Amplitude analysis and complete experiments for light baryon spectroscopy

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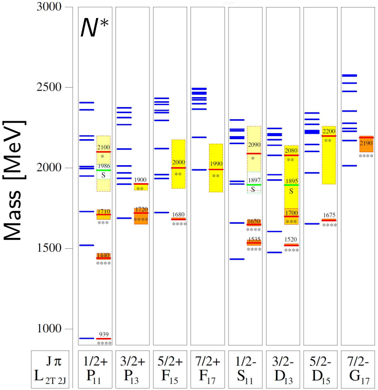
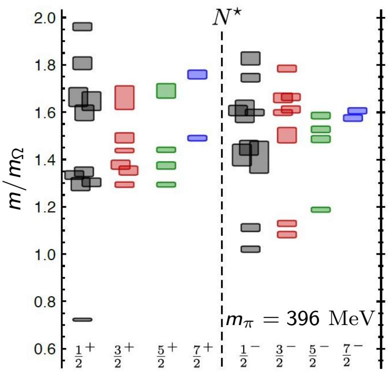
Why baryon spectroscopy?

Quark models



[Löring et al., EPJ A **10**, 395 (2001)]

Lattice QCD



[Edwards et al., PRD **84**, 074508 (2011)]

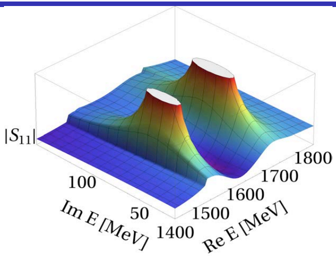
Further study baryon spectroscopy in order to:

- Solve the *missing resonance problem*
- Extract a precision-spectrum as a better basis for comparison for non-perturb. theoretical calculations (quark models, lattice QCD, UχPT, ...)
- (light) baryon spectrum is valuable input for modelling *final-state interactions*

Energy-dependent (ED) amplitude-analysis models

*) Reaction-theoretic models using *S-Matrix constraints* (analyticity, unitarity, crossing), often fit multiple reactions

*) Analytically continue (partial-wave) amplitudes to the resonance-poles in the complex energy-plane: 'pole-positions' $\Rightarrow$ (M^{*}, Γ);
'residues' $\Rightarrow$ branching fractions.



[Fig. courtesy of M. Döring]

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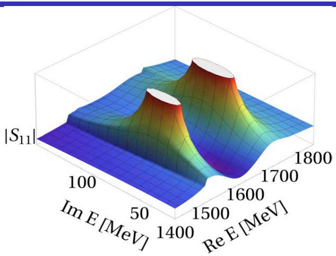

[Fig. courtesy of M. Döring]

Model	Type	Unitarity	non-resonant contrib.'s
Bonn-Gatchina	K-Matrix	2body & approx. 3body	Reggion-exchanges
JüBoW	dynamical coupled-channels	2body & approx. 3body	effective lagrangians
ANL-Osaka	dynamical coupled-channels	2body & approx. 3body	effective lagrangians
SAID	C.M. K-Matrix	2body & approx. 3body	phenomenological parametrizations
MAID	'Breit-Wigner + background'	2body (Watson-Theorem)	effective lagrangians & Reggion exchanges
Kent State University	K-Matrix	2body & approx. 3body	phenomenological parametrizations

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Procedure:

- ▷ ED amplitude-analysis groups perform and publish fits of large collections of datasets on (polarization) measurements (more on these later ...)
- ▷ The PDG Baryon-group filters these publications and assigns *star-ratings* to states:
  $\begin{cases} \ast\ast\ast\ast \text{ or } \ast\ast\ast\text{: established states,} \\ \ast\ast \text{ or } \ast\text{: controversial/claimed states; need confirmation ...} \end{cases}$
- ▷ Community uses listed data for further analyses . . .

Progress in terms of '$*$'-ratings

PDG 2002 vs. PDG 2020 (changes in red): [A. Thiel, F. Afzal & YW, PPNP 125, 103949 (2022), Tab. 16]

