Abel transforms: Transverse charge & EMT distributions

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Introduction

Modern Understanding on Nucleon form factors

- GPDs

$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle p', \sigma' | \overline{\psi}_q \left(-\frac{\lambda n}{2}\right) \not{n} \psi_q \left(\frac{\lambda n}{2}\right) | p, \sigma \rangle$$
$$= \boxed{H^q(x,\xi,t)} \overline{u}(p',\sigma') \not{n} u(p,\sigma) + \boxed{E^q(x,\xi,t)} \overline{u}(p',\sigma') \frac{i\sigma^{\mu\nu} n_\mu \Delta_\nu}{2M_B} u(p,\sigma)$$

- Mellin moments of the GPDs

- The first moments of the GPDs H & E yield the well-known EM form factors

$$\int_{-1}^{1} dx \sum_q H^q(x,\xi,t) = F_1(t), \quad \int_{-1}^{1} dx \sum_q E^q(x,\xi,t) = F_2(t)$$

- The second moments of the GPDs H & E give the gravitational (EMT) FFs (Ji's sum rules).

$$\int_{-1}^{1} dx x \sum_q H^q(x,\xi,t) = A^Q(t) + D^Q(t)\xi^2,$$

$$\int_{-1}^{1} dx x \sum_q E^q(x,\xi,t) = 2J^Q(t) - A^Q(t) - D^Q(t)\xi^2$$

D. Müller et al. Fortschr. Phys. 42 (1994).

X. D. Ji, PRL 78, PRD 55 (1997).

A. V. Radyushkin, PLB 380 (1996)

Critical view on Nucleon form factors

Traditional interpretation of the nucleon form factors

$$F_1(Q^2) = \int d^3x e^{i\mathbf{Q}\cdot\mathbf{x}} \rho(\mathbf{r}) \rightarrow \quad \rho(\mathbf{r}) = \sum \psi^\dagger(\mathbf{r})\psi(\mathbf{r})$$

Particle number fixed.

- This is valid for atoms and nuclei: $\frac{\delta r}{r} = \frac{m_e \alpha}{M} \sim 10^{-5}$

Crucial criticism on the traditional definition of the nucleon form factors.

- It is not valid anymore for the nucleon: $r \sim 0.8\,\mathrm{fm}$ $\quad \delta r \sim \frac{\hbar}{M_N c} \approx 0.2\,\mathrm{fm}$

  $\delta r / r \sim 0.25$

The nucleon is a **relativistic** particle!

Particle creation and annihilation inside a nucleon

⇒
- Validity of the nucleon 3D distributions was put into question.
- View on the nucleon form factors has been modernized.

M. Burkardt, PRD 62 (2000) [**66** (2002)]

Belitsky & Radyushkin, Phys.Rept. **418** (2005)

