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Basis-function approach to QCD of effective quarks

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Light Cone 2022,
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Bound states of quarks and gluons

$$H\,|\psi\rangle \quad = \quad E\,|\psi\rangle$$

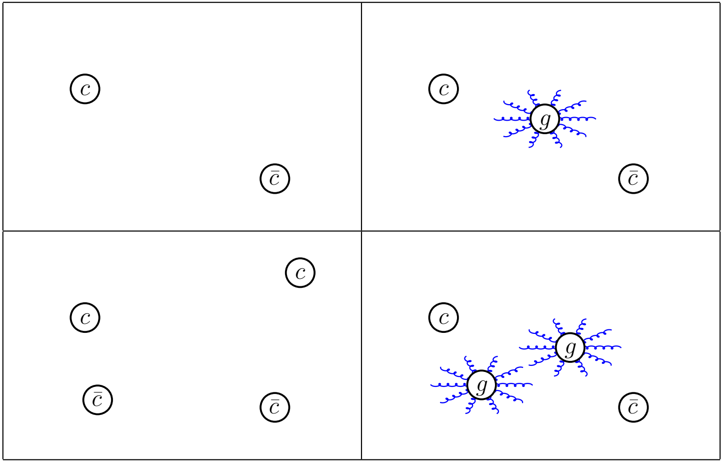
$$H \quad = \quad \hat{P}^+\hat{P}^- - \hat{\mathbf{P}}^2$$

Two challenges

- Fock space
- Renormalization

Two methods

- BLFQ
- RGPEP


c
$\bar{c}$
c
g
$\bar{c}$
c
c
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c
g
g
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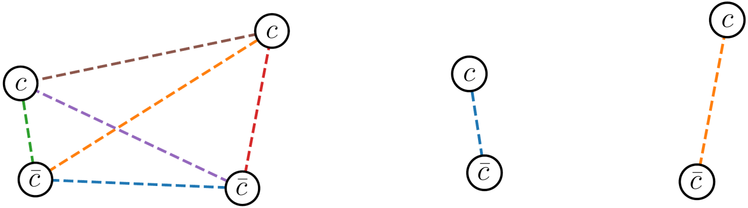
Example: $cc\bar{c}\bar{c}$ sector



Z.Kuang, K.Serafin, X.Zhao, J.P.Vary,
Phys.Rev.D 105 (2022) 094028

Momenta:

$$\mathbf{p} = (p^x,\ p^y) \text{ and } p^+ = p^t + p^z$$

Light-front helicity:

$$\sigma = \pm\frac{1}{2}$$

Color:

$$c = 1, 2, 3 = \begin{cases} r, g, b, & \text{for quarks} \\ \bar{r}, \bar{g}, \bar{b}, & \text{for antiquarks} \end{cases}$$

Fock space (only quarks, no spin, no color)

$$
\begin{aligned}
\mathcal{H}_0 &= \mathbb{C} \\
\mathcal{H}_1 &\approx L^2(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{C}) \\
\mathcal{H}_2 &= (\mathcal{H}_1 \otimes \mathcal{H}_1)_A \\
\mathcal{H}_3 &= (\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1)_A \\
\mathcal{H}_4 &= (\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1)_A \\
&\vdots
\end{aligned}
$$

$$
\mathcal{H}_{\text{Fock}} \quad = \quad (\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \dots)^c
$$

Fock space (only quarks, no spin, no color)

$$
\begin{array}{rl}
|0\rangle & \mathcal{H}_0 = \mathbb{C} \\
b_1^\dagger|0\rangle & \mathcal{H}_1 \approx L^2(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{C}) \\
b_1^\dagger b_2^\dagger|0\rangle & \mathcal{H}_2 = (\mathcal{H}_1 \otimes \mathcal{H}_1)_A \\
b_1^\dagger b_2^\dagger b_3^\dagger|0\rangle & \mathcal{H}_3 = (\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1)_A \\
b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger|0\rangle & \mathcal{H}_4 = (\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1)_A \\
\vdots &
\end{array}
$$