Particle	J^P	overall	PWA	$N\gamma$	$N\pi$	$\Delta\pi$	$N\sigma$	$N\eta$	ΛK	ΣK	$N\rho$	$N\omega$
N	$1/2^+$	****										
$N(1440)$	$1/2^+$	****	○ $\diamond_g$ ⋆ ▷	****	****	****	***	-			-	
$N(1520)$	$3/2^-$	****	○ ◇ ⋆ ▷	****	****	****	**	****			- - - -	
$N(1535)$	$1/2^-$	****	○ ◇ ⋆ ▷	****	****	***	*	****			- -	
$N(1650)$	$1/2^-$	****	○ ◇ ⋆ ▷	****	****	***	*	****	*- -	- -	- -	
$N(1675)$	$5/2^-$	****	○ ◇ ⋆ ▷	****	****	****	***	*	*	*	-	
$N(1680)$	$5/2^+$	****	○ ◇ ⋆ ▷	****	****	****	***	*	*	*	- - - -	
$N(1700)$	$3/2^-$	***	○ ▷	**	***	***	*	*	- -	-	-	
$N(1710)$	$1/2^+$	****	○ ◇ ▷	****	****	*-		***	**	*	*	*
$N(1720)$	$3/2^+$	****	○ ◇ ⋆ ▷	****	****	***	*	*	****	*	*-	*
$N(1860)$	$5/2^+$	**	▷	*	**		*	*				
$N(1875)$	$3/2^-$	***	○ ▷	**	**	*	**	*	*	*	*	*
$N(1880)$	$1/2^+$	***	○ ▷	**	*	**	*	*	**	**		**
$N(1895)$	$1/2^-$	****	○ ▷	****	*	*	*	****	**	**	*	*
$N(1900)$	$3/2^+$	****	○ ◇ ▷	****	**	**	*	*	**	**	-	*
$N(1990)$	$7/2^+$	**	○ ◇ ▷	**	**			*	*	*		
$N(2000)$	$5/2^+$	**	○ ⋆	**	*-	**	*	*	-	-	- -	*
$N(2040)$	$3/2^+$	*	▷		*							
$N(2060)$	$5/2^-$	***	○ $\diamond_g$ ▷	***	**	*	*	*	*	*	*	*
$N(2100)$	$1/2^+$	***	○ ▷	**	***	**	**	*	*		*	*
$N(2120)$	$3/2^-$	***	○ ▷	***	**	**	**		**	*		*
$N(2190)$	$7/2^-$	****	○ ◇ ⋆ ▷	****	****	****	**	*	**	*	*	*
$N(2220)$	$9/2^+$	****	○ ◇ ⋆	**	****			*	*	*		
$N(2250)$	$9/2^-$	****	○ ◇ ⋆ ▷	**	****			*	*	*		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮				

○: BnGa-2019, ◇: JüBo-2017 ('g' for dynamically generated), ⋆: SAID-MA19, ▷: KSU PWA

Progress in terms of 'multiplets'

$\hookrightarrow$ Work with spin-flavor $SU(6)$ symmetry and assign quarks to fundamental rep.

$$\mathbf{6} = \left(u \uparrow, d \uparrow, s \uparrow, u \downarrow, d \downarrow, s \downarrow \right)^T .$$

$\Rightarrow$ Baryon (super-) multiplet structure arises from decomposition into irrep.'s:

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_M \oplus \mathbf{70}_M \oplus \mathbf{20}_A .$$

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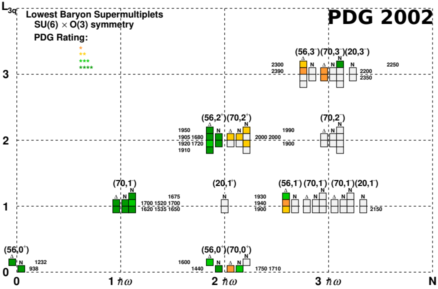
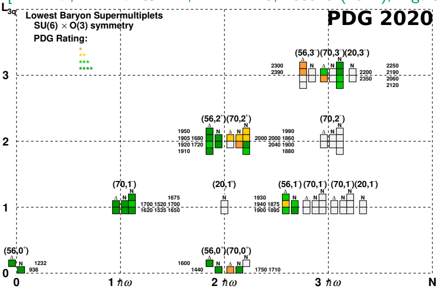
⇒ Assign PDG-resonances to multiplets: [cf. E. Klemt & B. Metsch, EPJ A 48, p. 127 (2012)]

[A. Thiel, F. Afzal & YW, PPNP 125, 103949 (2022), Figs 50,51]





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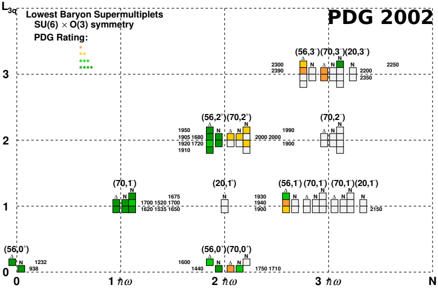

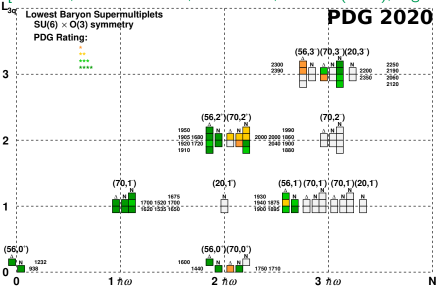

All this progress is great, but is there a way to tell when we are finished?

$\hookrightarrow$ There is: one needs a <u>coupled-channels</u> complete experiment.

Spin amplitudes

Generic case:

Photoproduction:

$\ast$) General meson-production reaction:

$$\mathcal{P} N \rightarrow \{\varphi_i\}\, B, \text{ with:}$$

- $\mathcal{P}$: 'probe'-particle $(\pi, \gamma, \gamma^{*}, \ldots)$,
- N: target-nucleon,
- $\{\varphi_i\}$: one or multiple meson(s),
- B: recoil (spin 1/2-) baryon.

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$*$) One can always expand:

$$\mathcal{T}_{fi} = \chi_B^\dagger \left[\sum_{k=1}^{N_{\mathcal{A}}} \boldsymbol{\kappa}_k b_k \left(\Omega_2^{n_f} \right) \right] \chi_N,$$

- $\kappa_i \left(\{ p_j \}, \{\sigma_i\} \right)$: spin-kinem. operators,
- $b_i \left(\Omega_2^{n_f} \right)$: spin ('transversity') ampl.'s,
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∗) The number of amplitudes $N_{\mathcal{A}}$ is determined from *spin-multiplicities*:

$$N_{\mathcal{A}} = \boldsymbol{n}_{\mathcal{P}} \boldsymbol{n}_N \boldsymbol{n}_{\varphi_1} \ldots \boldsymbol{n}_{\varphi_{\tilde{n}}} \boldsymbol{n}_B,$$

with additional factor of (1/2) in case of a $2 \rightarrow 2$ reaction (parity!)

