Computing GPDs in Lattice QCD: Recent Progress

(Andreas Metz, Temple University)

- Tremendous recent activity in studying parton structure of hadrons in lattice QCD (LQCD) through Euclidean correlators

- Largely triggered by work on quasi-PDFs (X. Ji, 1305.1539)

- Impact of approach(es) largest where experiments are difficult
  $\rightarrow$ for instance, generalized parton distributions (GPDs)

- Goals of presentation:
  1. brief overview of field
  2. discussion of recent work on LQCD calculation in asymmetric frames
     (S. Bhattacharya, C. Cichy, M. Constantinou, J. Dodson, X. Gao, A.M., S. Mukherjee, A. Scapellato, F. Steffens, Y. Zhao, 2209.05373)

supported by the NSF

Outline

- Quasi-PDFs and related quantities

- Light-cone GPDs

- Quasi-GPDs and related quantities: overview

- Quasi-GPDs and LQCD calculations in asymmetric frames
  - symmetric and asymmetric frames
  - non-uniqueness of quasi-PDFs
  - Lorentz-invariant amplitudes
  - definitions of quasi-GPDs
  - numerical results for the proton in LQCD

- Summary

Quasi-PDFs

- Light-cone unpolarized quark PDF (support: $-1 \leq x \leq 1$)

$$f_1(x) = \int \frac{dz^-}{4\pi} e^{ik\cdot z} \langle P|\bar{q}(-\tfrac{z}{2})\,\gamma^+\,\mathcal{W}(-\tfrac{z}{2},\tfrac{z}{2})\,q(\tfrac{z}{2})|P\rangle\Big|_{z^+=z_\perp=0}$$

  – correlator depends on time $t = z^0 = \frac{1}{\sqrt{2}}\,z^-$ $\rightarrow$ cannot be computed in LQCD

- Suggestion: consider quasi-PDF instead (Ji, 1305.1539) (support: $-\infty < x < \infty$)

$$f_{1,\mathrm{Q}(3)}(x, P^3) = \int \frac{dz^3}{4\pi} e^{ik\cdot z} \langle P|\bar{q}(-\tfrac{z}{2})\,\gamma^3\,\mathcal{W}_\mathrm{Q}(-\tfrac{z}{2},\tfrac{z}{2})\,q(\tfrac{z}{2})|P\rangle\Big|_{z^0=z_\perp=0}$$

  – (Euclidean) correlator depends on position z^3 $\rightarrow$ can be computed in LQCD
  – quasi-PDF depends on $x = k^3/P^3$, and on hadron momentum $P^3 = |\vec{P}|$
  – quasi-PDF and light-cone PDF contain same IR physics, but different UV physics
  – at large P^3, difference in UV behavior is dealt with via perturbative matching
  (Xiong et al, 1310.7471 / Stewart, Zhao, 1709.04933 / Izubuchi et al, 1801.03917 / ...)
  – LQCD calculations at finite P^3 $\rightarrow$ power corrections

- Generic structure of matching formula (scale-dependence omitted)

$$f_{1,\mathrm{Q}(3)}(x, P^3) = \int_{-1}^{1} \frac{dy}{|y|}\, C\left(\frac{x}{y}\right) f_1(y) + \mathcal{O}\left(\frac{M^2}{(P^3)^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{(P^3)^2}\right)$$

  - C is matching coefficient
  - several studies on power corrections exist
  - quasi-PDFs can be considered as "good lattice cross section" (Ma, Qiu, 1404.6860)

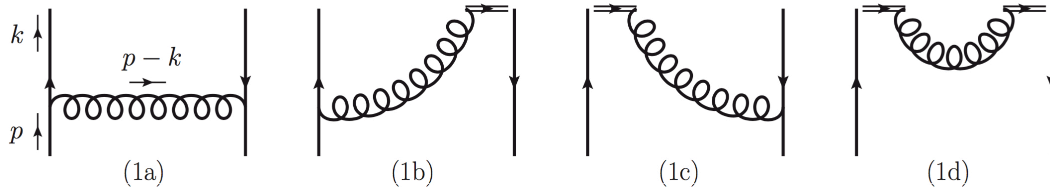
- Choosing γ^0 (instead of γ^3) for unpolarized quasi-PDF (e.g., Radyushkin, 1705.01488)

$$f_{1,\mathrm{Q}(0)}(x, P^3) = \int \frac{dz^3}{4\pi}\, e^{ik\cdot z}\, \langle P|\bar{q}(-\tfrac{z}{2})\, \gamma^0\, \mathcal{W}_{\mathrm{Q}}(-\tfrac{z}{2}, \tfrac{z}{2})\, q(\tfrac{z}{2})|P\rangle\Big|_{z^0 = z_\perp = 0}$$

  - in principle, any linear combination of γ^3 and γ^0 would work (except γ^-)
  - $f_{1,\mathrm{Q}(0)}$ may have smaller power corrections
  - $f_{1,\mathrm{Q}(0)}$ better behaved w.r.t. renormalization (Constantinou, Panagopoulos, 1705.11193)

- Several related suggestions for computing PDFs in LQCD
  (Braun, Müller, 0709.1348 / Ma, Qiu, 1404.6860 / Radyushkin, 1705.01488 / ...)

