_q$

- Our discussions apply to all massive systems, irrespective of their spin

Backup slides

Poincaré constraints (spin-0 or spin-1/2 targets)

$$\langle p|T_a^{\mu\nu}(0)|p\rangle = 2p^\mu p^\nu A_a(0) + 2M^2 g^{\mu\nu}\bar{C}_a(0)$$

$$\Rightarrow \quad \langle P_a^\mu\rangle = p^\mu A_a(0) + \frac{M^2}{p^0} g^{0\mu}\bar{C}_a(0)$$

Not a four-vector !
(unless state is massless)

Light-front version

$$\langle P_{a,\mathrm{LF}}^\mu\rangle = p^\mu A_a(0) + \frac{M^2}{p^+} g^{+\mu}\bar{C}_a(0) \quad \Rightarrow \quad \langle P_{a,\mathrm{LF}}^+\rangle = \underbrace{A_a(0)}_{=\langle x\rangle_a} p^+$$

$p^\pm = (p^0 \pm p^3)/\sqrt{2}$

Four-momentum sum rules

$$p^\mu = \langle P_q^\mu\rangle + \langle P_g^\mu\rangle \quad \Rightarrow \quad \boxed{\begin{cases} A_q(0) + A_g(0) = 1 \\ \bar{C}_q(0) + \bar{C}_g(0) = 0 \end{cases}}$$

[Ji (1998)]

From the experimental side, all we need is:
1. to measure *all* the GFFs and
2. to check the four-momentum sum rules

What is mass ?

Formal operator definition $P^\mu P_\mu = M^2$

Not additive and hence not suited for a mass decomposition

Expectation value

$$M = \frac{\langle p|P^\mu P_\mu|p\rangle}{\langle p|p\rangle}\,\frac{1}{M}$$

$$= \frac{\langle p|P^\mu|p\rangle}{\langle p|p\rangle}\,\underbrace{\frac{p_\mu}{M}}_{\text{CM four-velocity}}$$

$$= \frac{\langle p_{\text{rest}}|H|p_{\text{rest}}\rangle}{\langle p_{\text{rest}}|p_{\text{rest}}\rangle}$$

Proper inertia (i.e. rest-frame energy) of the system

A mass decomposition is fundamentally an energy decomposition $\quad H = \sum_a H_a$

MS-like renormalization

[Collins (1984)]

$$T^{\mu\nu} = \overline{\psi}\gamma^\mu iD^\nu\psi - F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}\,g^{\mu\nu}F^2$$

Finite operator!

Linearity property implies

[Makino, Suzuki (2014)]
[Hatta, Rajan, Tanaka (2018)]
[Tanaka (2019)]
[Rodini, Metz, Pasquini (2020)]
[Metz, Pasquini, Rodini (2021)]
[C.L., Metz, Pasquini, Rodini (2021)]

$$(T^{\mu\nu})_R = (\overline{\psi}\gamma^\mu iD^\nu\psi)_R - (F^{\mu\lambda}F^\nu{}_\lambda)_R + \frac{1}{4}\,g^{\mu\nu}(F^2)_R$$

$$H = \int \mathrm{d}^3r\,\underbrace{(T^{00})_R}_{(\overline{\psi}\gamma^0 iD^0\psi)_R + \frac{1}{2}(\vec{E}^2+\vec{B}^2)_R}$$

No anomalous contribution
(consistent with exact time translation symmetry)

Anomaly arises only via trace operation (hence « trace anomaly »)

$$g_{\mu\nu}(O^{\mu\nu})_R \neq (g_{\mu\nu}O^{\mu\nu})_R \qquad g_{\mu\nu}(T^{\mu\nu})_R = \frac{\beta}{2g}\,(F^2)_R + (1+\gamma_m)(\overline{\psi}m\psi)_R$$

Traceless operator is $$(O^{\mu\nu})_R - \frac{1}{4}\,g^{\mu\nu}\boxed{g_{\alpha\beta}(O^{\alpha\beta})_R}$$

Contains anomalous contributions

$$\left(O^{\mu\nu} - \frac{1}{4}\,g^{\mu\nu}g_{\alpha\beta}O^{\alpha\beta}\right)_R$$

Unjustified notation

$$\lim_{d\to 4}\left(O^{\mu\nu} - \frac{1}{d}\,g^{\mu\nu}g_{\alpha\beta}O^{\alpha\beta}\right)_R$$

Correct form

Mass decompositions

Trace decomposition

[Shifman, Vainshtein, Zakharov, PL78B (1978) 443]
[Donoghue, Golowich, Holstein, *Dynamics of the Standard Model* (1992)]
[Kharzeev, PISPF130 (1996) 105]

$$M = \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x \,(\tfrac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m \psi) \rangle}_{M - \sigma_q}$$