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*) Produce one pseudoscalar meson φ:

$$\gamma N \to \varphi B.$$

*) Then:
$$N_\mathcal{A} = \frac{1}{2} \left(\boldsymbol{n}_\gamma \boldsymbol{n}_N \boldsymbol{n}_\varphi \boldsymbol{n}_B \right) = \frac{1}{2} \left(2 \times 2 \times 1 \times 2 \right) = \underline{\underline{4}}.$$

*) We have:

$$\mathcal{T}_{fi} = \chi_B^\dagger [\boldsymbol{\kappa}_1 b_1 + \boldsymbol{\kappa}_2 b_2 + \boldsymbol{\kappa}_3 b_3 + \boldsymbol{\kappa}_4 b_4] \chi_N, \text{ where:}$$

- $b_i = b_i(W, \theta)$: transversity amplitudes.

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∗) $\{\boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_4\}$ are complicated, e.g.:

$$\boldsymbol{\kappa}_1 = \frac{(\hat{q}\cdot\hat{\epsilon})}{\sqrt{2}\sin^2\theta}\left[e^{-i\frac{\theta}{2}}\left(\hat{k}\cdot\hat{\boldsymbol{\sigma}}\right) - e^{i\frac{\theta}{2}}\left(\hat{q}\cdot\hat{\boldsymbol{\sigma}}\right)\right]$$

$\boldsymbol{\kappa}_2 = \ldots$, where:

- $\hat{k}, \hat{q}$: photon- and meson momentum,
- $\hat{\epsilon}$: γ-polarization,
- $\hat{\boldsymbol{\sigma}} = (\sigma_x, \sigma_y, \sigma_z)^T$: Pauli-matrices,
- W: CMS-energy,
- θ: CMS scattering-angle.

Polarization observables

<u>Generic case:</u>
Observables defined as *bilinear forms*:

$\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N_{\mathcal{A}}} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j$,

for $\alpha = 1, \ldots, N_{\mathcal{A}}^{2}$.
($\boldsymbol{c}^{\alpha}$: normaliz. factors)

*) The $\tilde{\Gamma}^{\alpha}$ are a set of complete, hermitean and unitary complex $N_{\mathcal{A}} \times N_{\mathcal{A}}$-matrices ('Clifford algebra')

*) The $\tilde{\Gamma}^{\alpha}$ can be decomposed into classes according to their *shape*

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Photoproduction observable	Class
$\sigma_0 = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$-\check{\Sigma} = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$	$\mathcal{S}$
$-\check{T} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$	
$\check{P} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\mathcal{O}^{a}_{1+} = \lvert b_1\rvert \lvert b_3\rvert \sin\phi_{13} + \lvert b_2\rvert \lvert b_4\rvert \sin\phi_{24} = \mathrm{Im}\left[b_3^{*} b_1 + b_4^{*} b_2\right] = -\check{G}$	
$\mathcal{O}^{a}_{1-} = \lvert b_1\rvert \lvert b_3\rvert \sin\phi_{13} - \lvert b_2\rvert \lvert b_4\rvert \sin\phi_{24} = \mathrm{Im}\left[b_3^{*} b_1 - b_4^{*} b_2\right] = \check{F}$	$a = \mathcal{BT}$
$\mathcal{O}^{a}_{2+} = \lvert b_1\rvert \lvert b_3\rvert \cos\phi_{13} + \lvert b_2\rvert \lvert b_4\rvert \cos\phi_{24} = \mathrm{Re}\left[b_3^{*} b_1 + b_4^{*} b_2\right] = -\check{E}$	
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$\mathcal{O}^{c}_{1+} = \lvert b_1\rvert \lvert b_2\rvert \sin\phi_{12} + \lvert b_3\rvert \lvert b_4\rvert \sin\phi_{34} = \mathrm{Im}\left[b_2^{*} b_1 + b_4^{*} b_3\right] = -\check{L}_{x'}$	
$\mathcal{O}^{c}_{1-} = \lvert b_1\rvert \lvert b_2\rvert \sin\phi_{12} - \lvert b_3\rvert \lvert b_4\rvert \sin\phi_{34} = \mathrm{Im}\left[b_2^{*} b_1 - b_4^{*} b_3\right] = -\check{T}_{z'}$	$c = \mathcal{TR}$
$\mathcal{O}^{c}_{2+} = \lvert b_1\rvert \lvert b_2\rvert \cos\phi_{12} + \lvert b_3\rvert \lvert b_4\rvert \cos\phi_{34} = \mathrm{Re}\left[b_2^{*} b_1 + b_4^{*} b_3\right] = -\check{L}_{z'}$	
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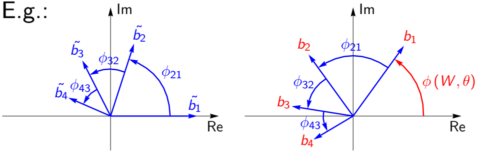
The complete-experiment analysis (CEA)

CEA: What are the minimal subsets of the observables $\mathcal{O}^{\alpha}$, which allow for the unique extraction of the amplitudes b_i up to one unknown overall phase $\phi\left(\Omega_2^{n_f}\right)$?