- Making the essence of the quasi-PDF approach explicit
  - 1-loop real diagrams for matching calculation



  - sample result, with nonzero gluon mass m_g as IR regulator

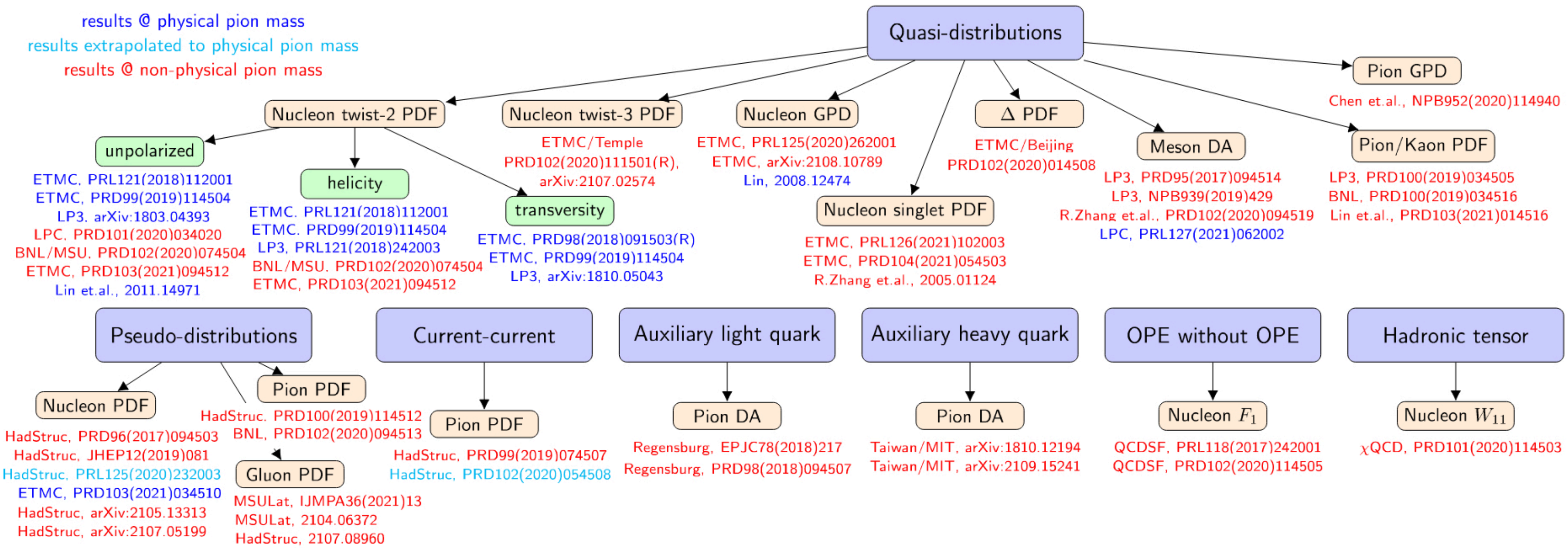
$$f_1^{(1a)} = \frac{\alpha_s C_F}{2\pi}(1-x)\left(\ln\frac{\mu^2}{x\,m_g^2} - 2\right)$$

$$f_{1,\mathrm{Q}}^{(1a)} = \frac{\alpha_s C_F}{2\pi}\begin{cases}(1-x)\,\ln\dfrac{x-1}{x} - 1 & x<0\\[2ex] (1-x)\,\ln\dfrac{4(1-x)p_3^2}{m_g^2} + x & 0<x<1\\[2ex] (1-x)\,\ln\dfrac{x}{x-1} + 1 & x>1\end{cases}$$

$\rightarrow$ light-cone and quasi-PDF have same $\ln m_g^2$ singularity, which is essence of quasi-PDF approach

Dynamical Progress of LQCD Calculations

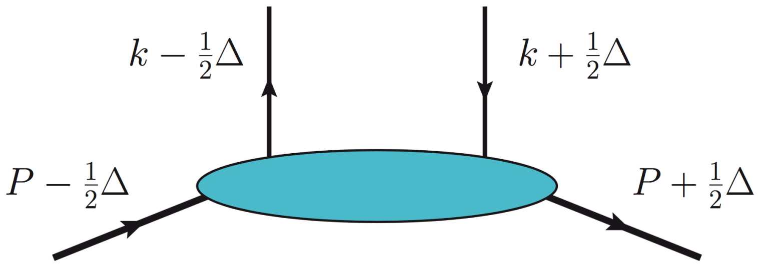
- LQCD calculations of x-dependence of PDFs and related quantities, using Euclidean correlators (compilation by Cichy, 2110.07440)



- Also many theoretical studies available

Light-Cone GPDs

- GPD correlator: graphical representation



$$P = \frac{p + p'}{2} \qquad \Delta = p' - p$$

- Correlator for light-cone GPDs of quarks

$$F^{[\Gamma]}(x, \Delta) = \int \frac{dz^-}{4\pi}\, e^{ik \cdot z}\, \langle p' | \bar{q}(-\tfrac{z}{2})\, \Gamma\, \mathcal{W}(-\tfrac{z}{2}, \tfrac{z}{2})\, q(\tfrac{z}{2}) | p \rangle \Big|_{z^+ = 0, \vec{z}_\perp = \vec{0}_\perp}$$

correlator parameterized through GPDs $X(x, \xi, t)$

$$x = \frac{k^+}{P^+} \qquad \xi = \frac{p^+ - p'^+}{p^+ + p'^+} = -\frac{\Delta^+}{2P^+} \qquad t = \Delta^2$$

- Kinematic relation

$$t = -\frac{1}{1 - \xi^2}\,(4\xi^2 M^2 + \vec{\Delta}_\perp^2)$$

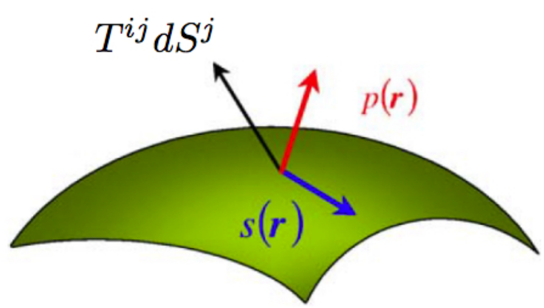
- Main motivations for GPDs
  - impact parameter distributions (Burkardt, hep-ph/0005108 / ...)

$$\mathrm{GPD}(x, \xi = 0, \vec{\Delta}_T) \quad \overset{\mathcal{F}.\mathcal{T}.}{\longleftrightarrow} \quad f(x, \vec{b}_T)$$

  - spin sum rule and orbital angular momentum (Ji, hep-ph/9603249)

$$J_q = \int_{-1}^{1} dx\, x\, (H_q + E_q)\Big|_{t=0} \qquad\qquad J_g = \int_{0}^{1} dx\, (H_g + E_g)\Big|_{t=0}$$

  - closely related: "mechanical properties" (pressure, shear) inside nucleon (Polyakov, Shuvaev, hep-ph/0207153 / ...)