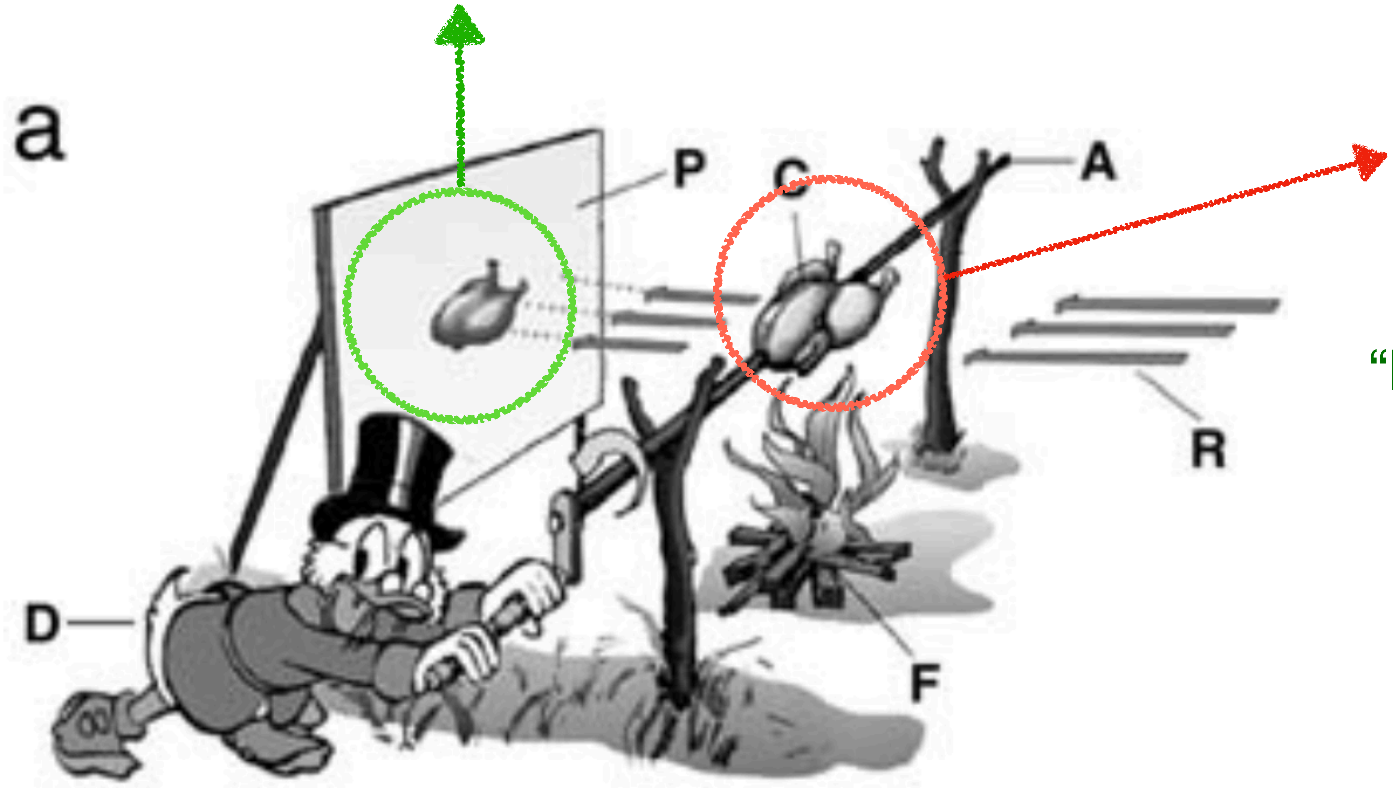
G.A. Miller, PRL **99** (2007)

C. Lorce, PRL **125** (2020)

R. L. Jaffe, PRD **103** (2021)

Abel & Radon transforms & Nucleon tomography

2D transverse distributions in the IMF (QM probabilistic)



3D distributions in the BF
(Quasi-probabilistic)

"Electron tomography, edited by J. Frank"

- Abel transformation maps 3D distributions of a particle with spin 0 or 1/2 at rest onto 2D transverse plane in the IMF.
  (Radon transform is required for that with higher spin.)

3D distributions ⟺ 2D distributions

This is the subject of the present talk.

M. Burkardt, PRD 62 (2000) [66 (2002)]

G. A. Miller, PRL. 99 (2007).

Carlson & Vanderhaeghen, PRL 100 (2008)

C. Lorce, PRL 125 (2020).

Panteleeva & Polyakov, ArXiv: 2102.10902

Mechanical properties
of
Baryons

Gravitational form factors

- EMT current in QCD & GFFs

Kobzarev et al. 1962; Pagels, 1966

$$T_q^{\mu\nu} = \frac{1}{4}\overline{\psi}_q\left(-i\overleftarrow{\mathcal{D}}^\mu\gamma^\nu - i\overleftarrow{\mathcal{D}}^\nu\gamma^\mu + i\overrightarrow{\mathcal{D}}^\mu\gamma^\nu + i\overrightarrow{\mathcal{D}}^\nu\gamma^\mu\right)\psi_q - g^{\mu\nu}\overline{\psi}_q\left(-\frac{i}{2}\overleftarrow{\not{\mathcal{D}}} + \frac{i}{2}\overrightarrow{\not{\mathcal{D}}} - m_q\right)\psi_q,$$

$$T_g^{\mu\nu} = F^{a,\mu\eta}\, F^{a,}{}_{\eta}{}^{\nu} + \frac{1}{4}\, g^{\mu\nu} F^{a,\kappa\eta}\, F^{a,}{}_{\kappa\eta}.$$

D(Druck)-term

Weiss & Polyakov, 1999

$$\langle p'|T^{\mu\nu}(0)|p\rangle = \bar{u}(p')\left[A^a(t)\frac{P^\mu P^\nu}{M_N} + J^a(t)\frac{iP^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho}{2M_N} + D^a(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{4M_N} + M_N\bar{c}^a(t)g^{\mu\nu}\right]u(p)$$