*) Analysis operates on each 'bin' in $\Omega_2^{n_f}$ individually.

*) Disregard measurement uncertainty!

E.g.:



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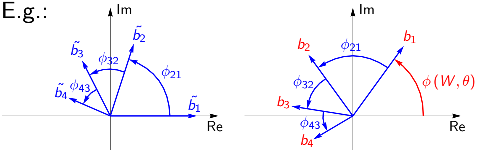

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- ∗) Disregard measurement uncertainty!
- ∗) Initial standard assumption: the moduli $|b_1|, |b_2|, \ldots, |b_N|$ are already known from a certain subset of $N_{\mathcal{A}}$ 'diagonal' observables.
  $\Rightarrow$ Have to determine a minimal set of relative phases $\phi_{ij} := \phi_i - \phi_j$ $\left(b_j = |b_j|\, e^{i\phi_j}\right)$

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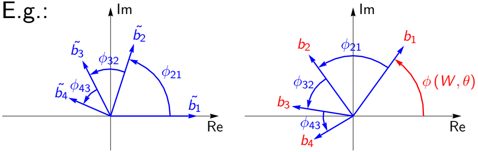

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*) From 'heuristic' arguments: complete sets have minimal length of $2N_{\mathcal{A}}$ observables.
$\hookrightarrow$ We know how many observables we have to select. But which ones?

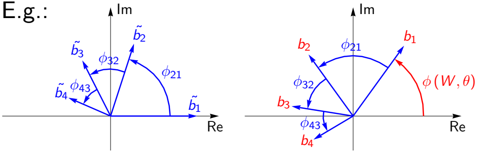
The complete-experiment analysis (CEA)

CEA: What are the minimal subsets of the observables $\mathcal{O}^{\alpha}$, which allow for the unique extraction of the amplitudes b_i up to one unknown overall phase $\phi\left(\Omega_2^{n_f}\right)$?

∗) Analysis operates on each 'bin' in $\Omega_2^{n_f}$ individually.

∗) Disregard measurement uncertainty!

E.g.:



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∗) Simplest (formal) solution: $\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}^{\alpha}_{ij} b_j$ can be 'inverted' (using the *completeness* of the $\tilde{\Gamma}$-matrices):

$$b_i^{*} b_j = \frac{1}{\tilde{N}} \sum_{\alpha=1}^{N_{\mathcal{A}}^2} \left(\tilde{\Gamma}^{\alpha}_{ij}\right)^{*} \left(\frac{\mathcal{O}^{\alpha}}{\boldsymbol{c}^{\alpha}}\right) .$$

$\Rightarrow$ Obtain moduli from $|b_i| = \sqrt{b_i^{*} b_i}$ and rel.-phases from a 'minimal' set of $b_i^{*} b_j$
$\Rightarrow$ Obtain (quite large) over-complete set $\{\mathcal{O}^{\alpha}\}$ determined via the RHS

Solution: Moravcsik's Theorem

From [YW, P. Kroenert, F. Afzal, A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020)], based on [Moravcsik, J. Math. Phys. **26**, 211 (1985).]:

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'Geometrical (graphical) analog': Represent every amplitude $b_1, \ldots, b_N$ by a *point* and every product $b_j^* b_i$, or rel.-phase ϕ_{ij}, by a *line connecting points* 'i' and 'j'.

Furthermore: $\hookrightarrow$ Represent every $\mathrm{Re}\left[b_i^* b_j\right] \propto \cos\phi_{ij}$ by a *solid line*,

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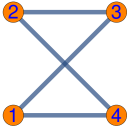

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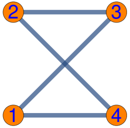

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(ii) the graph has to have an *odd* number of dashed lines, as well as *any* number of solid lines:

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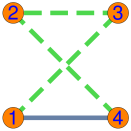
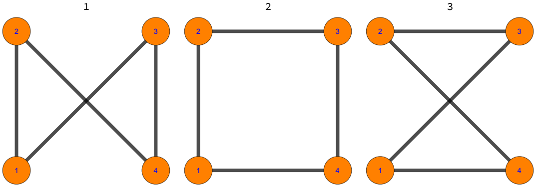


is complete ✓

For $N_{\mathcal{A}} = 4$, one gets $\frac{(N_{\mathcal{A}}-1)!}{2} = \frac{3!}{2} = 3$ possible graph-topologies :



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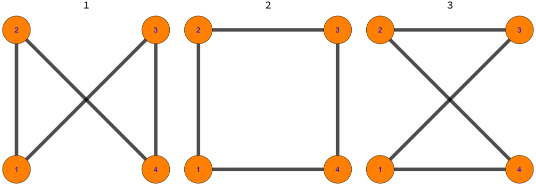

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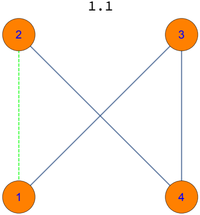

$\longrightarrow \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{34}, \cos\phi_{13}\}$