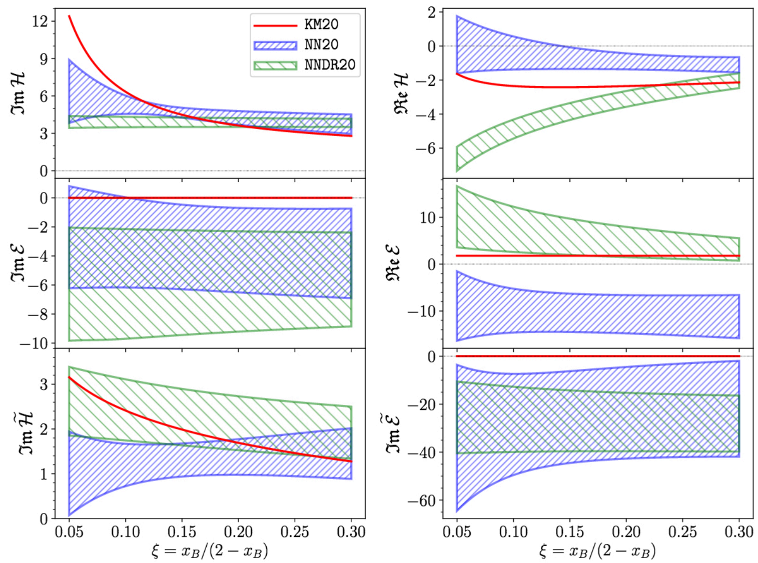
- Key processes: deep-virtual Compton scattering, hard exclusive meson production, ...

- Example of observable: Compton form factor

$$\mathcal{H}(\xi,t) = \sum_q e_q^2 \int_{-1}^{1} dx\, H^q(x,\xi,t) \left(\frac{1}{\xi - x - i\varepsilon} - \frac{1}{\xi + x - i\varepsilon} \right) + \mathcal{O}(\alpha_s)$$

extracted CFFs ($t = -0.2\,\mathrm{GeV}^2$)

(Čuić, Kumerički, Schäfer, 2007.00029)



- extraction of CFFs difficult
- how to get GPDs from CFFs, that is, how to solve DVCS inverse problem ? (information on x lost in observable CFFs)
  (see, for instance, Bertone et al, 2104.03836)
- existing GPD parameterizations have (significant) model dependence
- in principle, data from other processes could help, but by no means easy

$\rightarrow$ information on GPDs from LQCD (beyond moments) would be very helpful

Definition of Quasi-GPDs

- (Spatial) correlator for quasi-GPDs of quarks (Ji, 1305.1539)

$$F_{\mathrm{Q}}^{[\Gamma]}(x, \Delta; P^3) = \int \frac{dz^3}{4\pi} \, e^{ik \cdot z} \, \langle p' | \bar{q}(-\tfrac{z}{2}) \, \Gamma \, \mathcal{W}_{\mathrm{Q}}(-\tfrac{z}{2}, \tfrac{z}{2}) \, q(\tfrac{z}{2}) | p \rangle \Big|_{z^0 = z_\perp = 0}$$

- Definition of quasi-GPDs for twist-2 vector GPDs H and E

$$F_{\mathrm{Q}}^{[\gamma^0]}(x, \Delta; P^3) \; = \; \frac{1}{2P^0} \, \bar{u}(p') \Big[\gamma^0 \, H_{\mathrm{Q}(0)} + \frac{i\sigma^{0\mu} \Delta_\mu}{2M} \, E_{\mathrm{Q}(0)} \Big] u(p)$$

$$F_{\mathrm{Q}}^{[\gamma^3]}(x, \Delta; P^3) \; = \; \frac{1}{2P^3} \, \bar{u}(p') \Big[\gamma^3 \, H_{\mathrm{Q}(3)} + \frac{i\sigma^{3\mu} \Delta_\mu}{2M} \, E_{\mathrm{Q}(3)} \Big] u(p)$$

  - in forward limit, definitions of quasi-GPDs reduce to definitions of quasi-PDFs
  - quasi-GPDs depend on

$$x = \frac{k^3}{P^3} \neq \frac{k^+}{P^+} \qquad\qquad \xi \qquad\qquad t = \Delta^2 \qquad\qquad P^3$$

