

Amplitude analysis and complete experiments for light baryon spectroscopy

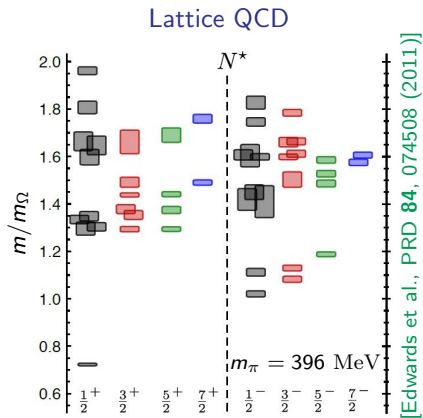
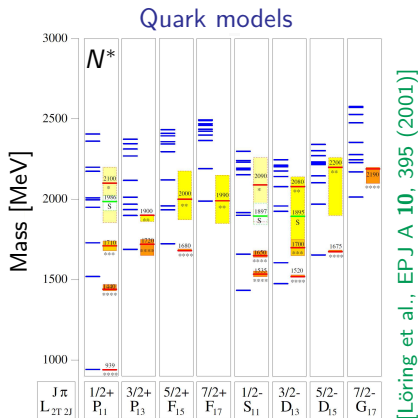
Yannick Wunderlich

HISKP, University of Bonn

July 12, 2022



Why baryon spectroscopy?

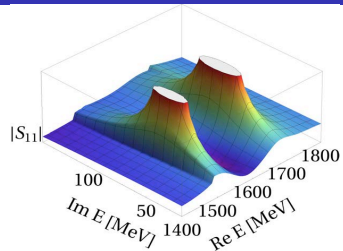


Further study baryon spectroscopy in order to:

- Solve the *missing resonance problem*
- Extract a precision-spectrum as a better basis for comparison for non-perturb. theoretical calculations (quark models, lattice QCD, $U_\chi\text{PT}$, ...)
- (light) baryon spectrum is valuable input for modelling *final-state interactions*

Energy-dependent (ED) amplitude-analysis models

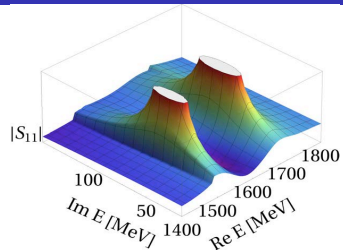
- *) Reaction-theoretic models using *S-Matrix constraints* (analyticity, unitarity, crossing), often fit multiple reactions
- *) Analytically continue (partial-wave) amplitudes to the resonance-poles in the complex energy-plane: 'pole-positions' $\Rightarrow (M^*, \Gamma)$; 'residues' $\Rightarrow$ branching fractions.



[Fig. courtesy of M. Döring]

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Model	Type	Unitarity	non-resonant contrib.'s
Bonn-Gatchina JüBoW	<i>K</i> -Matrix dynamical coupled-channels	2body 2body & approx. 3body	Reggion-exchanges effective lagrangians
ANL-Osaka	dynamical coupled-channels	2body & approx. 3body	effective lagrangians
SAID	C.M. <i>K</i> -Matrix	2body	phenomenological parametrizations
MAID	'Breit-Wigner + background'	2body (Watson-Theorem)	effective lagrangians & Reggion exchanges
Kent State University	<i>K</i> -Matrix	2body	phenomenological parametrizations

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Procedure:

- ▶ ED amplitude analysis groups perform and publish fits of large collections of datasets on (polarization) measurements (more on these later ...)
- ▶ The PDG Baryon-group filters these publications and assigns *star-ratings* to states:
 - { *** or **: established states,
 - { ** or *: controversial/claimed states; need confirmation ...
- ▶ Community uses listed data for further analyses ...

Progress in terms of '*' -ratings

PDG 2002 vs. PDG 2020 (changes in red): [A. Thiel, F. Afzal & YW, PPNP 125, 103949 (2022), Tab. 16]