δg^{00} → Mass: $\sum_a A^a(0) = 1$

δg^{0i} → Spin: $\sum_a J^a(0) = \frac{1}{2}$

δg^{ij} → Deformation of space = mechanical properties of the nucleon

Non-conservation piece of EMT FFs

Pressure & Shear-force distributions (pressure anisotropy)

Pressure & Shear-force distributions

$$T^a_{ij}(\boldsymbol{r},\sigma',\sigma) = p^a(r)\delta^{ij}\delta_{\sigma'\sigma} + s^a(r)\left(\frac{r^ir^j}{r^2} - \frac{1}{3}\delta^{ij}\right)\delta_{\sigma'\sigma}$$

- 3D Shear-force density in the BF

$$s^a(r) = -\frac{1}{4M_B} r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\tilde{D}^a(r)$$

- 3D Pressure density in the BF   M.V. Polyakov, PLB555 (2003)

$$p^a(r) = \frac{1}{6M_B}\frac{1}{r^2}\frac{1}{dr}r^2\frac{d}{dr}\tilde{D}^a(r) - M_B\int\frac{d^3\Delta}{(2\pi)^3}e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{r}}\bar{c}^a(t)$$

$$\tilde{D}^a(r) = \int\frac{d^3\Delta}{(2\pi)^3}e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{r}}D^a(t)$$

- This term is related to forces between quark and gluon subsystems (Polyakov & Son, 2018).
- It contributes to gluon and quark parts of energy density (mass decomposition). (Lorce, 2018)
- It vanishes for Goldstone bosons (P. Schweitzer & M.V. Polyakov, 2019).

Abel transforms

- Abel transform from 3D in the BF to 2D in the IMF (Also invertible)

$$\mathcal{E}(x_\perp) = 2\int_{x_\perp}^{\infty} \left(\varepsilon(r) + \frac{3}{2}p(r) + \frac{1}{4m}\partial^2\left[\tilde{A}(r) - 2\tilde{J}(r)\right]\right)\frac{r\,dr}{\sqrt{r^2 - x_\perp^2}}$$

$$\rho_J^{(2D)}(x_\perp) = 3\int_{x_\perp}^{\infty} \frac{\rho_J(r)}{r}\frac{x_\perp^2\,dr}{\sqrt{r^2 - x_\perp^2}}$$

$$\mathcal{S}(x_\perp) = \int_{x_\perp}^{\infty} \frac{s(r)}{r}\frac{x_\perp^2\,dr}{\sqrt{r^2 - x_\perp^2}}$$

$$\frac{1}{2}\mathcal{S}(x_\perp) + \mathcal{P}(x_\perp) = \frac{1}{2}\int_{x_\perp}^{\infty}\left(\frac{2}{3}s(r) + p(r)\right)\frac{r\,dr}{\sqrt{r^2 - x_\perp^2}}$$

Abel, J. Reine und Angew. Math. 1 (1826)

- Abel transform is used for tomography of spherically symmetric systems (spin 0 & 1/2 hadrons).
- For non-spherical objects (spin > 1/2), the Radon transform comes into play.

Panteleeva & Polyakov, PRD 104 (2021)

J. Y. Kim & HChK, ArXiv 2105.10279

Equivalence of the 3D BF & 2D LF distributions

- Von Laue Conditions

A. Freese and G. A. Miller, PRD 103 (2021)

$$\int_0^\infty dr\ r^2 p(r) = 0 \quad \Longleftrightarrow \quad \int d^2x_\perp \mathcal{P}(x_\perp) = 0$$

- Local stability Conditions

$$\frac{2}{3}s(r) + p(r) > 0 \quad \Longleftrightarrow \quad \frac{1}{2}\mathcal{S}(x_\perp) + \mathcal{P}(x_\perp) > 0$$

Geometric factor

$$\Omega_d = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}$$

- Mechanical radius

$$\langle x_\perp^2 \rangle_{\text{mech}} = \frac{\int d^2x_\perp x_\perp^2 \left(\frac{1}{2}\mathcal{S}(x_\perp) + \mathcal{P}(x_\perp)\right)}{\int d^2x_\perp \left(\frac{1}{2}\mathcal{S}(x_\perp) + \mathcal{P}(x_\perp)\right)} = \frac{4D(0)}{\int_{-\infty}^0 dt D(t)} = \frac{2}{3}\langle r^2 \rangle_{\text{mech}}$$