Studies on Quasi-GPDs and Related Quantities: Overview

- Matching calculations for quasi-GPDs

  Ji, Schäfer, Xiong, Zhang, 1506.00248 / Xiong, Zhang, 1509.08016 /

  Liu et al, 1902.00307 / Ma, Pang, Zhang, 2202.07116

- Model calculations and model-independent results

  Bhattacharya, Cocuzza, AM, 1808.01437 / 1903.05721 / Ma, Zhu, Lu, 1912.12816 /

  Luo, Sun, 2005.09832 / Shastry, Broniowski, Ruiz Arriola, 2209.02619

- GPDs via pseudo distributions

  Radyushkin, 1909.08474

- LQCD calculation of pion GPDs using quasi-GPD approach

  Chen, Lin, Zhang, 1904.12376

- LQCD calculation of nucleon GPDs using quasi-GPD approach

  Alexandrou et al (ETMC) 2008.10573 / 2108.10789 / Lin, 2008.12474 / 2112.07519

- LQCD calculation of GPDs in asymmetric frames

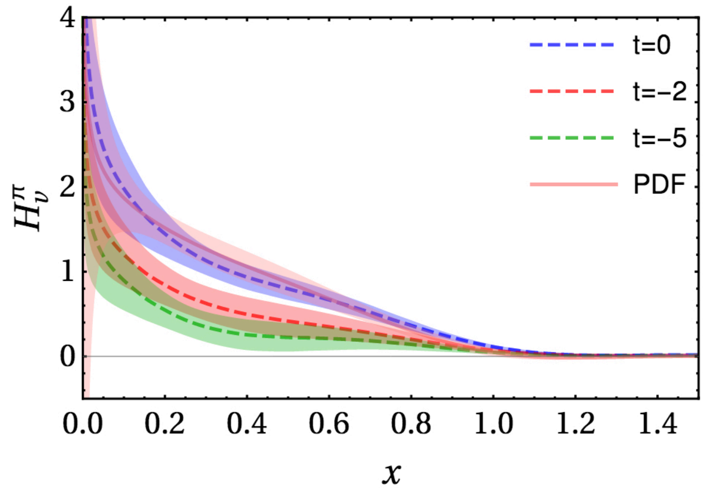
  Bhattacharya et al, 2209.05373

GPDs: Some Results from LQCD

- Pioneering LQCD calculations for pion and nucleon GPDs

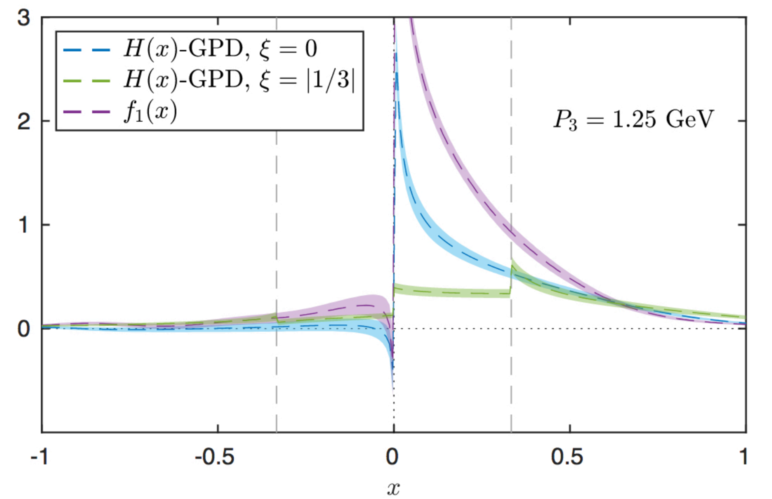
pion GPD H $(m_\pi = 140\,\mathrm{MeV})$
$(\xi = 0)$

(Chen, Lin, Zhang, 1904.12376)



proton GPD H $(m_\pi = 260\,\mathrm{MeV})$
$(\xi = 0 \text{ and } \xi \neq 0\,,\ t = -0.69\,\mathrm{GeV}^2)$

(Alexandrou et al (ETMC), 2008.10573)



  - pioneering calculations quite encouraging
  - more results have been published

Symmetric and Asymmetric Frames

- Comparing symmetric and asymmetric frames
  - momenta for the sink (final state)

$$\vec{P}_s^{\,\mathrm{sink}} = \left(\tfrac{\Delta^1}{2},\ \tfrac{\Delta^2}{2},\ P^3\right) \qquad\qquad \vec{P}_a^{\,\mathrm{sink}} = \left(0,\ 0,\ P^3\right)$$

  - LQCD calculations in asymmetric frame advantageous because:
    (i) in symmetric frame, full new calculation for each momentum transfer Δ
    (ii) larger variety of momentum transfers possible for asymmetric frame

- Lorentz-transformation (LT) between frames
  - transverse boost, in direction of transverse momentum transfer
  - β of LT for transverse momentum transfer in x-direction

$$\beta = \frac{\Delta_a^1}{E_a + E_a'}$$

  - kinematics of asymmetric frame fixes kinematics of corresponding symmetric frame

Quasi-PDFs and Lorentz-Invariant Amplitudes (spin-0)

- Definition of quasi-PDFs not unique (Wilson line suppressed)

$$f_{1,\mathrm{Q}(3)}(x, P^3) = \int \frac{dz^3}{4\pi}\, e^{ik\cdot z}\, \langle P|\bar{q}(-\tfrac{z}{2})\, \gamma^3\, q(\tfrac{z}{2})|P\rangle\Big|_{z^0=z_\perp=0}$$

$$f_{1,\mathrm{Q}(0)}(x, P^3) = \frac{P^3}{P^0}\int \frac{dz^3}{4\pi}\, e^{ik\cdot z}\, \langle P|\bar{q}(-\tfrac{z}{2})\, \gamma^0\, q(\tfrac{z}{2})|P\rangle\Big|_{z^0=z_\perp=0}$$

$$f_{1,\mathrm{Q}(+)}(x, P^3) = \frac{P^3}{P^+}\int \frac{dz^3}{4\pi}\, e^{ik\cdot z}\, \langle P|\bar{q}(-\tfrac{z}{2})\, \gamma^+\, q(\tfrac{z}{2})|P\rangle\Big|_{z^0=z_\perp=0}$$

$$\int dx\, f_{1,\mathrm{Q}}(x, P^3) = \int dx\, f_1(x) \qquad \text{(norm preserved)}$$

  - in principle, infinitely many definitions possible
  - as $P^3 \to \infty$, which definition converges fastest to the light-cone PDF ?