Particle	J^P	overall	PWA	$N\gamma$	$N\pi$	$\Delta\pi$	$N\sigma$	$N\eta$	ΛK	ΣK	$N\rho$	$N\omega$
N	$1/2^+$	****										
$N(1440)$	$1/2^+$	****	○ ◇ g^* ▷	****	****	****	***	-			-	
$N(1520)$	$3/2^-$	****	○ ◇ ★ ▷	****	****	****	**	****			- - - -	
$N(1535)$	$1/2^-$	****	○ ◇ ★ ▷	****	****	***	*	****			- -	
$N(1650)$	$1/2^-$	****	○ ◇ ★ ▷	****	****	***	*	****	* _ -	- -	- -	
$N(1675)$	$5/2^-$	****	○ ◇ ★ ▷	****	****	****	***	*	*	*	-	
$N(1680)$	$5/2^+$	****	○ ◇ ★ ▷	****	****	****	***	*	*	*	- - - -	
$N(1700)$	$3/2^-$	***	○ ▷	**	***	**	*	*	- -	-	-	
$N(1710)$	$1/2^+$	***	○ ◇ ▷	****	****	* _		***	**	*	*	*
$N(1720)$	$3/2^+$	****	○ ◇ ★ ▷	****	****	***	*	*	****	*	* _	*
$N(1860)$	$5/2^+$	**	▷	*	**		*	*				
$N(1875)$	$3/2^-$	***	○ ▷	**	**	*	**	*	*	*	*	*
$N(1880)$	$1/2^+$	**	○ ▷	**	*	**	*	*	**	**		**
$N(1895)$	$1/2^-$	****	○ ▷	****	*	*	*	****	**	**	*	*
$N(1900)$	$3/2^+$	****	○ ◇ ▷	****	**	**	*	*	**	**	-	*
$N(1990)$	$7/2^+$	**	○ ◇ ▷	**	**			*	*	*		
$N(2000)$	$5/2^+$	**	○ ★	**	* _	**	*	*	-	-	- -	*
$N(2040)$	$3/2^+$	*	▷		*							
$N(2060)$	$5/2^-$	***	○ ◇ g^* ▷	***	**	*	*	*	*	*	*	*
$N(2100)$	$1/2^+$	**	○ ▷	**	***	**	**	*	*		*	*
$N(2120)$	$3/2^-$	***	○ ▷	***	**	**	**		**	*		*
$N(2190)$	$7/2^-$	****	○ ◇ ★ ▷	****	****	****	**	*	**	*	*	*
$N(2220)$	$9/2^+$	****	○ ◇ ★	**	****			*	*	*		
$N(2250)$	$9/2^-$	****	○ ◇ ★ ▷	**	****			*	*	*		
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

○: BnGa-2019, ◇: JüBo-2017 ('g' for dynamically generated), ★: SAID-MA19, ▷: KSU PWA

Progress in terms of 'multiplets'

↪ Work with spin-flavor $SU(6)$ symmetry and assign quarks to fundamental rep.

$$\mathbf{6} = (u \uparrow, d \uparrow, s \uparrow, u \downarrow, d \downarrow, s \downarrow)^T.$$

⇒ Baryon (super-) multiplet structure arises from decomposition into irrep.'s:

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_M \oplus \mathbf{70}_M \oplus \mathbf{20}_A.$$

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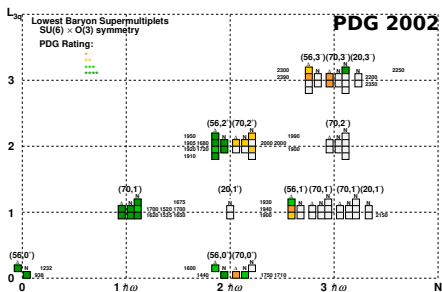
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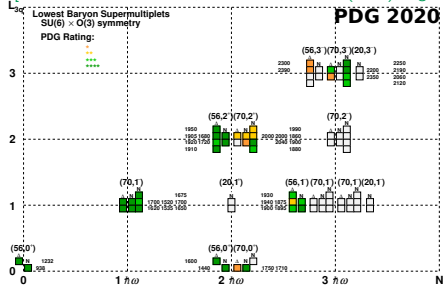
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[A. Thiel, F. Afzal & YW, PPNP 125, 103949 (2022), Figs 50,51]



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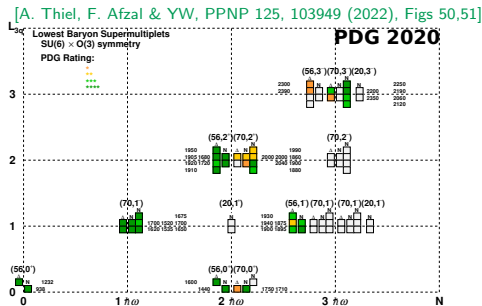
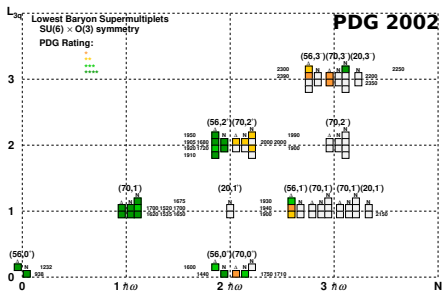
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All this progress is great, but is there a way to tell when we are finished?

↪ There is: one needs a coupled-channels complete experiment.

Spin amplitudes

Generic case:

Photoproduction:

*) General meson-production reaction:

$$\mathcal{P}N \rightarrow \{\varphi_i\} B, \text{ with:}$$

- $\mathcal{P}$: 'probe'-particle ($\pi, \gamma, \gamma^*, \dots$),
- N : target-nucleon,
- $\{\varphi_i\}$: one or multiple meson(s),
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*) One can always expand:

$$\mathcal{T}_{fi} = \chi_B^\dagger \left[\sum_{k=1}^{N_A} \kappa_k b_k (\Omega_2^{n_f}) \right] \chi_N,$$

- $\kappa_i (\{p_j\}, \{\sigma_i\})$: spin-kinem. operators,
- $b_i (\Omega_2^{n_f})$: spin ('transversity') ampl.'s,
- $\Omega_2^{n_f}$: phase-space for $2 \rightarrow n_f$ -reaction.