- D(Druck)-terms

$$D(0) = -\frac{4M_N}{15}\int d^3r r^2 s(r) = m\int d^3r r^2 p(r) \quad \Longleftrightarrow \quad D(0) = -m\int d^2x_\perp x_\perp^2 \mathcal{S}(x_\perp) = 4m\int d^2x_\perp x_\perp^2 \mathcal{P}(x_\perp)$$

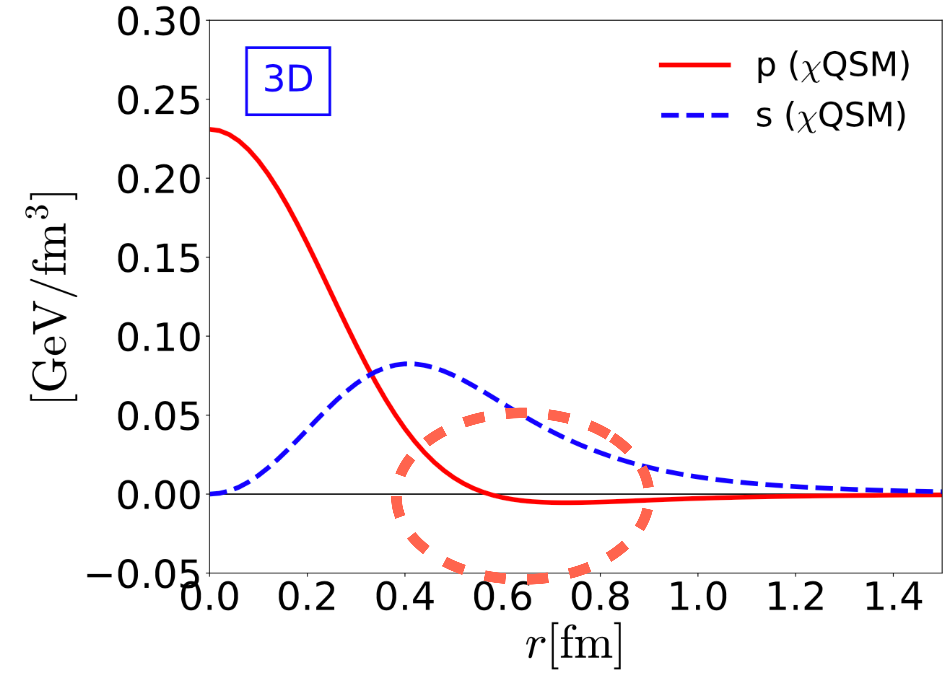
Panteleeva & Polyakov, PRD 104 (2021)

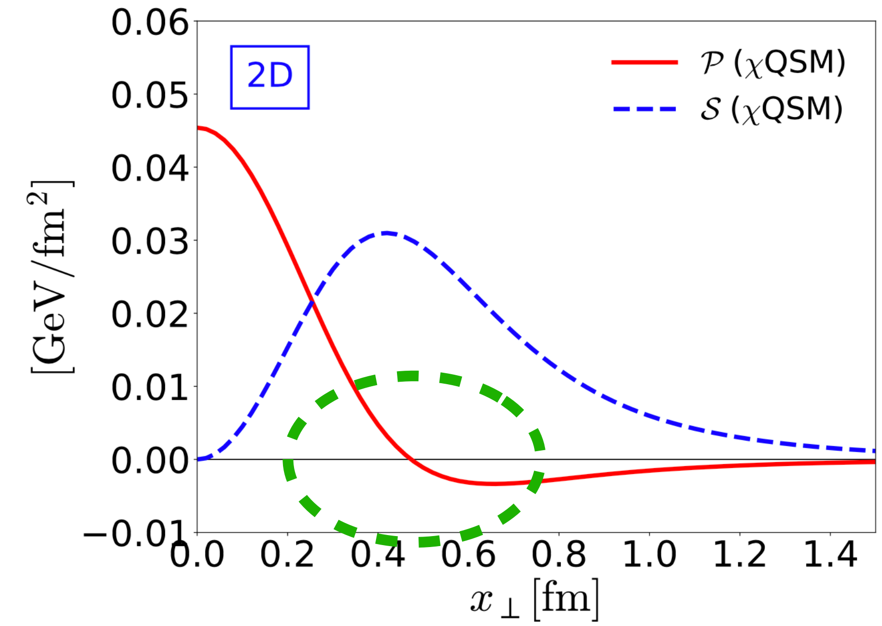
The 3D & 2D pressure & shear-force densities