- Matrix element of bi-local operator in terms of Lorentz-invariant amplitudes

$$\langle P|\bar{q}(-\tfrac{z}{2})\, \gamma^\mu\, q(\tfrac{z}{2})|P\rangle = P^\mu A_1(z\cdot P, z^2) + z^\mu M^2 A_2(z\cdot P, z^2)$$

  - decomposition follows from constraints due to parity, Hermiticity and time-reversal

- PDFs in terms of invariant amplitudes

$$f_1(x) = P^+ \int \frac{dz^-}{4\pi}\, e^{ik\cdot z} \Big[A_1\Big]_{z^+=z_\perp=0}$$

$$f_{1,\mathrm{Q}(3)}(x, P^3) = P^3 \int \frac{dz^3}{4\pi}\, e^{ik\cdot z} \Big[A_1 + M^2 \tfrac{z^3}{P^3}\, A_2\Big]_{z^0=z_\perp=0}$$

$$f_{1,\mathrm{Q}(0)}(x, P^3) = P^3 \int \frac{dz^3}{4\pi}\, e^{ik\cdot z} \Big[A_1\Big]_{z^0=z_\perp=0}$$

$$f_{1,\mathrm{Q}(+)}(x, P^3) = P^3 \int \frac{dz^3}{4\pi}\, e^{ik\cdot z} \Big[A_1 + M^2 \tfrac{z^+}{P^+}\, A_2\Big]_{z^0=z_\perp=0}$$

  - from perspective of matrix elements, quasi-PDF with γ^+ seems natural
  - from perspective of amplitudes, quasi-PDF with γ^0 seems natural
  - difference between these two definitions is power-suppressed term $\sim A_2$ (and vanishes upon integration over x)
  - quasi-PDF with γ^0 may converge faster to light-cone PDF (amplitude A_1 for both)
  - final answer to question about fastest convergence most likely depends on the non-perturbative dynamics (that is, numerical values of the A_i) and on x

Quasi-GPDs and Lorentz-Invariant Amplitudes (spin-0)

- Definition of quasi-GPD through matrix elements
  - traditional definition in symmetric frame; motivated by the PDF case

$$H_{\mathrm{Q}}^{s} \sim \mathcal{F}.\mathcal{T}.\ \langle p'|\bar{q}(-\tfrac{z}{2})\,\gamma^{0}\,q(\tfrac{z}{2})|p\rangle_{s}$$

  - one may also consider the same matrix element in the asymmetric frame

$$\begin{aligned} H_{\mathrm{Q}}^{a} &\sim \mathcal{F}.\mathcal{T}.\ \langle p'|\bar{q}(-\tfrac{z}{2})\,\gamma^{0}\,q(\tfrac{z}{2})|p\rangle_{a} \\ &= H_{\mathrm{Q}}^{s} + \text{power-suppressed terms} \end{aligned}$$

- Lorentz-invariant amplitudes

$$\langle p'|\bar{q}(-\tfrac{z}{2})\,\gamma^{\mu}\,q(\tfrac{z}{2})|p\rangle = P^{\mu}A_{1} + z^{\mu}M^{2}A_{2} + \Delta^{\mu}A_{3}$$

$$\text{with}\ \ A_{i} = A_{i}(z\cdot P, z\cdot\Delta, \Delta^{2}, z^{2})$$

  - amplitudes can be used to relate matrix elements in different frames

- Lorentz-invariant definition of quasi-GPDs
  - light-cone GPD H depends on linear combination of A_1 and A_3

$$H \sim \mathcal{F}.\mathcal{T}.\left(A_1 + \frac{z \cdot \Delta}{z \cdot P} A_3\right)_{z^+ = z_\perp = 0}$$

  - new (Lorentz-invariant) definition of quasi-GPD

$$H_{\mathrm{Q}}^{\mathrm{L.I.}} \sim \mathcal{F}.\mathcal{T}.\left(A_1 + \frac{z \cdot \Delta}{z \cdot P} A_3\right)_{z^0 = z_\perp = 0}$$

  - $H_{\mathrm{Q}}^{\mathrm{L.I.}}$, H_{Q}^{s}, H_{Q}^{a} identical, modulo power-suppressed terms
  - $H_{\mathrm{Q}}^{\mathrm{L.I.}}$ may converge fastest to the light-cone GPD (but see above caveats)

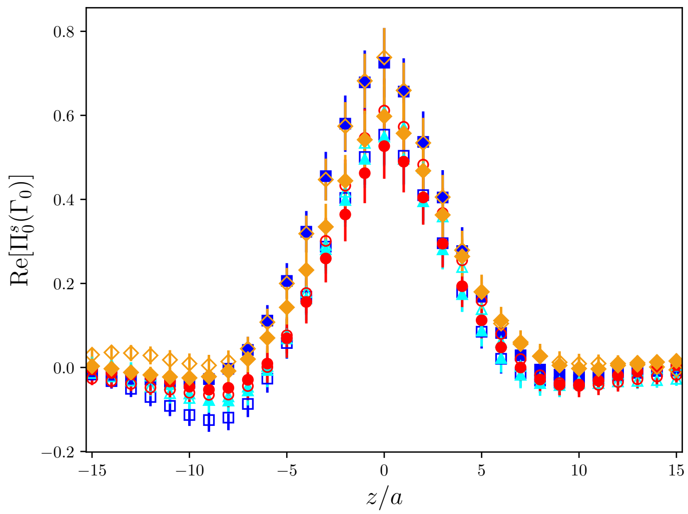
- Most important result: all (three) quasi-GPDs can be related to matrix elements in the asymmetric frame