$$
\mathcal{H}_{\mathrm{Fock}} = (\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \dots)^c
$$

$$
\begin{aligned}
\left\{b_1, b_2^\dagger\right\} &= \delta_{c_1,c_2}\, \delta_{\sigma_1,\sigma_2}\, 4\pi p_1^+ \delta(p_1^+ - p_2^+)\ (2\pi)^2 \delta^2(\mathbf{p}_1 - \mathbf{p}_2) \\
\{b_1, b_2\} &= 0 \\
b_1|0\rangle &= 0
\end{aligned}
$$

Basis light-front quantization

(Vary et al. 2010)

$$B_i^\dagger = \frac{1}{\sqrt{P^+}} \int \frac{d^2q}{(2\pi)^2} \, \Psi_{n_i}^{m_i}(\mathbf{q}) \, b_i^\dagger \left(p_i^+, \mathbf{p}_i\right)$$

$$\left\{B_i, B_j^\dagger\right\} = \delta_{n_i,n_j} \, \delta_{m_i,m_j} \, \delta_{k_i,k_j} \, \delta_{\sigma_i,\sigma_j} \, \delta_{c_i,c_j}$$

$$\langle 0|b \; B^\dagger|0\rangle = \sqrt{P^+}\Psi\left(\frac{\mathbf{p}}{\sqrt{x}}\right)$$

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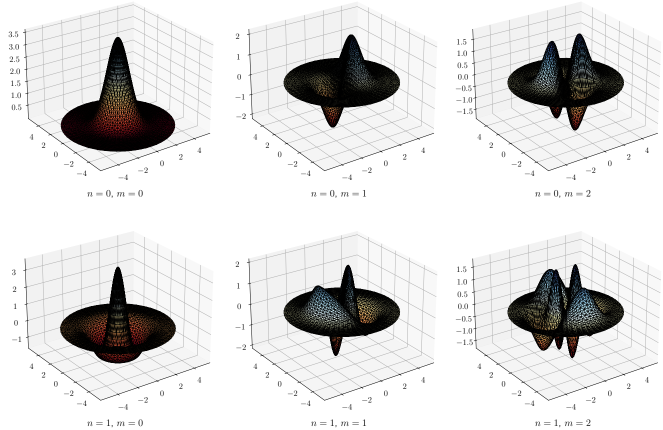

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$$\mathbf{q} = \frac{\mathbf{p}_i}{\sqrt{x_i}} \,, \qquad x_i = \frac{p_i^+}{P^+} \,,$$

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$$\mathbf{q} = \frac{\mathbf{p}_i}{\sqrt{x_i}} \,, \qquad x_i = \frac{p_i^+}{P^+} \,,$$

$$\begin{aligned} \mathcal{M}_{12}^2 &= \frac{m_1^2 + \mathbf{p}_1^2}{x_1} + \frac{m_2^2 + \mathbf{p}_2^2}{x_2} - (\mathbf{p}_1 + \mathbf{p}_2)^2 \\ &= \frac{m_1^2}{x_1} + \frac{m_2^2}{x_2} + \mathbf{q}_1^2 + \mathbf{q}_2^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \; . \end{aligned}$$

$$\begin{aligned} H_{\text{holographic}} &= \mathbf{q}_{12}^2 + \kappa^4 \mathbf{s}^2 \\ \mathbf{s} &= -i \frac{\partial}{\partial \mathbf{q}_{12}} \end{aligned}$$

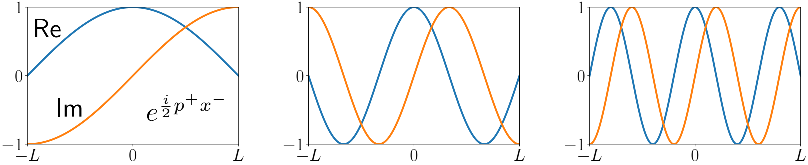
$$\begin{aligned} \mathbf{P} &= \mathbf{q}_1 \cos\xi + \mathbf{q}_2 \sin\xi \\ \mathbf{q}_{12} &= \mathbf{q}_1 \sin\xi - \mathbf{q}_2 \cos\xi \end{aligned}$$