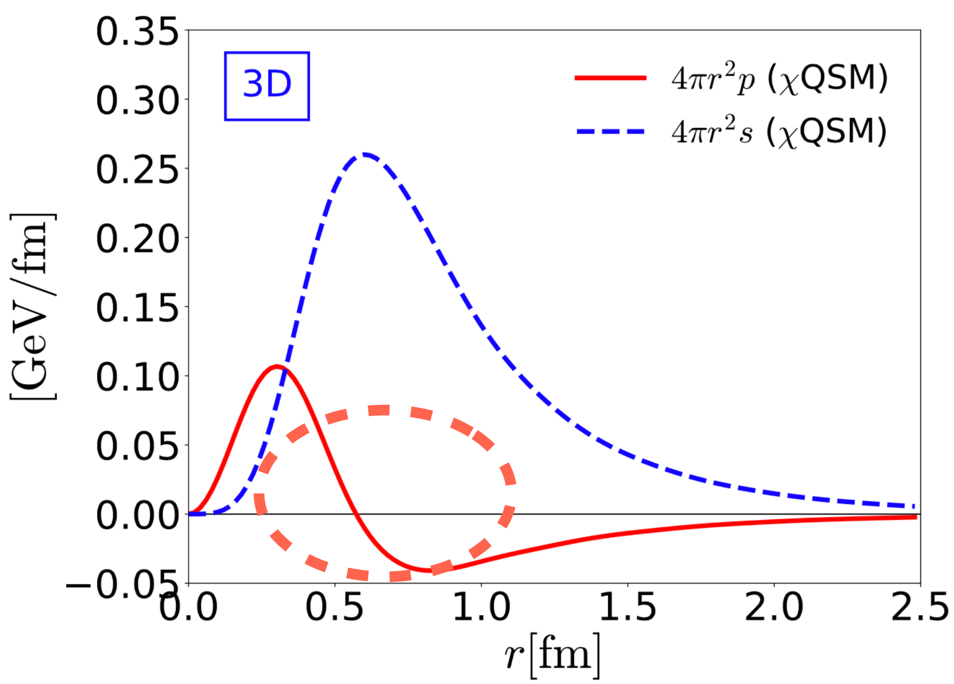
3D in BF ⟷ 2D in IMF χQSM

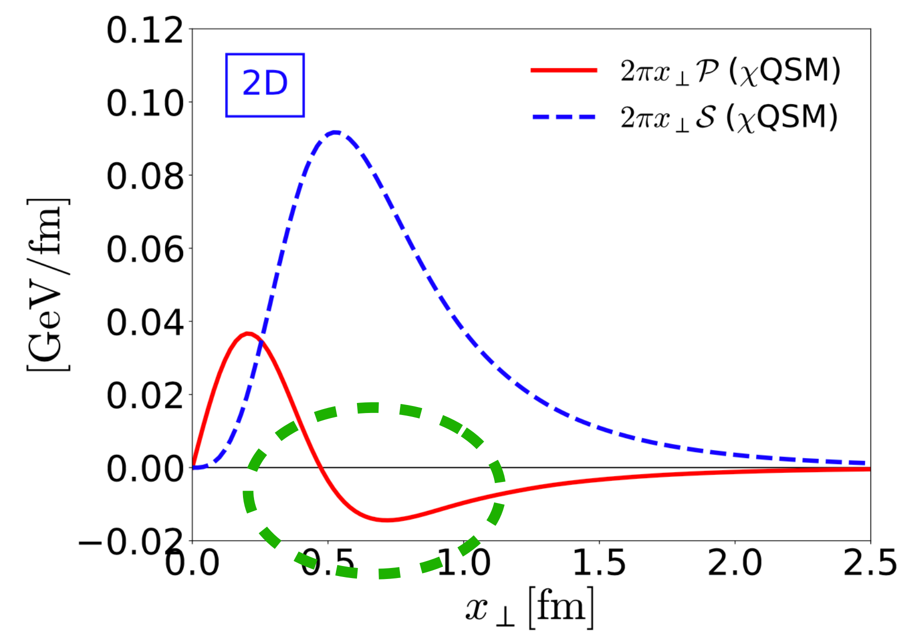
Abel transforms

$$\int_0^\infty dr\ r^2 p(r) = 0 \quad \longleftrightarrow \quad \int d^2x_\perp \mathcal{P}(x_\perp) = 0$$