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*) The number of amplitudes N_A is determined from *spin-multiplicities*:

$$N_A = \mathbf{n}_{\mathcal{P}} \mathbf{n}_N \mathbf{n}_{\varphi_1} \dots \mathbf{n}_{\varphi_{\bar{n}}} \mathbf{n}_B,$$

with additional factor of (1/2) in case of a $2 \rightarrow 2$ reaction (parity!)

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Photoproduction:

- *) Produce one pseudoscalar meson φ :

$$\gamma N \rightarrow \varphi B.$$

- *) Then: $N_A = \frac{1}{2} (\mathbf{n}_\gamma \mathbf{n}_N \mathbf{n}_\varphi \mathbf{n}_B)$
 $= \frac{1}{2} (2 \times 2 \times 1 \times 2) = \underline{\underline{4}}.$

- *) We have:

$$\mathcal{T}_{fi} = \chi_B^\dagger [\kappa_1 b_1 + \kappa_2 b_2 + \kappa_3 b_3 + \kappa_4 b_4] \chi_N, \text{ where:}$$

- $b_i = b_i(W, \theta)$: transversity amplitudes.

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- *) $\{\kappa_1, \dots, \kappa_4\}$ are complicated, e.g.:

$$\kappa_1 = \frac{(\hat{q} \cdot \hat{\epsilon})}{\sqrt{2} \sin^2 \theta} \left[e^{-i\frac{\theta}{2}} (\hat{k} \cdot \hat{\sigma}) - e^{i\frac{\theta}{2}} (\hat{q} \cdot \hat{\sigma}) \right]$$

$\kappa_2 = \dots$, where:

- $\hat{k}, \hat{q}$: photon- and meson momentum,
- $\hat{\epsilon}$: γ -polarization,
- $\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$: Pauli-matrices,
- W : CMS-energy,
- θ : CMS scattering-angle.

Polarization observables

Generic case:

Observables defined as
bilinear forms:

$$\mathcal{O}^\alpha = \mathbf{c}^\alpha \sum_{i,j=1}^{N_{\mathcal{A}}} b_i^* \tilde{\Gamma}_{ij}^\alpha b_j,$$

for $\alpha = 1, \dots, N_{\mathcal{A}}^2$.

($\mathbf{c}^\alpha$: normaliz. factors)

- *) The $\tilde{\Gamma}^\alpha$ are a set of complete, hermitean and unitary complex $N_{\mathcal{A}} \times N_{\mathcal{A}}$ -matrices ('Clifford algebra')
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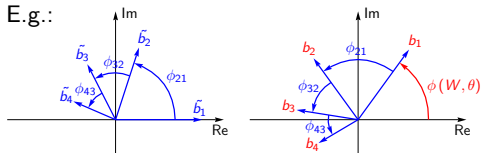
Photoproduction observable	Class
$\sigma_0 = \frac{1}{2} (b_1 ^2 + b_2 ^2 + b_3 ^2 + b_4 ^2)$ $-\check{\Sigma} = \frac{1}{2} (b_1 ^2 + b_2 ^2 - b_3 ^2 - b_4 ^2)$ $-\check{T} = \frac{1}{2} (- b_1 ^2 + b_2 ^2 + b_3 ^2 - b_4 ^2)$ $\check{P} = \frac{1}{2} (- b_1 ^2 + b_2 ^2 - b_3 ^2 + b_4 ^2)$	S
$\mathcal{O}_{1+}^a = b_1 b_3 \sin \phi_{13} + b_2 b_4 \sin \phi_{24} = \text{Im} [b_3^* b_1 + b_4^* b_2] = -\check{G}$ $\mathcal{O}_{1-}^a = b_1 b_3 \sin \phi_{13} - b_2 b_4 \sin \phi_{24} = \text{Im} [b_3^* b_1 - b_4^* b_2] = \check{F}$ $\mathcal{O}_{2+}^a = b_1 b_3 \cos \phi_{13} + b_2 b_4 \cos \phi_{24} = \text{Re} [b_3^* b_1 + b_4^* b_2] = -\check{E}$ $\mathcal{O}_{2-}^a = b_1 b_3 \cos \phi_{13} - b_2 b_4 \cos \phi_{24} = \text{Re} [b_3^* b_1 - b_4^* b_2] = \check{H}$	$a = \mathcal{B}\mathcal{T}$
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The complete-experiment analysis (CEA)

CEA: What are the minimal subsets of the observables $\mathcal{O}^\alpha$, which allow for the unique extraction of the amplitudes b_i up to one unknown overall phase $\phi(\Omega_2^{n_f})$?