BLFQ

- Longitudinal box

$$x^- \in [-L, L]\ , \quad p^+ = \frac{2\pi}{L} k\ , \qquad \begin{array}{ll} k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots & \text{for fermions} \\ k = 1, 2, 3, \ldots & \text{for bosons} \end{array}$$

- Finite resolution $\sum_i k_i = K$, $P^+ = \frac{2\pi K}{L}$



$k = \frac{1}{2}$ $\qquad$ $k = \frac{3}{2}$ $\qquad$ $k = \frac{5}{2}$

- Basis:

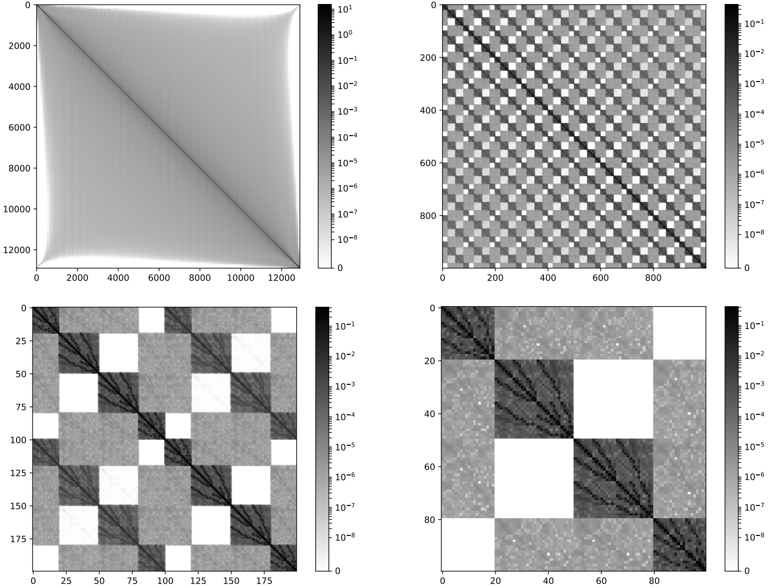
$$\begin{aligned}
\mathcal{H}_1 &= \mathrm{Span}\left(\left\{B_1^\dagger|0\rangle : n_1 \in \mathbb{N},\ m_1 \in \mathbb{Z},\ k_1 \in \mathbb{N} + \frac{1}{2}\right\}\right) \\
\mathcal{H}_2 &= \mathrm{Span}\left(\left\{B_1^\dagger B_2^\dagger|0\rangle : n_1, n_2 \in \mathbb{N},\ m_1, m_2 \in \mathbb{Z},\ k_1, k_2 \in \mathbb{N} + \frac{1}{2}\right\}\right) \\
&\vdots
\end{aligned}$$

- Truncation

$$\sum_i (2n_i + |m_i| + 1) \leq N_{\max}$$

- Fixed P^+: $\sum_i k_i = K = \mathrm{const}$
- M-scheme $M_J = \sum_i (m_i + \sigma_i)$
- Color singlets
- Center of mass motion decouples in truncated basis

Matrix of the Hamiltonian $\langle 0|D_{2'}B_{1'}\ H\ B_1^\dagger D_2^\dagger|0\rangle$



Advantages

- Scalable with the number of particles
- One can take advantage of growth of computing power of supercomputers

Challenges

- Inifinitely many Fock sectors produce factorial growth of the basis dimension with N_{max}
- Renormalization of divergent interactions

Effective particles

(Głazek 2012)

- Effective particles of size s,

$$\begin{aligned} \psi_t &= U_t\, \psi\, U_t^\dagger \;, && \text{(effective field)} \\ b_t^\dagger &= U_t\, b^\dagger\, U_t^\dagger \;, && \text{(effective particle)} \end{aligned}$$

where $t = s^4$ and U_t is a unitary operator.