J. Y. Kim & HChK, ArXiv 2105.10279

Radii of the proton

$$\langle x_\perp^2\rangle_{\rm mass} < \langle x_\perp^2\rangle_{\rm mech} < \langle x_\perp^2\rangle_{\rm charge} < \langle x_\perp^2\rangle_J \quad (\text{2D } \chi\text{QSM})$$

$$\langle r^2\rangle_{\rm mech} < \langle r^2\rangle_{\rm mass} < \langle r^2\rangle_{\rm charge} < \langle r^2\rangle_J \quad (\text{3D } \chi\text{QSM})$$

$$\langle x_\perp^2\rangle_{\rm mass} = \frac{1}{m}\int d^2x_\perp x_\perp^2 \mathcal{E}(x_\perp) = \frac{2}{3}\langle r^2\rangle_{\rm mass} + \frac{D(0)}{m^2} \qquad (D(0)<0)$$

Note that 2D mass radius is smaller than the 3D one.

$\langle x_\perp^2\rangle_{\rm mass}(\rm fm^2)$	$\langle x_\perp^2\rangle_J\ (\rm fm^2)$	$\langle x_\perp^2\rangle_{\rm mech}\ (\rm fm^2)$	$\langle x_\perp^2\rangle_{\rm charge}\ (\rm fm^2)$
0.39	1.19	0.42	0.58

$\langle r^2\rangle_{\rm mass}\ (\rm fm^2)$	$\langle r^2\rangle_J\ (\rm fm^2)$	$\langle r^2\rangle_{\rm mech}\ (\rm fm^2)$	$\langle r^2\rangle_{\rm charge}\ (\rm fm^2)$
0.66	1.49	0.63	0.86

J. Y. Kim & HChK, ArXiv 2105.10279

Stability conditions

- Conservation of the statice EMT current → Global & local stability conditions

$$\partial^i T_{ij} = \frac{r_j}{r}\left[\frac{2}{3}\frac{\partial s(r)}{\partial r} + \frac{2s(r)}{r} + \frac{\partial p(r)}{\partial r}\right] = 0$$