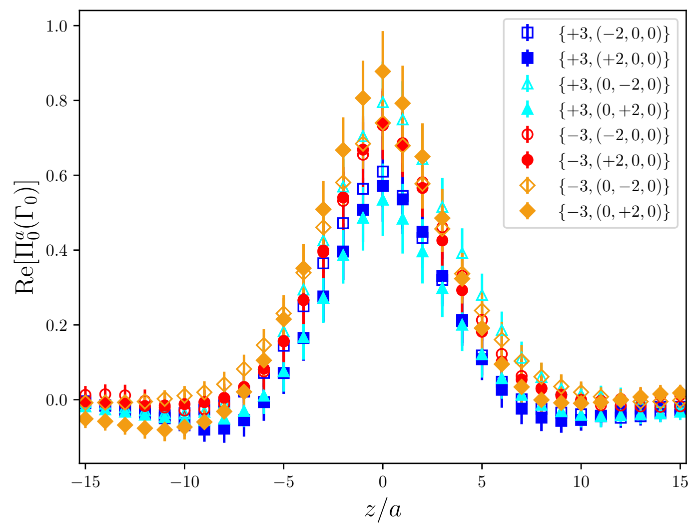
- Extension of formalism for spin-$\frac{1}{2}$ hadron $\rightarrow$ 8 Lorentz-invariant amplitudes
  (compare also Meissner, AM, Schlegel, 0906.5323 / Rajan, Engelhardt, Liuti, 1709.05770)

Numerical Results for Proton

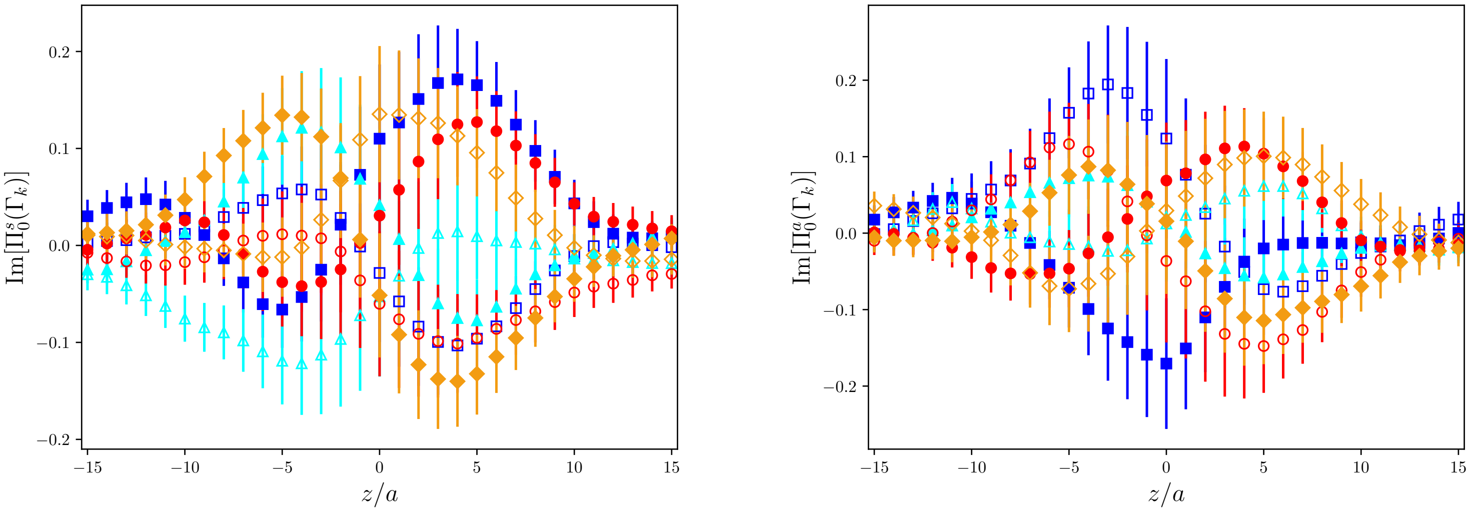
- Key parameters
  - $m_\pi = 260\,\mathrm{MeV}$
  - $P^3 = 1.25\,\mathrm{GeV}$
  - symmetric and asymmetric frame $(|t_s| = 0.69\,\mathrm{GeV}^2,\ |t_a| = 0.64\,\mathrm{GeV}^2)$
  - vanishing skewness, $\xi = 0$
  - 8 momentum configurations $\{(P^3, -P^3),\ (\Delta^1, -\Delta^1),\ (\Delta^2, -\Delta^2)\}$