- Rewrite the theory in terms of effective particles

$$\underbrace{H_{t=0}(b, b^\dagger, d, d^\dagger, a, a^\dagger)}_{\text{initial Hamiltonian}} = \underbrace{H_t(b_t, b_t^\dagger, d_t, d_t^\dagger, a_t, a_t^\dagger)}_{\text{effective Hamiltonian}}$$

- Instead of U_t it is more convenient to write the equation that governs scale evolution of the Hamiltonian:

$$\begin{aligned} \frac{d}{dt}\mathcal{H}_t &= \left[[\mathcal{H}_f, \tilde{\mathcal{H}}_t], \mathcal{H}_t\right] \;, \qquad U_t = T\exp\left(-\int_0^t d\tau \left[\mathcal{H}_f, \tilde{\mathcal{H}}_\tau\right]\right) \;, \\ \mathcal{H}_t &= H_t(b, b^\dagger, d, d^\dagger, a, a^\dagger) \;. \end{aligned}$$

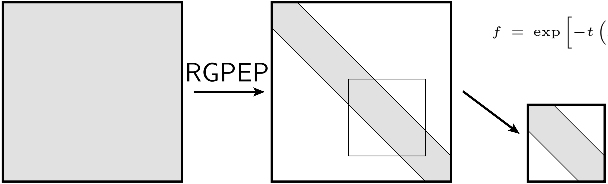
Renormalization group procedure

(Głazek, Wilson 1993, Głazek 2012)

- The first guess for $H_{t=0}$ usually gives divergent results for observables
- Condition for finiteness of the observables of $H_{t=0}$ is equivalent with the demand that all matrix elements of H_t are UV-finite

	Matrix elements	Observables	
$H_{\text{canonical}}$	Finite	Divergent	
H_{initial}	Divergent	Finite	$H_{\text{initial}} = H_{\text{canonical}} + CT$
$H_{t>0}$	Finite	Finite	

- H_t is then called renormalized Hamiltonian



$$f = \exp\left[-t\left(\mathcal{M}^2_{\text{out}} - \mathcal{M}^2_{\text{in}}\right)^2\right]$$

Advantages of combining RGPEP and BLFQ

- RGPEP and BLFQ approaches share many similarities
- RGPEP should provide better effective potentials for BLFQ
- Effective Hamiltonians require smaller matrices to be diagonalized
  - Effective Hamiltonians are practically band diagonal
  - Effective gluons are expected to be massive
  - Justification for Fock sector truncation
- BLFQ is suitable for handling very large bases and many particles
- BLFQ provides nonperturbative solutions to truncated Hamiltonians
  - Help in development of RGPEP
  - Calculations with many Fock sectors can be done

Heavy quarkonium problem

$$H_t \,|\psi_t\rangle \;=\; M^2 \,|\psi_t\rangle$$

$$|\psi_t\rangle \;=\; \left|Q_t\bar{Q}_t\right\rangle + \left|Q_t\bar{Q}_tG_t\right\rangle + \left|Q_t\bar{Q}_tG_tG_t\right\rangle + \ldots$$

$$\downarrow$$

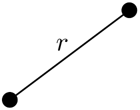
$$H_{\mathrm{eff}\;t} \left|\mathbb{Q}_t\bar{\mathbb{Q}}_t\right\rangle \;=\; M^2 \left|\mathbb{Q}_t\bar{\mathbb{Q}}_t\right\rangle$$

Plan

- Solve 2nd-order effective Hamiltonian for heavy quarkonium using BLFQ
- Compute 4th-order effective Hamiltonian for heavy quarkonium using RGPEP
- Solve 4th-order effective Hamiltonian for heavy quarkonium using BLFQ