- Von Laue condition: Global stability condition

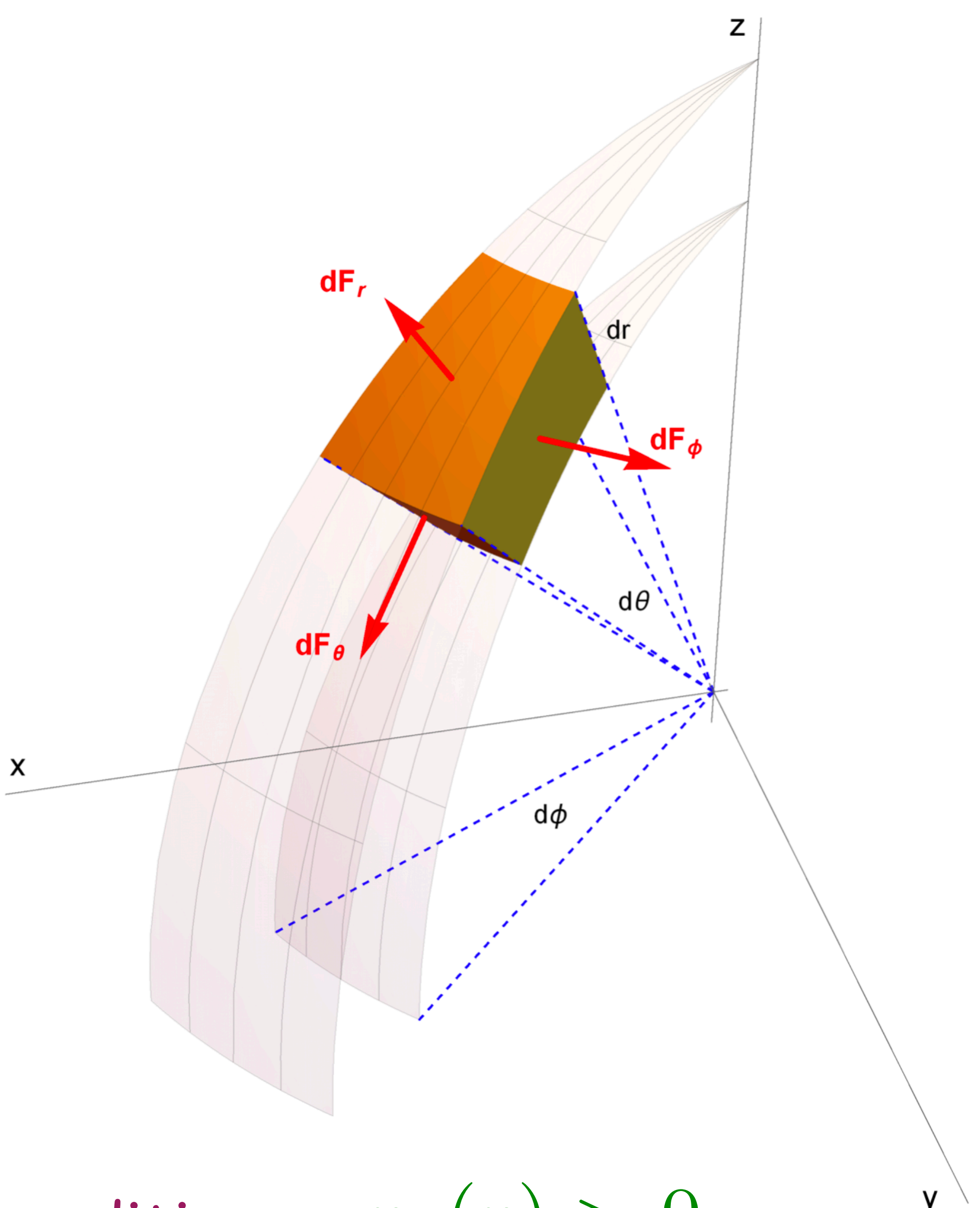
$$\int_0^\infty dr\; r^2 p(r) = 0$$

$$dF^i_{(r,\theta,\phi)} = T^{ij} dS_{(r,\theta,\phi)} \boldsymbol{e}^j_{(r,\theta,\phi)}$$

$$p_r(r) := \frac{dF_r}{dS_r} = \frac{2}{3}s(r) + p(r),$$

$$p_\theta(r) := \frac{dF_\theta}{dS_\theta} = -\frac{1}{3}s(r) + p(r),$$

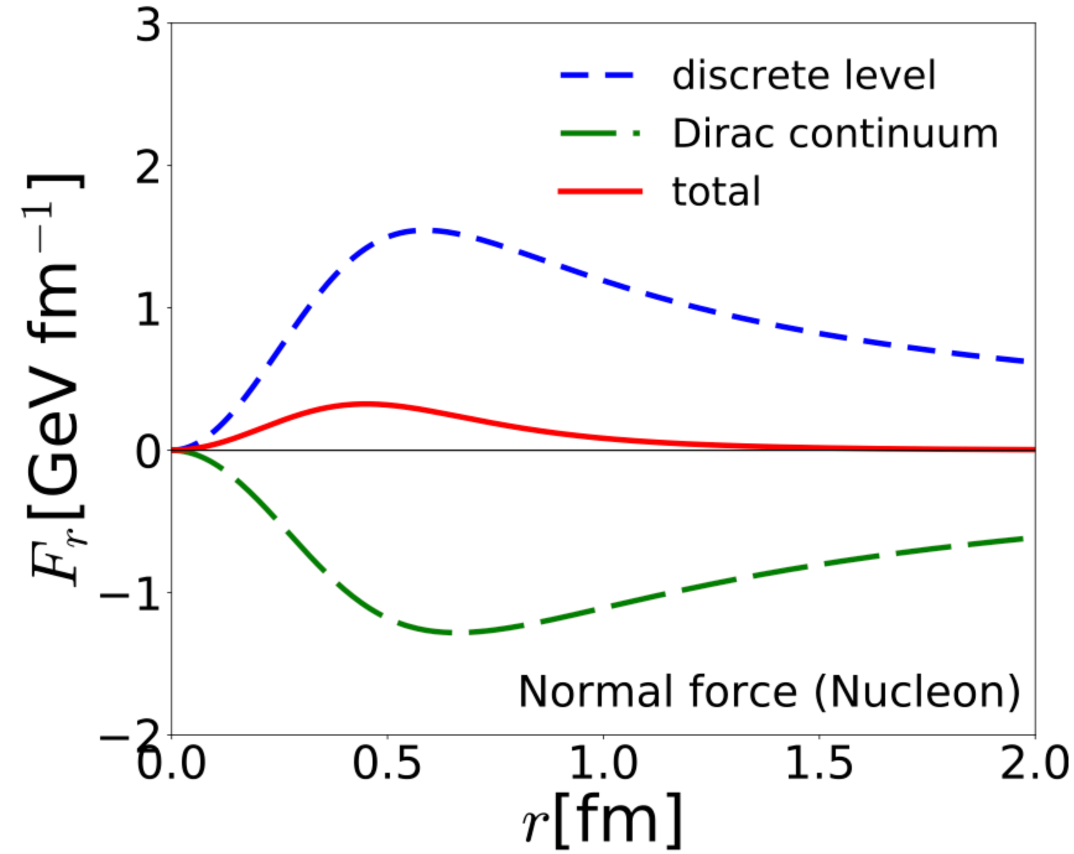
$$p_\phi(r) := \frac{dF_\phi}{dS_\phi} = -\frac{1}{3}s(r) + p(r)$$