$\Rightarrow$ Map relative-phases to observables:

$$|b_1|\,|b_2|\sin\phi_{12} = (1/2)\left[-\check{L}_{x'} - \check{T}_{z'}\right],\ |b_2|\,|b_4|\cos\phi_{24} = (1/2)\left[-\check{E} - \check{H}\right],$$
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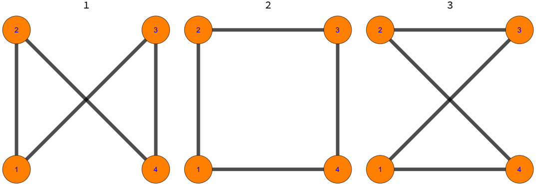

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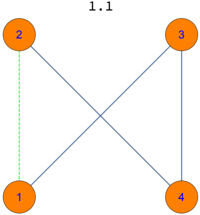

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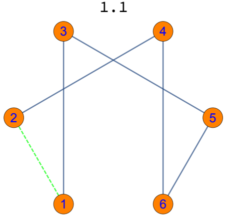
$\Rightarrow$ For $\underline{N_{\mathcal{A}} = 4}$: obtain 12 Moravcsik-complete sets of length $10 > 2N_{\mathcal{A}} = 8$.

Moravcsik's Th. applied to electroproduction ($N_{\mathcal{A}} = 6$)

$*$) From the first of 60 topologies, construct example (1.1) (fully complete):



$\longrightarrow \quad \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{46}, \cos\phi_{56}, \cos\phi_{35}, \cos\phi_{13}\}$

$\hookrightarrow$ In the same way as before, extract the 'Moravcsik-complete' set (combined with 'diagonal' obs.'s $\left\{ R_{T}^{00}, \ {}^{c}R_{TT}^{00}, R_{T}^{0y}, R_{T}^{y'0}, R_{L}^{00}, R_{L}^{0y} \right\}$):

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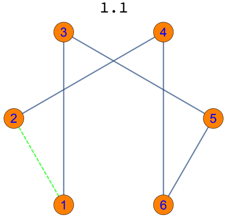

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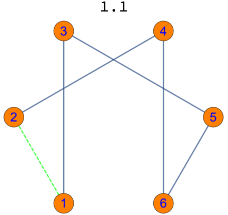

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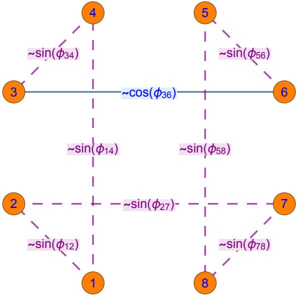
$\hookrightarrow$ What about problems with large numbers of amplitudes (i.e. $N_{\mathcal{A}} > 6$)?

Cases with larger numbers of $N_{\mathcal{A}} > 6$ amplitudes

Two-meson photoproduction

*) 8 amplitudes vs. 64 observables

*) Typical complete graph:



[P. Kroenert et al. (2021)]

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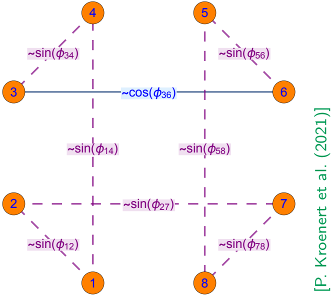

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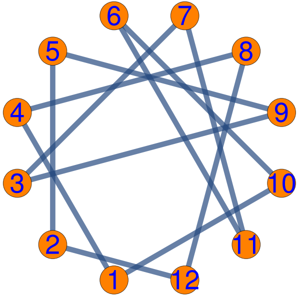

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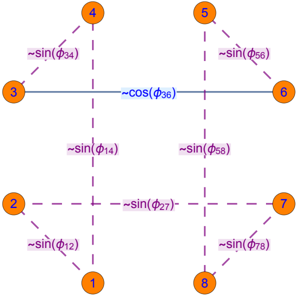

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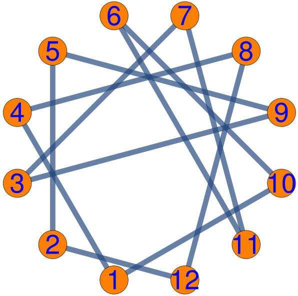

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Observation: Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N_{\mathcal{A}}$ observables, for problems with $N_{\mathcal{A}} \geq 4$ amplitudes

Reductions obtained using new 'directional' graphs

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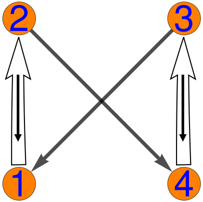

$\Leftrightarrow$ complete photoproduction-set ($2N = 8$ obs.'s in combination with 4 'diagonal' obs.'s $\left\{ \sigma_0, \check{\Sigma}, \check{T}, \check{P} \right\}$):

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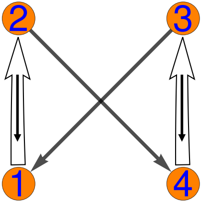

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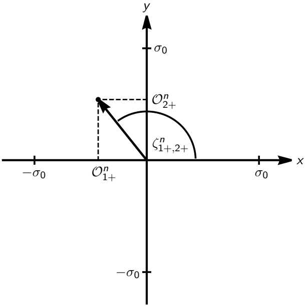

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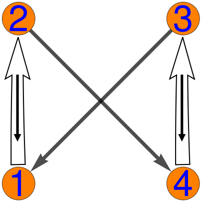

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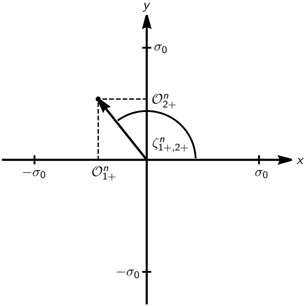

$\hookrightarrow$ Confirm min. sets of $2N_{\mathcal{A}} = 8$ for photoproduction ;
Obtain new sets of $2N_{\mathcal{A}} = 12$ for e^--production!
For $N_{\mathcal{A}} > 6$, sets still (slightly) over-complete ...