- *) Analysis operates on each 'bin' in $\Omega_2^{n_f}$ individually.
- *) Disregard measurement uncertainty!

E.g.:

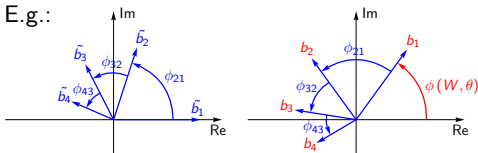


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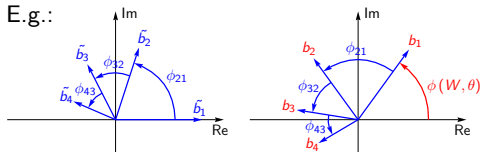


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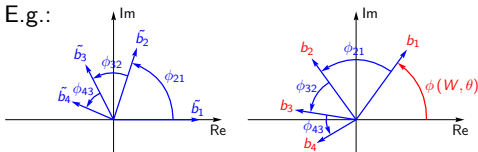
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- *) Simplest (formal) solution: $\mathcal{O}^\alpha = \mathbf{c}^\alpha \sum_{i,j=1}^N b_i^* \tilde{\Gamma}_{ij}^\alpha b_j$ can be 'inverted' (using the *completeness* of the $\tilde{\Gamma}$ -matrices):

$$b_i^* b_j = \frac{1}{N} \sum_{\alpha=1}^{N_A^2} \left(\tilde{\Gamma}_{ij}^\alpha \right)^* \left(\frac{\mathcal{O}^\alpha}{\mathbf{c}^\alpha} \right).$$
 $\Rightarrow$ Obtain moduli from $|b_i| = \sqrt{b_i^* b_i}$ and rel.-phases from a 'minimal' set of $b_i^* b_j$
 $\Rightarrow$ Obtain (quite large) over-complete set $\{\mathcal{O}^\alpha\}$ determined via the RHS

Solution: Moravcsik's Theorem

From [YW, P. Kroenert, F. Afzal, A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020)],
based on [Moravcsik, J. Math. Phys. **26**, 211 (1985).]:

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'Geometrical (graphical) analog': Represent every amplitude $b_1, \dots, b_N$ by a *point*
and every product $b_j^* b_i$, or rel.-phase ϕ_{ij} , by a *line connecting points 'i' and 'j'*.
Furthermore: $\hookrightarrow$ Represent every $\text{Re}[b_i^* b_j] \propto \cos \phi_{ij}$ by a *solid line*,
 $\hookrightarrow$ Represent every $\text{Im}[b_i^* b_j] \propto \sin \phi_{ij}$ by a *dashed line*.

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'Geometrical (graphical) analog': Represent every amplitude $b_1, \dots, b_N$ by a *point*
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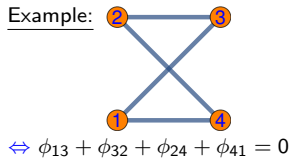
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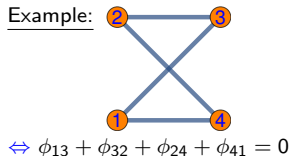
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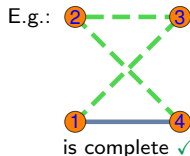
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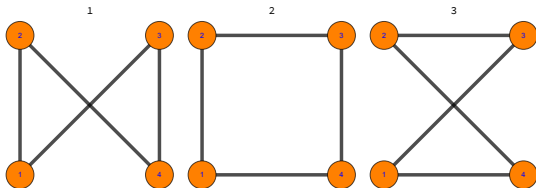
(ii) the graph has to have an *odd* number of dashed lines,
as well as *any* number of solid lines:

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Moravcsik's Th. applied to photoproduction ($N_A = 4$)

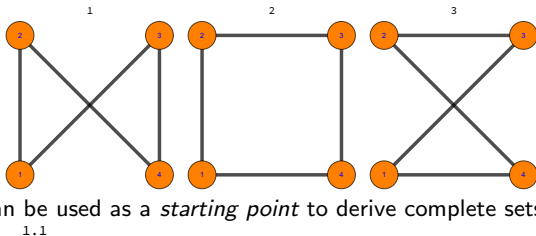
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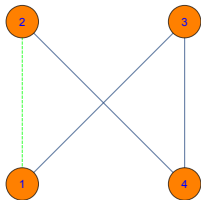
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e.g., example 1.1:
 (fully complete)



→ $\{\sin \phi_{12}, \cos \phi_{24}, \cos \phi_{34}, \cos \phi_{13}\}$

⇒ Map relative-phases to observables:

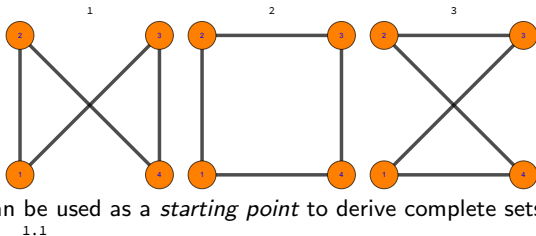
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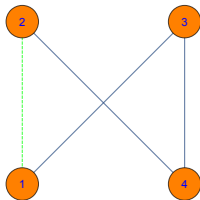
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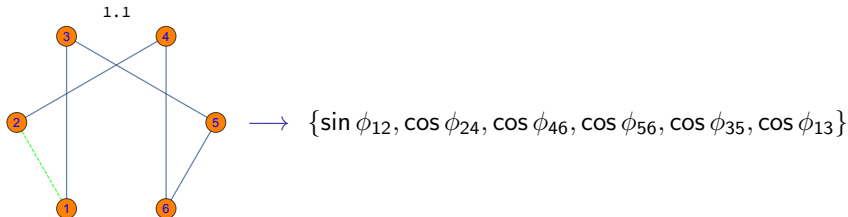
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⇒ For $N_A = 4$: obtain 12 Moravcsik-complete sets of length $10 > 2N_A = 8$.

Moravcsik's Th. applied to electroproduction ($N_A = 6$)

*) From the first of 60 topologies, construct example (1.1) (fully complete):



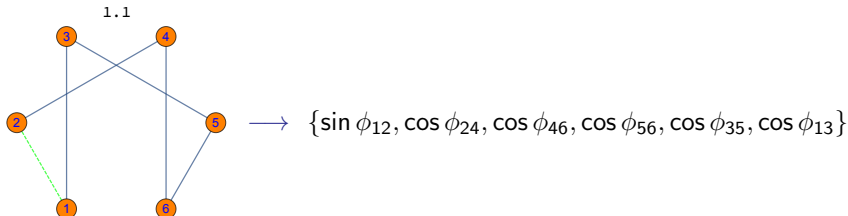
↪ In the same way as before, extract the 'Moravcsik-complete' set (combined with 'diagonal' obs.'s $\{R_T^{00}, {}^c R_{TT}^{00}, R_T^{0y}, R_T^{y'0}, R_L^{00}, R_L^{0y}\}$):

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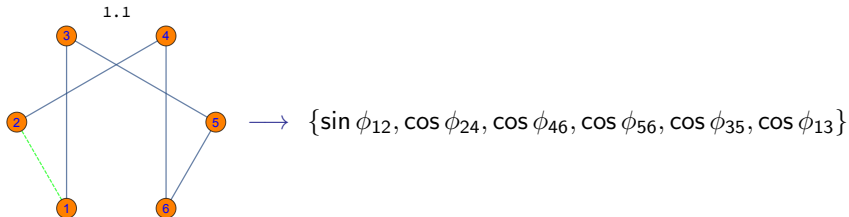
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- 64 non-redundant Moravcsik-complete sets composed of 13 observables
 - ↪ Only one observable more than the minimal number of $2N_A = 12$ observables!

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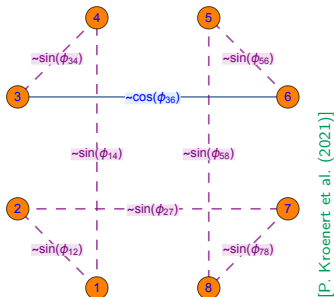
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↪ What about problems with large numbers of amplitudes (i.e. $N_A > 6$)?

Cases with larger numbers of $N_A > 6$ amplitudes

Two-meson photoproduction

- *) 8 amplitudes vs. 64 observables
- *) Typical complete graph:

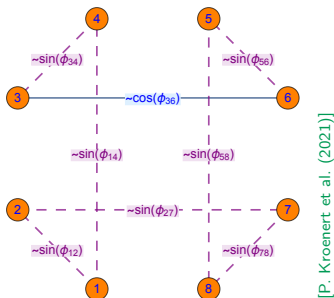


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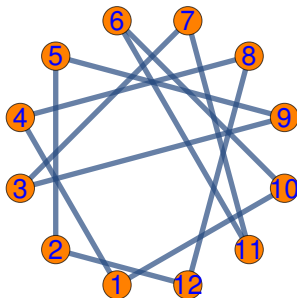
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- *) 12 amp.'s vs. 244 observables
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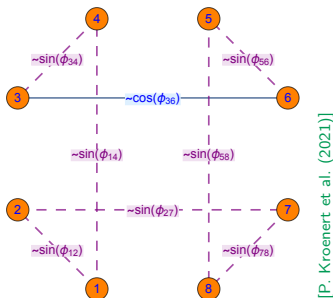