- Matrix elements: examples



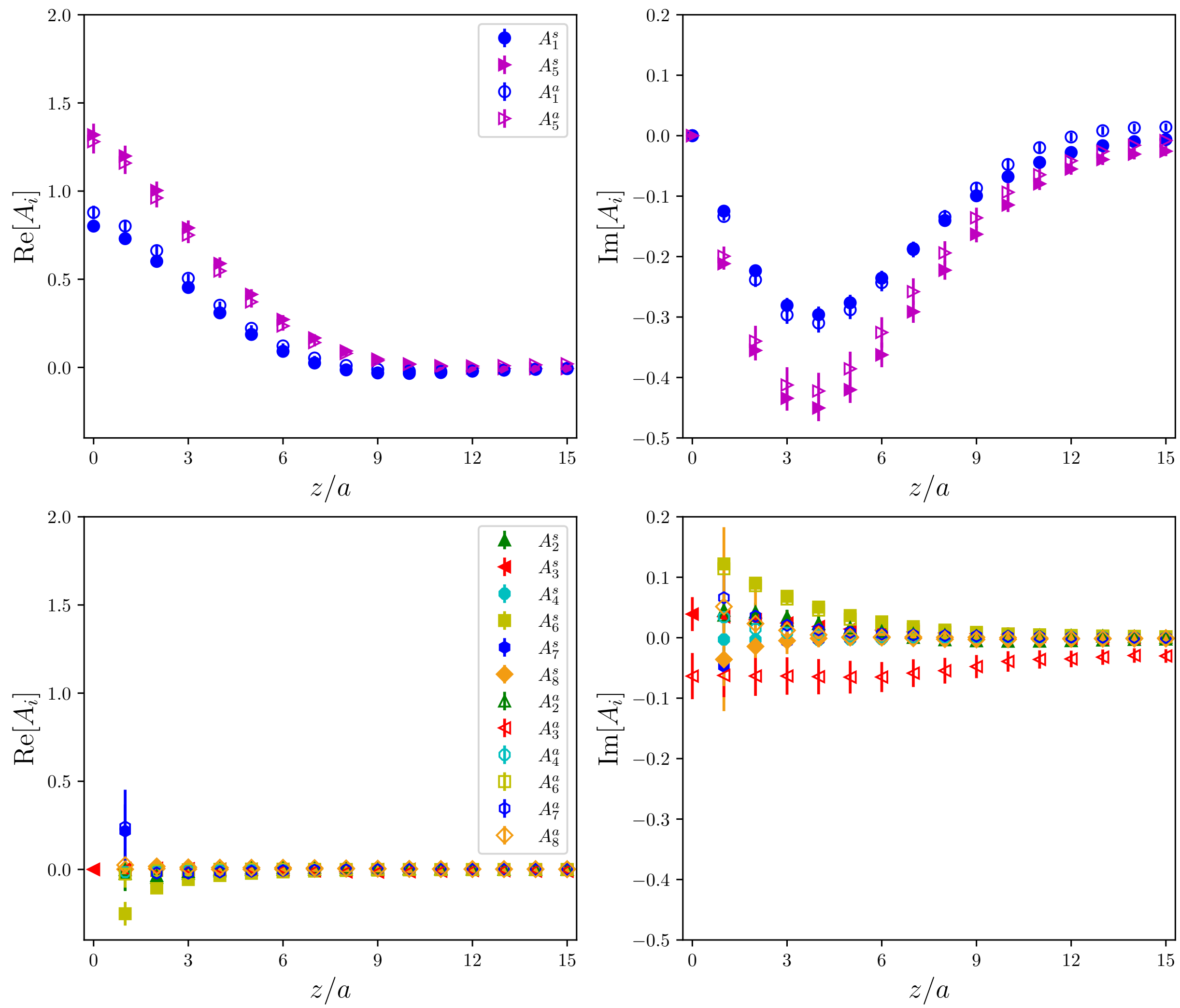


  - shown are statistical uncertainties
  - agreement within errors for different momentum configurations



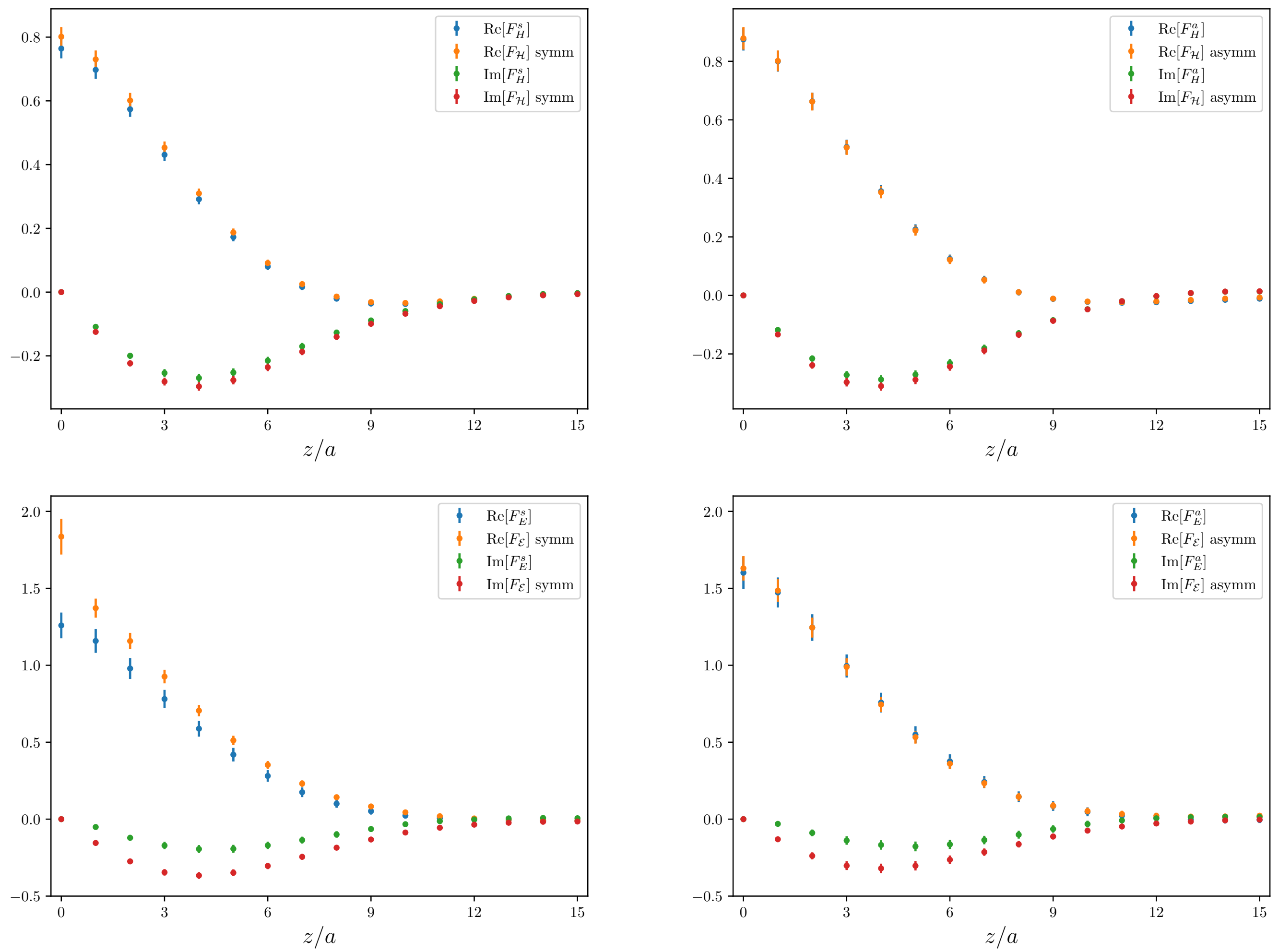
- some matrix elements are noisy
- but still agreement within errors for different momentum configurations