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We have new *standard-approaches* in place to determine complete experiments, and thus $|b_i|$ and ϕ_{ij}, for reactions with any number of amplitudes $N_{\mathcal{A}}$!

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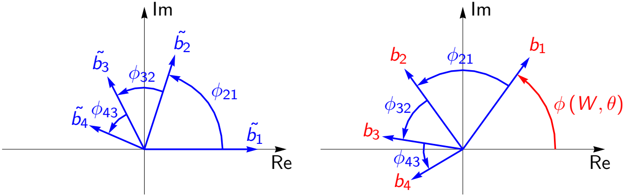
However: the unknown phase $\phi\left(\Omega_2^{n_f}\right)$ remains a problem, because:



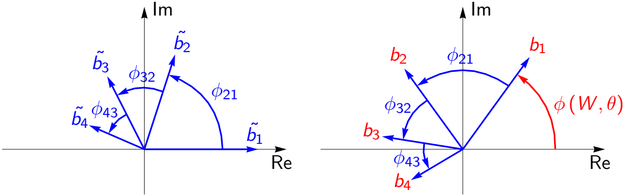
$$\mathcal{T}_{fi}\left(\Omega_2^{n_f}\right) = \chi_B^{\dagger}\left[\sum_{k=1}^{N_{\mathcal{A}}} \boldsymbol{\kappa}_k b_k\left(\Omega_2^{n_f}\right)\right]\chi_N = \chi_B^{\dagger}\left[\sum_{k=1}^{N_{\mathcal{A}}} \boldsymbol{\kappa}_k e^{i\phi_{fi}\left(\Omega_2^{n_f}\right)}\tilde{b}_k\left(\Omega_2^{n_f}\right)\right]\chi_N$$

$$= e^{i\phi_{fi}\left(\Omega_2^{n_f}\right)}\chi_B^{\dagger}\left[\sum_{k=1}^{N_{\mathcal{A}}} \boldsymbol{\kappa}_k \tilde{b}_k\left(\Omega_2^{n_f}\right)\right]\chi_N = e^{i\phi_{fi}\left(\Omega_2^{n_f}\right)}\tilde{\mathcal{T}}_{fi}\left(\Omega_2^{n_f}\right),$$

$\Rightarrow$ Complete experiment fixes only 'internal spin-structure' $\tilde{\mathcal{T}}_{fi}$!

Physical significance of the overall phase

We have new *standard-approaches* in place to determine complete experiments, and thus $|b_i|$ and ϕ_{ij}, for reactions with <u>any</u> number of amplitudes $N_{\mathcal{A}}$!
<u>However:</u> the unknown phase $\phi\left(\Omega_2^{n_f}\right)$ remains a problem, because:



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$\Rightarrow$ Complete experiment fixes only 'internal spin-structure' $\tilde{\mathcal{T}}_{fi}$!
The phase has *physical significance*, because:

- (Perfect) knowledge of the phase $\Rightarrow$ project-out partial waves to all orders!
- *Analytically continue* part. waves away from physical region $\Rightarrow$ *unique* resonance-poles!

Measuring the overall phase

Measure the overall phase in scattering experiments using vortex beams
$\equiv$ beams of particles with intrinsic orbital angular momentum $\langle L_z \rangle = \hbar \ell$ along the axis of propagation (i.e. z-axis) cf. [Ivanov, Phys. Rev. D **85**, 076001 (2012)], [Ivanov, arXiv:2205.00412 [hep-ph] (2022)]

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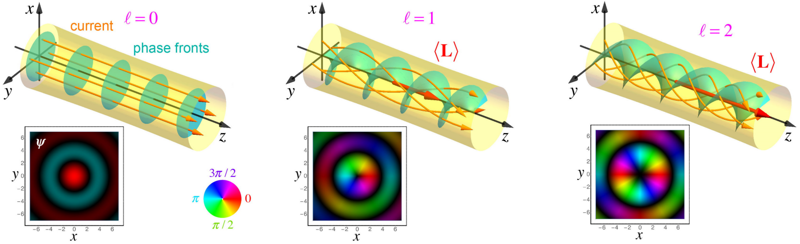

[arXiv:2205.00412, Fig.1]

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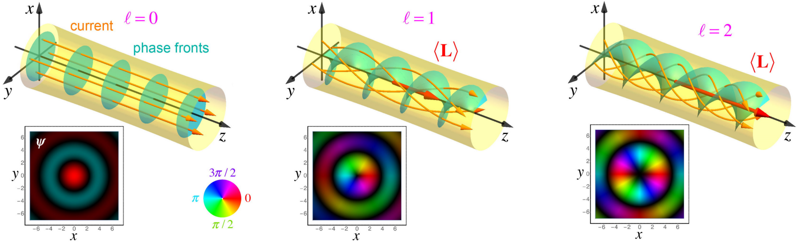