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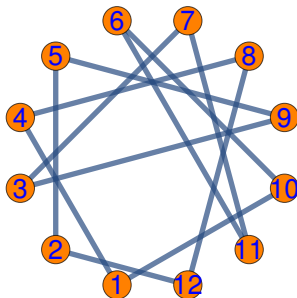
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Observation: Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N_A$ observables, for problems with $N_A \geq 4$ amplitudes

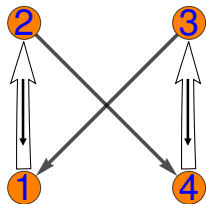
Reductions obtained using new 'directional' graphs

- One can improve the situation using new kind of graphs, containing additional *directional information*. [YW, Phys. Rev. C **104**, no.4, 045203 (2021)]

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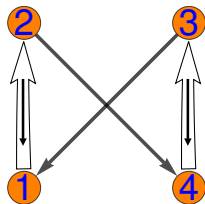


⇔ complete photoproduction-set ($2N = 8$ obs.'s in combination with 4 'diagonal' obs.'s $\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\}$): $\{\mathcal{O}_{2+}^a, \mathcal{O}_{2-}^a, \mathcal{O}_{1+}^c, \mathcal{O}_{2-}^c\} = \{\check{E}, \check{H}, \check{L}_{x'}, \check{T}_{x'}\}$.

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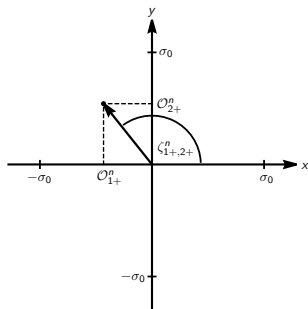
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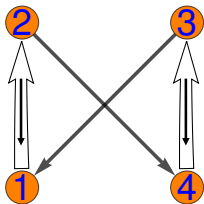
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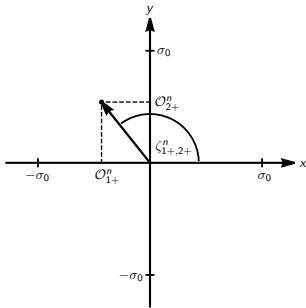
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→ Confirm min. sets of $2N_A = 8$ for photoproduction ;
Obtain new sets of $2N_A = 12$ for e^- -production!
 For $N_A > 6$, sets still (slightly) over-complete ...

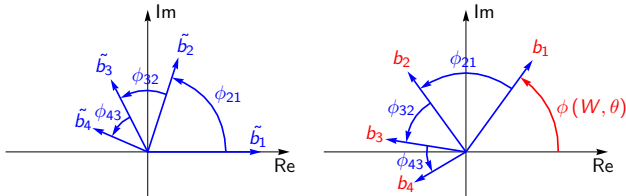
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We have new *standard-approaches* in place to determine complete experiments, and thus $|b_i|$ and ϕ_{ij} , for reactions with any number of amplitudes N_A !

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However: the unknown phase $\phi(\Omega_2^{nf})$ remains a problem, because:



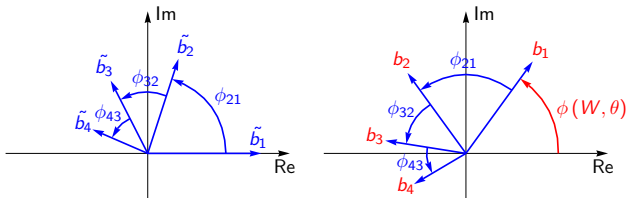
$$\begin{aligned} \mathcal{T}_{fi}(\Omega_2^{nf}) &= \chi_B^\dagger \left[\sum_{k=1}^{N_A} \kappa_k b_k(\Omega_2^{nf}) \right] \chi_N = \chi_B^\dagger \left[\sum_{k=1}^{N_A} \kappa_k e^{i\phi_{fi}(\Omega_2^{nf})} \tilde{b}_k(\Omega_2^{nf}) \right] \chi_N \\ &= e^{i\phi_{fi}(\Omega_2^{nf})} \chi_B^\dagger \left[\sum_{k=1}^{N_A} \kappa_k \tilde{b}_k(\Omega_2^{nf}) \right] \chi_N = e^{i\phi_{fi}(\Omega_2^{nf})} \tilde{\mathcal{T}}_{fi}(\Omega_2^{nf}), \end{aligned}$$

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$\Rightarrow$ Complete experiment fixes only 'internal spin-structure' $\tilde{\mathcal{T}}_{fi}$!

The phase has *physical significance*, because:

- (Perfect) knowledge of the phase $\Rightarrow$ project-out partial waves to all orders!
- *Analytically continue* part. waves away from physical region $\Rightarrow$ *unique* resonance-poles!