Requires 1 independent input

Energy decomposition

[C.L., EPJC78 (2018) 120]
[Metz, Pasquini, Rodini, PRD102 (2020) 114042]

$$M = \underbrace{\langle \int \mathrm{d}^3x \,(T^{00}_q - \overline{\psi} m \psi) \rangle}_{[A_q(0) + \bar{C}_q(0)]\, M - \sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, T^{00}_g \rangle}_{[A_g(0) + \bar{C}_g(0)]\, M}$$

$$A_q(0) + A_g(0) = 1$$
$$\bar{C}_q(0) + \bar{C}_g(0) = 0$$

$$A_q(0) = \langle x \rangle_q$$
$$A_q(0) + 4\bar{C}_q(0) = c_1 + c_2\, \frac{\sigma_q}{M}$$

Scheme and scale-dependent !

Requires 2 independent inputs

Ji's decomposition

[Ji, PRL74 (1995) 1071]
[Ji, PRD52 (1995) 271]

$$M = \underbrace{\langle \int \mathrm{d}^3x \,(\bar{T}^{00}_q - \tfrac{3}{4}\, \overline{\psi} m \psi) \rangle}_{\frac{3}{4}[A_q(0)\, M - \sigma_q]} + \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, \bar{T}^{00}_g \rangle}_{\frac{3}{4} A_g(0)\, M} + \underbrace{\langle \int \mathrm{d}^3x\, \tfrac{1}{4} (\tfrac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m \psi) \rangle}_{\frac{1}{4}(M - \sigma_q)}$$

Requires 2 independent inputs

$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{4}\, g^{\mu\nu} T^{\alpha}_{\ \alpha}$$

How does one measure contributions to mass?

[Metz, Pasquini, Rodini (2021)]

DIS $\langle x_q \rangle = \int \mathrm{d}x\, x\, f_q(x) = \frac{\langle p|(T_q^{++})_R(0)|p\rangle}{2(p^+)^2} = \boxed{A_q(0) = 0.586 \pm 0.013}$

Momentum sum rule $A_q(0) + A_g(0) = 1 \quad \Rightarrow \quad \boxed{A_g(0) = 0.414 \pm 0.013}$

Quark mass $\sigma_{u+d+s} = \frac{\langle p|(\overline{\psi} m \psi)_R|p\rangle}{2M} \approx 80\,\mathrm{MeV}$

Trace sum rule
Mass sum rule+virial theorem

$$M = \frac{\langle p|(T^\mu{}_\mu)_R|p\rangle}{2M} = \frac{\langle p|\frac{\beta}{2g}(F^2)_R + (1+\gamma_m)(\overline{\psi} m \psi)_R|p\rangle}{2M}$$

Scheme-dependent calculable coefficients

$$\Rightarrow \quad \frac{\langle p|(F^2)_R|p\rangle}{2M} = \frac{2g}{\beta}\left[M - (1+\gamma_m)\sigma_q\right]$$

Quark EMT trace $\frac{\langle p|g_{\mu\nu}(T_q^{\mu\nu})_R|p\rangle}{2M} = \frac{\langle p|x(F^2)_R + (1+y)(\overline{\psi} m \psi)_R|p\rangle}{2M} = \left[A_q(0) + 4\boxed{\bar{C}_q(0)}\right] M$

Virial theorem $\bar{C}_q(0) + \bar{C}_g(0) = 0 \quad \Rightarrow \quad \boxed{\bar{C}_g(0)} = -\bar{C}_q(0)$

Virial theorem (field theory)

[Landau, Lifshitz, *The Classical Theory of Fields* (1951)]
[Deser, PLB64 (1976) 463]
[Dudas, Pirjol, PLB260 (1991) 186]
[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]

$$G = \sum_k \vec{p}_k \cdot \vec{r}_k \quad \Rightarrow \quad \int \mathrm{d}^3x\, T^{0i} x^i$$

$$\begin{aligned}
\frac{\mathrm{d}G}{\mathrm{d}t} &= \int \mathrm{d}^3x\, \partial_0 T^{0i} x^i \\
&= \int \mathrm{d}^3x\, (\partial_\mu T^{\mu i}) x^i - \int \mathrm{d}^3x\, (\partial_k T^{ki}) x^i \\
&= \int \mathrm{d}^3x\, \vec{\mathcal{F}} \cdot \vec{x} + \sum_i \int \mathrm{d}^3x\, T^{ii}
\end{aligned}$$