[arXiv:2205.00412, Fig.1]

Proposal [Ivanov, Phys. Rev. D **85**, 076001 (2012)]: for the example $\gamma p \to \pi p$, consider double-twisted γp-collision (i.e. both γ and p are in a [Bessel] vortex-state)

$\Rightarrow$ Measure *azimuthal asymmetry* $A = \frac{\Delta\sigma}{\sigma}$ ('sine-weighted' c.s. $\Delta\sigma$; non-weighted σ)

$\Rightarrow$ Then, one has $\boxed{A = \frac{d\phi\left(\theta_{\gamma\pi}^{\mathrm{LAB}}\right)}{d\theta_{\gamma\pi}^{\mathrm{LAB}}} \cdot P}$, with an 'analyzing power' P.

$\Rightarrow$ For insanely good accuracy and statistics, integration yields: $\underline{\phi\left(\theta_{\gamma\pi}^{\mathrm{LAB}}\right) + \mathcal{C}}$.

Vortex-beams at the GeV-scale maybe feasible within 10-20 years [Ivanov, priv. comm. (2022)]

Unitarity can constrain the overall phase

Consider case of elastic $2 \rightarrow 2$-scattering of spinless particles and solve the 'CEA':

$$\sigma_0(W,\theta) = \left|\mathcal{T}(W,\theta)\right|^2 \Leftrightarrow \left|\mathcal{T}(W,\theta)\right| = +\sqrt{\sigma_0(W,\theta)}.$$

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$$\sqrt{\sigma_0 \left(\mathrm{c}_{fi}\right)} \sin\left[\phi\left(\mathrm{c}_{fi}\right)\right] = \frac{\tau}{4\pi} \int d\Omega_{\mathrm{i'}} \sqrt{\sigma_0\left(\mathrm{c}_{fi'}\right)} \sqrt{\sigma_0\left(\mathrm{c}_{i'i}\right)} \cos\left[\phi\left(\mathrm{c}_{fi'}\right) - \phi\left(\mathrm{c}_{i'i}\right)\right], \ \{\mathrm{c}_{fi} \equiv \cos\theta_{fi}, \ldots\}.$$

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$$\sqrt{\sigma_0\left(\mathrm{c}_{fi}\right)}\underbrace{\sin\left[\phi\left(\mathrm{c}_{fi}\right)\right]}_{\text{linear in } e^{i\phi}} = \frac{\tau}{4\pi}\int d\Omega_{\mathrm{i'}}\sqrt{\sigma_0\left(\mathrm{c}_{fi'}\right)}\sqrt{\sigma_0\left(\mathrm{c}_{i'i}\right)}\underbrace{\cos\left[\phi\left(\mathrm{c}_{fi'}\right) - \phi\left(\mathrm{c}_{i'i}\right)\right]}_{\text{'quadratic' in } e^{i\phi}},\ \left\{\mathrm{c}_{fi} \equiv \cos\theta_{fi}, \ldots\right\}.$$

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$*$) One can show [A. Martin, Nuovo Cim. A 59, 131 (1969)]: the solution for $\phi(\cos\theta)$ is ***unique*** iff. the amplitude satisfies the 'Martin-bound'

$$\boxed{\sin\phi\left(c_{fi}\right) < \sin\mu := \max\left(\int \frac{d\Omega_{i'}}{4\pi}\frac{\left|\mathcal{T}\left(c_{fi'}\right)\right|\left|\mathcal{T}\left(c_{i'i}\right)\right|}{\left|\mathcal{T}\left(c_{fi}\right)\right|}\right) < 0.79}.$$

Uniqueness theorems under certain *bounds* have also been found for:

- 2 coupled scalar 2-body channels [R. F. Alvarez-Estrada et al., Nucl. Phys. B **88**, 289 (1975)]
- elastic πN-scattering [R. F. Alvarez-Estrada et al., Nucl. Phys. B **54**, 237 (1973)]
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$*$) In case the Martin-bound is violated, there exists only *one discrete ambiguity*, the <u>'Crichton-ambiguity' $\left\{\phi^{\mathrm{I}}(c_{fi}), \phi^{\mathrm{II}}(c_{fi})\right\}$</u> [J. H. Crichton, Nuovo Cim. A **45**, 256 (1966)]

[A. Martin and J. M. Richard, Phys. Rev. D **101**, no.9, 094014 (2020)]

Finale: the 'coupled-channels complete experiment'

Consider *channel-space* $\{|\pi N\rangle, |\gamma N\rangle, |\pi\pi N\rangle\}$, i.e.:

$$(\mathcal{T}_{fi}) = \left[\begin{array}{ccc} \mathcal{T}_{\pi N,\pi N} & \mathcal{T}_{\pi N,\gamma N} & \mathcal{T}_{\pi N,\pi\pi N} \\ \mathcal{T}_{\gamma N,\pi N} & \mathcal{T}_{\gamma N,\gamma N} \simeq 0 & \mathcal{T}_{\gamma N,\pi\pi N} \\ \mathcal{T}_{\pi\pi N,\pi N} & \mathcal{T}_{\pi\pi N,\gamma N} & \mathcal{T}_{\pi\pi N,\pi\pi N} \end{array} \right].$$