Measuring the overall phase

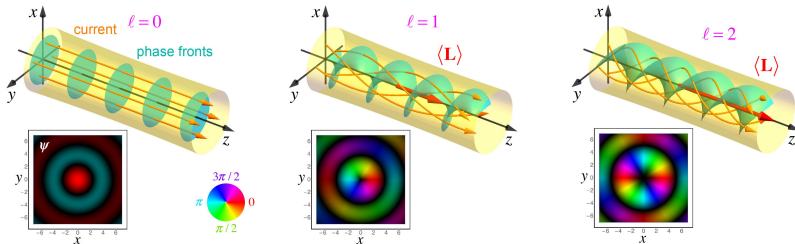
Measure the overall phase in scattering experiments using vortex beams

≡ beams of particles with intrinsic orbital angular momentum $\langle L_z \rangle = \hbar \ell$ along the axis of propagation (i.e. z-axis) cf. [Ivanov, Phys. Rev. D **85**, 076001 (2012)], [Ivanov, arXiv:2205.00412 [hep-ph] (2022)]

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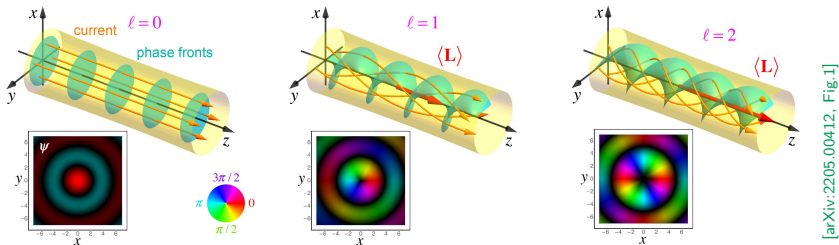


[arXiv:2205.00412, Fig.1]

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Proposal [Ivanov, Phys. Rev. D **85**, 076001 (2012)]: for the example $\gamma p \rightarrow \pi p$, consider double-twisted γp -collision (i.e. both γ and p are in a [Bessel] vortex-state)

⇒ Measure *azimuthal asymmetry* $A = \frac{\Delta\sigma}{\sigma}$ ('sine-weighted' c.s. $\Delta\sigma$; non-weighted σ)

⇒ Then, one has $A = \frac{d\phi(\theta_{\gamma\pi}^{\text{LAB}})}{d\theta_{\gamma\pi}^{\text{LAB}}} \cdot P$, with an 'analyzing power' P .

⇒ For insanely good accuracy and statistics, integration yields: $\phi(\theta_{\gamma\pi}^{\text{LAB}}) + \mathcal{C}$.

Vortex-beams at the GeV-scale maybe feasible within 10-20 years [Ivanov, priv. comm. (2022)]

Unitarity can constrain the overall phase

Consider case of elastic $2 \rightarrow 2$ -scattering of spinless particles and solve the 'CEA':

$$\sigma_0(W, \theta) = |\mathcal{T}(W, \theta)|^2 \Leftrightarrow |\mathcal{T}(W, \theta)| = +\sqrt{\sigma_0(W, \theta)}.$$

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Consider case of elastic $2 \rightarrow 2$ -scattering of spinless particles and solve the 'CEA':

$$\sigma_0(W, \theta) = |\mathcal{T}(W, \theta)|^2 \Leftrightarrow |\mathcal{T}(W, \theta)| = +\sqrt{\sigma_0(W, \theta)}.$$

- *) Use the elastic $2 \rightarrow 2$ unitarity-equation (operator-form: $\hat{T} - \hat{T}^\dagger = i\hat{T}^\dagger \hat{T}$) for $\mathcal{T} = |\mathcal{T}| e^{i\phi} = |\mathcal{T}| \sqrt{\sigma_0}$ to solve for the overall phase (W -dependence implicit):
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- *) One can show [A. Martin, Nuovo Cim. A 59, 131 (1969)]: the solution for $\phi(\cos \theta)$ is unique iff. the amplitude satisfies the 'Martin-bound'

$$\sin \phi(c_{fi}) < \sin \mu := \max \left(\int \frac{d\Omega_{i'}}{4\pi} \frac{|\mathcal{T}(c_{fi'})| |\mathcal{T}(c_{i'i})|}{|\mathcal{T}(c_{fi})|} \right) < 0.79.$$

Uniqueness theorems under certain *bounds* have also been found for:

- 2 coupled scalar 2-body channels [R. F. Alvarez-Estrada et al., Nucl. Phys. B 88, 289 (1975)]
- elastic πN -scattering [R. F. Alvarez-Estrada et al., Nucl. Phys. B 54, 237 (1973)]
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- *) In case the Martin-bound is violated, there exists only *one discrete ambiguity*, the 'Crichton-ambiguity' $\{\phi^I(c_{fi}), \phi^{II}(c_{fi})\}$ [J. H. Crichton, Nuovo Cim. A 45, 256 (1966)]

[A. Martin and J. M. Richard, Phys. Rev. D 101, no.9, 094014 (2020)]

Finale: the 'coupled-channels complete experiment'