Using gluon mass ansatz and nonrelativistic limit

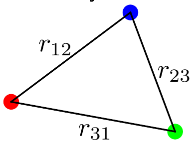
Meson



$$V(r) = \text{Smeared}\left(-\frac{4}{3}\frac{\alpha}{r}\right) + kr^2$$

S.D.Glazek, M.Gómez-Rocha, J.More, K.Serafin, Phys.Lett.B773 (2017) 172

Baryon



$$V_{ij} = \text{Smeared}\left(-\frac{2}{3}\frac{\alpha}{r_{ij}}\right) + k_{ij}r_{ij}^2$$

K.Serafin, M.Gómez-Rocha, J.More, S.D.Glazek, Eur.Phys.J.C78 (2018) 964

- Nothing depends on the gluon mass.
- Rotationally symmetric potential.
- Oscillator frequencies depend on the size of effective particles.
- Spin-dependent interactions are neglected.

Using gluon mass ansatz and nonrelativistic limit

$$\frac{\vec{k}^2}{m}\,\phi(\vec{k}\,) + \int \frac{d^3k'}{(2\pi)^3}\,V(\vec{k},\vec{k}')\,\phi(\vec{k}') - \frac{m}{4}\omega^2\frac{\partial^2}{\partial\vec{k}^2}\phi(\vec{k}\,) = \left(B + \frac{B^2}{4m}\right)\phi(\vec{k}\,)$$

- Light-Front equation in disguise.
- B is the binding energy ($M = 2m + B$).
- Coulomb potential,

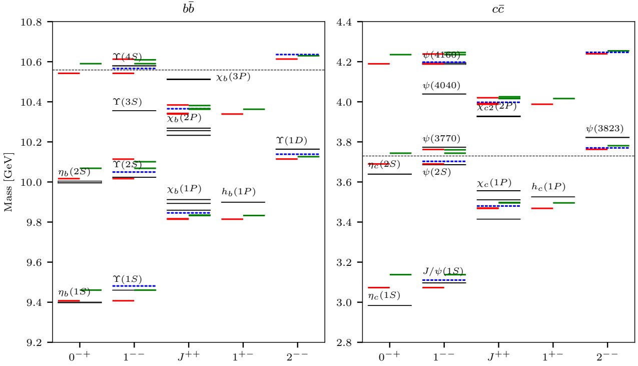
$$V(\vec{k},\vec{k}') = -\frac{4}{3}\,\frac{4\pi\alpha}{\left(\vec{k}-\vec{k}'\right)^2 + m_g^2}e^{-16\,t\left(k^2-k'^2\right)^2}\ .$$

The exponential form factor smears the interaction (effective particles have nonzero size).

Meson spectrum

m_b	$\omega_{b\bar{b}}$	$\alpha_{b\bar{b}}$	$\lambda_{b\bar{b}}$	m_c	$\omega_{c\bar{c}}$	$\alpha_{c\bar{c}}$	$\lambda_{c\bar{c}}$
4.698 GeV	0.2688 GeV	0.2664	4.258 GeV	1.460 GeV	0.3216 GeV	0.3926	1.944 GeV

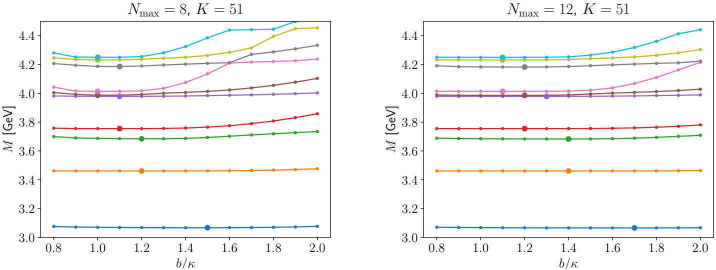
$\lambda = 1/s$



- black – experiment, no form factor, with form factor, reference (analytical formula) (K. Serafin, M. Gómez-Rocha, J. More, S. Głazek, EPJC 78, 964, 2018).
- $N_{\text{max}} = 14,\ K = 111,\ m_g = 2$ MeV
- Harmonic oscillator + Coulomb