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$\hookrightarrow$ Measure individual complete experiments with perfect *phase-space coverage and overlap* among individual reactions (complete exp.'s determinable using *graphs*):

Reaction	Example complete experiment (yields $\lvert b_i \rvert$ & ϕ_{ij})
$\pi N \rightarrow \pi N$ ($N_{\mathcal{A}} = 2$)	$\sigma_0, \hat{P}, \hat{R}, \hat{A}$
$\pi N \rightarrow \pi\pi N$ ($N_{\mathcal{A}} = 4$)	$\sigma_0, \check{P}_y, \check{P}_z, \check{P}_{x'}, \check{P}_{y'}, \check{\mathcal{O}}_{yy'}, \check{\mathcal{O}}_{zy'}, \check{\mathcal{O}}_{yz'}$
$\gamma N \rightarrow \pi N$ ($N_{\mathcal{A}} = 4$)	$\sigma_0, \check{\Sigma}, \check{T}, \check{P}, \check{E}, \check{H}, \check{L}_{x'}, \check{T}_{x'}$
$\gamma N \rightarrow \pi\pi N$ ($N_{\mathcal{A}} = 8$)	$\sigma_0, \check{P}_y, \check{P}_{y'}, \check{\mathcal{O}}^{\odot}_{yy'}, \check{\mathcal{O}}_{yy'}, \check{P}^{\odot}_{y'}, \check{P}^{\odot}_{y}, I^{\odot}, \check{P}_x, \check{P}_z, \check{P}_{x'}, \check{P}^{s}_{x}, \check{P}^{\odot}_{x}, \check{P}^{c}_{z}, \check{P}^{\odot}_{z}, \check{P}^{\odot}_{x'}$

$\Rightarrow$ For these 4 reactions, we have $\mathcal{T}_{fi} = e^{i\phi_{fi}} \tilde{\mathcal{T}}_{fi}$, with $\tilde{\mathcal{T}}_{fi}$ fixed.

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$\Rightarrow$ Missing phase-information $e^{i\phi_{fi}}$ fixed and resonance-spectrum (hopefully) unique!

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Issues: - Can we assume perfect time-reversal inv., to relate $3 \rightarrow 2$ to $2 \rightarrow 3$ processes?

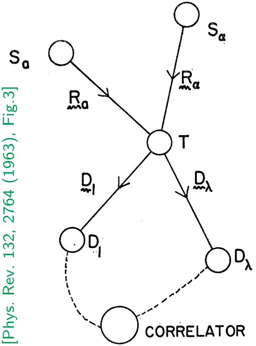
- $3 \rightarrow 3$-process $\pi\pi N \rightarrow \pi\pi N$ unmeasurable. Does this hurt the proposal?

Thank You!

Additional Slides

The Hanbury-Brown and Twiss experiment

Measure the overall phase via intensity correlations in a *Hanbury-Brown and Twiss-type* experiment [Goldberger, Lewis & Watson, Phys. Rev. 132, 2764 (1963)]



[Phys. Rev. 132, 2764 (1963), Fig.3]

- $*$) Two sources, S_a and S_α, emitting beam-particles
- $*$) One single irradiated target T
- $*$) Two spatially separated detectors, D_l and D_λ
- $*$) A CORRELATOR, which registers only in case D_l and D_λ count in *coincidence*

$\hookrightarrow$ The correlator counting-rate contains an isolatable term, which is proportional to:

$$\mathrm{Re}\,[\mathcal{T}_{\lambda\leftarrow\alpha}\mathcal{T}^*_{l\leftarrow\alpha}\mathcal{T}_{l\leftarrow a}\mathcal{T}^*_{\lambda\leftarrow a}].$$

$*$) Assuming $|\mathcal{T}_{f\leftarrow i}|$ as known, one can measure $\cos(\Gamma)$, where:

$$\Gamma := \phi(\boldsymbol{g}_{la}) - \phi(\boldsymbol{g}_{l\alpha}) + \phi(\boldsymbol{g}_{\lambda\alpha}) - \phi(\boldsymbol{g}_{\lambda a}),\ \text{with } \boldsymbol{g}_{la} \equiv \hat{R}_a - \hat{D}_l,\ \ldots.$$

$*$) *Varying positions* of detectors and sources, do measurements for *many* angles:

$$\left\{\boldsymbol{g}^{(1)}_{la}, \boldsymbol{g}^{(2)}_{la}, \ldots\right\},\ \left\{\boldsymbol{g}^{(1)}_{l\alpha}, \boldsymbol{g}^{(2)}_{l\alpha}, \ldots\right\},\ \left\{\boldsymbol{g}^{(1)}_{\lambda\alpha}, \boldsymbol{g}^{(2)}_{\lambda\alpha}, \ldots\right\},\ \left\{\boldsymbol{g}^{(1)}_{\lambda a}, \boldsymbol{g}^{(2)}_{\lambda a}, \ldots\right\}.$$

$\hookrightarrow$ Extract: $\phi\left(g^{(\nu+1)}_{la}\right) - \phi\left(g^{(\nu)}_{la}\right) \equiv \delta\phi(\nu) \longrightarrow$ Overall phase: $\phi(\boldsymbol{g})$.