Consider *channel-space* $\{|\pi N\rangle, |\gamma N\rangle, |\pi\pi N\rangle\}$, i.e.:

$$(\mathcal{T}_{fi}) = \begin{bmatrix} \mathcal{T}_{\pi N, \pi N} & \mathcal{T}_{\pi N, \gamma N} & \mathcal{T}_{\pi N, \pi\pi N} \\ \mathcal{T}_{\gamma N, \pi N} & \mathcal{T}_{\gamma N, \gamma N} \simeq 0 & \mathcal{T}_{\gamma N, \pi\pi N} \\ \mathcal{T}_{\pi\pi N, \pi N} & \mathcal{T}_{\pi\pi N, \gamma N} & \mathcal{T}_{\pi\pi N, \pi\pi N} \end{bmatrix}.$$

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→ Measure individual complete experiments with perfect *phase-space coverage and overlap* among individual reactions (complete exp.'s determinable using *graphs*):

Reaction	Example complete experiment (yields $ b_i $ & ϕ_{ij})
$\pi N \rightarrow \pi N$ ($N_A = 2$)	$\sigma_0, \hat{P}, \hat{R}, \hat{A}$
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$\gamma N \rightarrow \pi\pi N$ ($N_A = 8$)	$\sigma_0, \check{P}_y, \check{P}_{y'}, \check{O}_{yy'}, \check{O}_{yy'}, \check{P}_{y'}, \check{P}_y^\ominus, I^\ominus, \check{P}_x, \check{P}_z, \check{P}_{x'}, \check{P}_x^s, \check{P}_x^\ominus, \check{P}_z^c, \check{P}_z^\ominus, \check{P}_{x'}^\ominus$

→ For these 4 reactions, we have $\mathcal{T}_{fi} = e^{i\phi_{fi}} \tilde{\mathcal{T}}_{fi}$, with $\tilde{\mathcal{T}}_{fi}$ fixed.

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→ Fit at least two (or more) complementary ED models (BnGa, JüBo, ...), which have to have *as good unitarity-constraints as possible*, to this database

⇒ Missing phase-information $e^{i\phi_{fi}}$ fixed and resonance-spectrum (hopefully) unique!

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Issues: - Can we assume perfect time-reversal inv., to relate $3 \rightarrow 2$ to $2 \rightarrow 3$ processes?

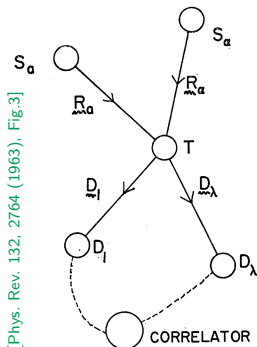
- $3 \rightarrow 3$ -process $\pi\pi N \rightarrow \pi\pi N$ unmeasurable. Does this hurt the proposal?

Thank You!

Additional Slides

The Hanbury-Brown and Twiss experiment

Measure the overall phase via intensity correlations in a *Hanbury-Brown and Twiss-type* experiment
 [Goldberger, Lewis & Watson, Phys. Rev. 132, 2764 (1963)]



- *) Two sources, S_a and S_α , emitting beam-particles
 - *) One single irradiated target T
 - *) Two spatially separated detectors, D_l and D_λ
 - *) A CORRELATOR, which registers only in case D_l and D_λ count in *coincidence*
- The correlator counting-rate contains an isolatable term, which is proportional to:

$$\text{Re} [\mathcal{T}_{\lambda \leftarrow \alpha} \mathcal{T}_{l \leftarrow \alpha}^* \mathcal{T}_{l \leftarrow a} \mathcal{T}_{\lambda \leftarrow a}^*].$$

- *) Assuming $|\mathcal{T}_{f \leftarrow i}|$ as known, one can measure $\cos(\Gamma)$, where:
 $\Gamma := \phi(\mathbf{g}_{la}) - \phi(\mathbf{g}_{l\alpha}) + \phi(\mathbf{g}_{\lambda\alpha}) - \phi(\mathbf{g}_{\lambda a})$, with $\mathbf{g}_{la} \equiv \hat{R}_a - \hat{D}_l, \dots$
 - *) Varying positions of detectors and sources, do measurements for *many* angles:
 $\{\mathbf{g}_{la}^{(1)}, \mathbf{g}_{la}^{(2)}, \dots\}, \{\mathbf{g}_{l\alpha}^{(1)}, \mathbf{g}_{l\alpha}^{(2)}, \dots\}, \{\mathbf{g}_{\lambda\alpha}^{(1)}, \mathbf{g}_{\lambda\alpha}^{(2)}, \dots\}, \{\mathbf{g}_{\lambda a}^{(1)}, \mathbf{g}_{\lambda a}^{(2)}, \dots\}.$
- Extract: $\phi(\mathbf{g}_{la}^{(\nu+1)}) - \phi(\mathbf{g}_{la}^{(\nu)}) \equiv \delta\phi(\nu) \rightarrow$ Overall phase: $\phi(\mathbf{g})$.