b-dependence

- In the limit $N_{\text{max}} \to \infty$ there should be no dependence on b

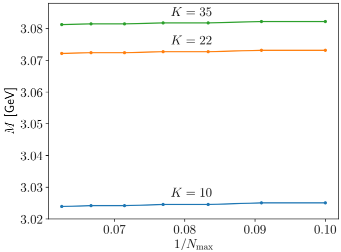


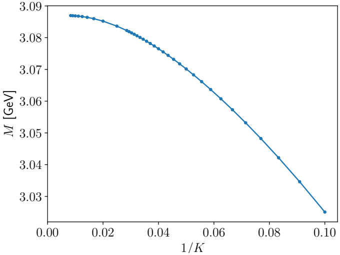
$\kappa = \sqrt{2m\omega}$

- The bigger N_{max} the weaker the dependence of energies on b
- In practice some choices of b are much better for convergence than other
- What b to choose?

N_{max}- and K-dependence

- Dependence of the masses on N_{max} is rather weak. Here, for the ground state of charmonium:





- Dependence of the mass on K is much stronger for small K

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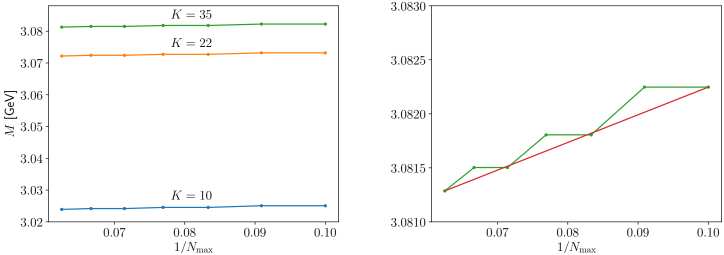

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m_g-dependence

- "Positronium" (Wiecki et al.):

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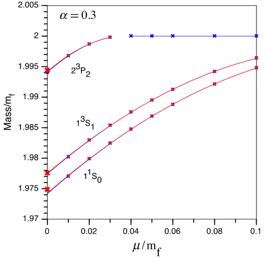
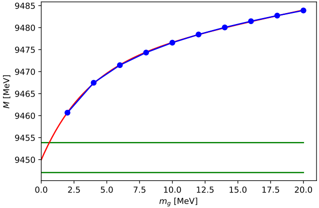
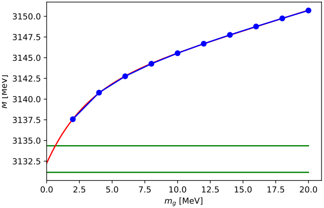


- Botommonium ($N_{\text{max}} = 10,\ K = 111$)

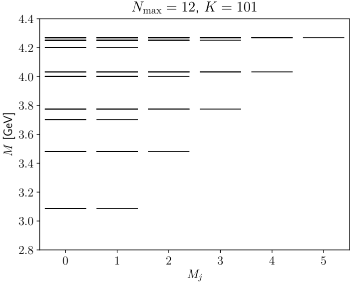


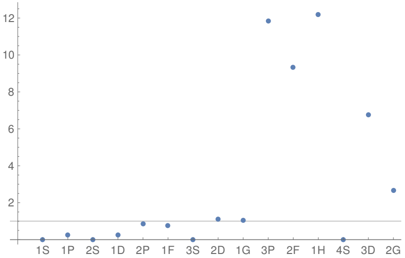
- Charmonium ($N_{\text{max}} = 10,\ K = 111$)



Rotational symmetry (harmonic oscillator + Coulomb)

$N_{\text{max}} = 10,\ K = 111$





To do

- Use full 2nd-order effective interaction (being done)
- Calculate effective Hamiltonian using RGPEP up to 4th order in the coupling constant
- Compute form factors and structure functions

Other possible developments

- Calculations including higher Fock sectors explicitly
- Baryons, tetraquarks, hybrids...

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Thank you for attention