Four-force density $\mathcal{F}^\nu \equiv \partial_\mu T^{\mu\nu}$

Classical

$$\boxed{[\![\sum_i \int \mathrm{d}^3x\, T^{ii}]\!] = -[\![\int \mathrm{d}^3x\, \vec{\mathcal{F}} \cdot \vec{x}]\!]} \quad = 0$$

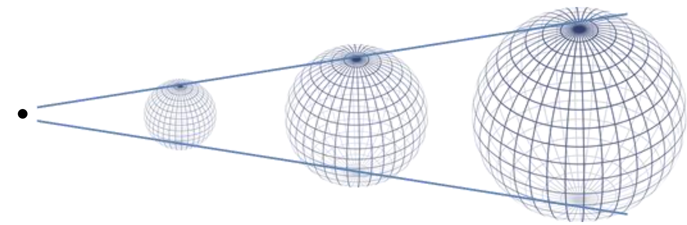
Isolated system
(von Laue condition)

Quantum
(stationary state)

$$\boxed{\langle \sum_i \int \mathrm{d}^3x\, T^{ii} \rangle = -\langle \int \mathrm{d}^3x\, \vec{\mathcal{F}} \cdot \vec{x} \rangle} \quad = 0$$

Virial theorem (field theory)

Physical interpretation $\vec{x} \mapsto (1+\delta\kappa)\vec{x}$



$$\left.\begin{aligned} \delta H &= -\delta W \\ \delta H &= \frac{1}{i}\,[H,G]\,\delta\kappa = -\frac{\mathrm{d}G}{\mathrm{d}t}\,\delta\kappa \end{aligned}\right\} \Rightarrow \quad \delta W = \frac{\mathrm{d}G}{\mathrm{d}t}\,\delta\kappa$$

$$= \int \mathrm{d}^3x\, \underbrace{\partial_0 T^{0i}}_{\text{Local force}} \delta x^i$$

$$\delta\vec{x} = \delta\kappa\,\vec{x}$$

$$\delta(\mathrm{d}^3x) = 3\delta\kappa\,\mathrm{d}^3x$$

$$= \underbrace{\int \mathrm{d}^3x\, \vec{\mathcal{F}}\cdot\delta\vec{x}}_{\text{Work on the system}} + \underbrace{\int \delta(\mathrm{d}^3x)\, \overbrace{\tfrac{1}{3}\sum_i T^{ii}}^{\text{Isotropic pressure}}}_{\text{Work by the system}}$$

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

Energy density (T^{00})

Momentum density (T^{01}, T^{02}, T^{03})

Energy flux (T^{10}, T^{20}, T^{30})

Momentum flux (T^{ij})

Shear stress

Normal stress (pressure)

Virial theorem $\boxed{\langle \delta W \rangle = 0}$ Mechanical equilibrium !

Virial theorem (field theory)

Generalization to spacetime dilatations $j_D^\mu = T^{\mu\nu} x_\nu$

$$D = \int \mathrm{d}^3x\, j_D^0 = Ht - G$$

$$e^{i\kappa D}\phi(x)e^{-i\kappa D} = e^{\kappa d_\phi}\phi(e^\kappa x)$$

$$\Leftrightarrow \quad \frac{1}{i}[\phi(x), D] = (x_\mu \partial^\mu + d_\phi)\phi(x)$$

$$H = \int \mathrm{d}^3x\, T^{00}$$

$$G = \int \mathrm{d}^3x\, T^{0i}x^i$$

$$\left.\begin{aligned} \frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{i}[D,H] + \frac{\partial D}{\partial t} \\ \frac{1}{i}[P^i, D] &= -\frac{1}{i}[P^i, G] = P^i \end{aligned}\right\} \Rightarrow \boxed{\frac{1}{i}[P^\mu, D] = \underbrace{P^\mu}_{\text{Spacetime rescaling}} - \underbrace{g^{0\mu}\frac{\mathrm{d}D}{\mathrm{d}t}}_{\text{Dilatation breaking}}}$$

Virial theorem

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = 0$$

$$\underbrace{\langle H \rangle}_{\text{Temporal rescaling}} = \left\langle \frac{\mathrm{d}D}{\mathrm{d}t} \right\rangle = \underbrace{\langle H \rangle}_{\text{Temporal dilatation breaking}} - \underbrace{\left\langle \frac{\mathrm{d}G}{\mathrm{d}t} \right\rangle}_{\text{Spatial dilatation breaking}}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \int \mathrm{d}^3x\, T^\mu{}_\mu + \int \mathrm{d}^3x\, \mathcal{F}^\mu x_\mu$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i \int \mathrm{d}^3x\, T^{ii} + \int \mathrm{d}^3x\, \vec{\mathcal{F}}\cdot\vec{x}$$

No new information from temporal dilatations !