- 8 Lorentz-invariant amplitudes



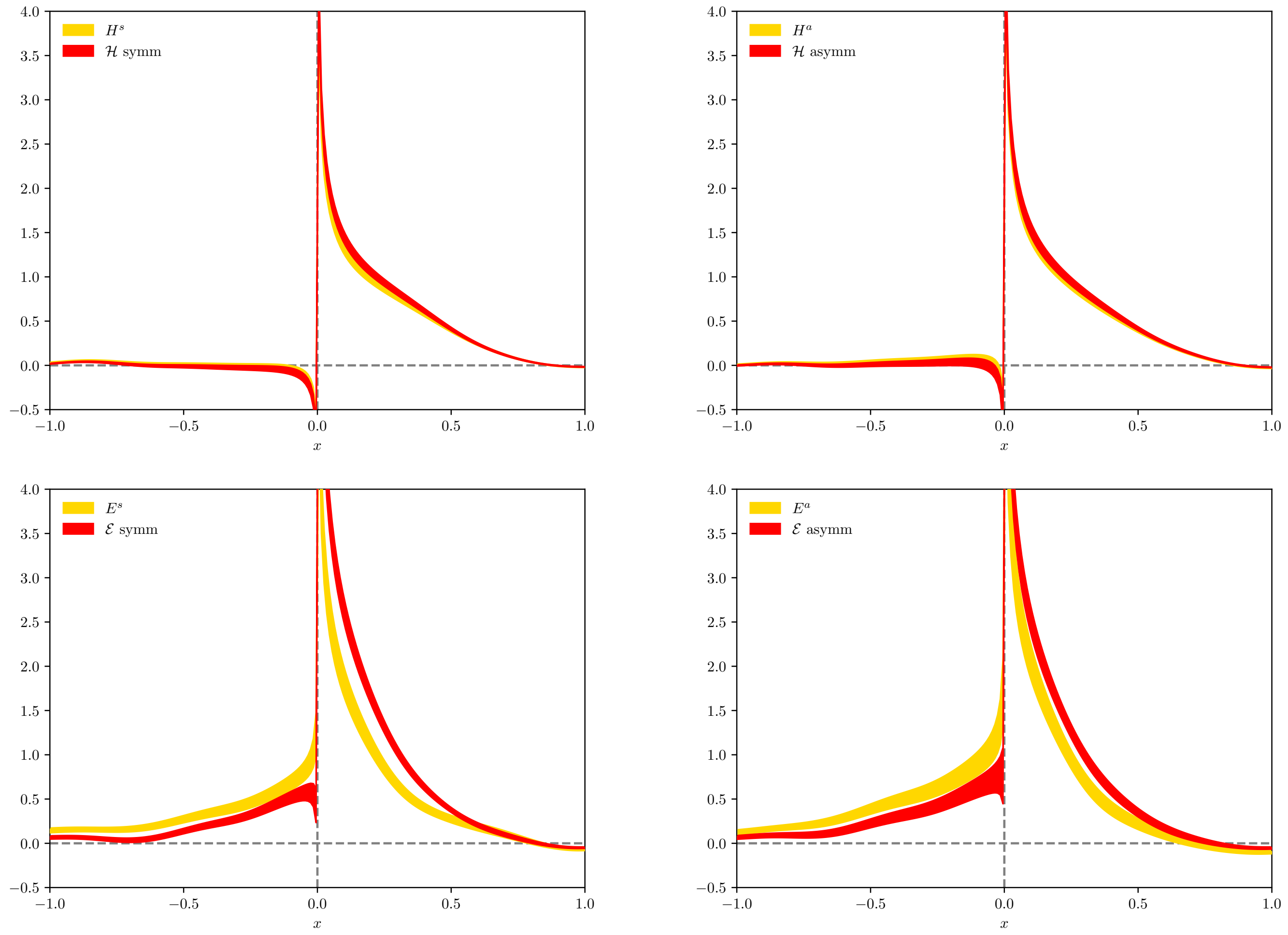
  - amplitudes from matrix elements in symmetric and asymmetric frames
  - agreement within errors $\longrightarrow$ nontrivial check of numerics
  - interestingly, two amplitudes (much) larger than all the others

- Quasi-GPDs in position space ($F_H^s \sim H_\mathrm{Q}^s,\ F_H^a \sim H_\mathrm{Q}^a,\ F_\mathcal{H} \sim H_\mathrm{Q}^{\mathrm{L.I.}}$)



  - for H, very similar results for all definitions, despite (power-suppressed) differences
  - some visible differences for E
  - for E, smallest noise when using Lorentz-invariant definition (not shown)

- Light-cone GPDs



  - agreement and differences in position space reflected in light-cone GPDs
  - differences could be considered part of the systematic uncertainties
  - calculations at higher P^3 will (significantly) reduce those uncertainties

Summary

- Tremendous recent activity in studying parton structure of hadrons in LQCD through Euclidean correlators

- First encouraging results exist for GPDs

- Previous LQCD calculations of GPDs have used symmetric frame

- Asymmetric frame more advantageous for LQCD studies

- Formalism for computing quasi-GPDs through matrix elements in asymmetric frame
  - Lorentz-invariant amplitudes can be used for transformation between frames
  - we also employ amplitudes for new Lorentz-invariant definition of quasi-GPDs
  - definition of quasi-GPDs not unique — differences through power-suppressed terms
  - numerical results for proton, related to the GPDs H and E