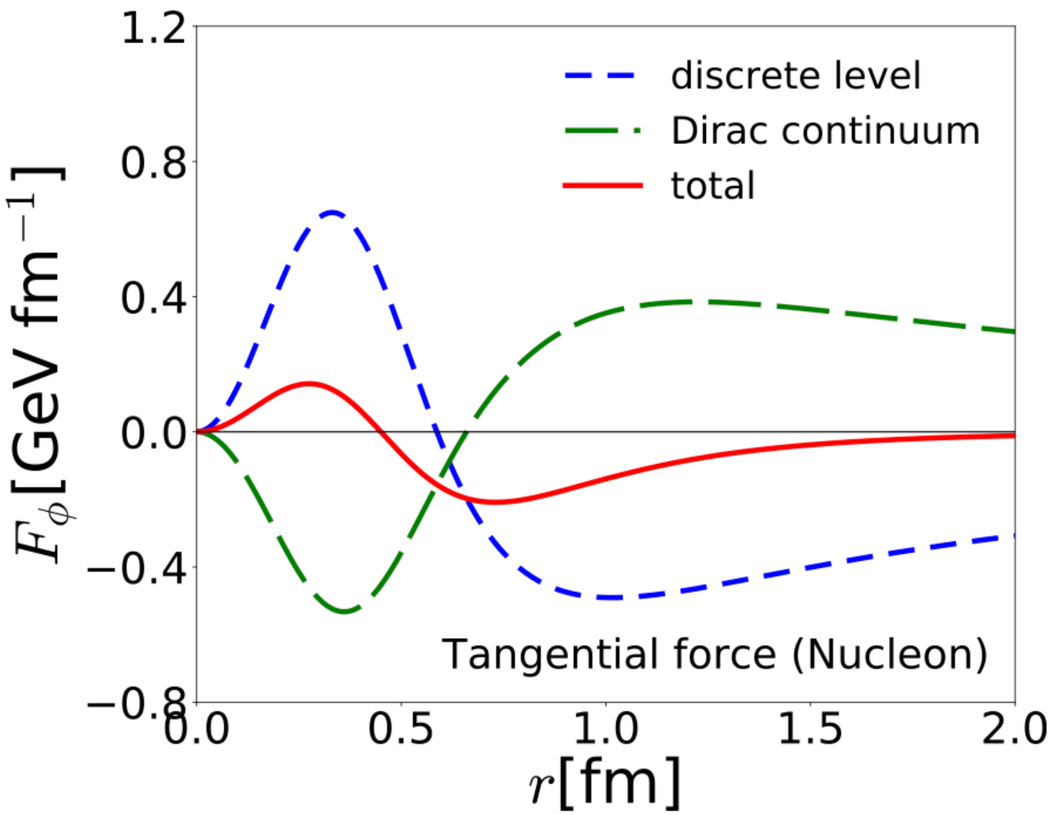


Local stability condition $\quad p_r(r) > 0$

A. Perevalova, M. V. Polyakov and P. Schweitzer, PRD 94 (2016)

3D force fields & local stability





- Normal force is always positive:

$$p_r(r) > 0 \quad \longrightarrow \quad F_r(r) > 0$$

The discrete level overcomes the Dirac continuum.

- Tangential force should at least have one nodal point.

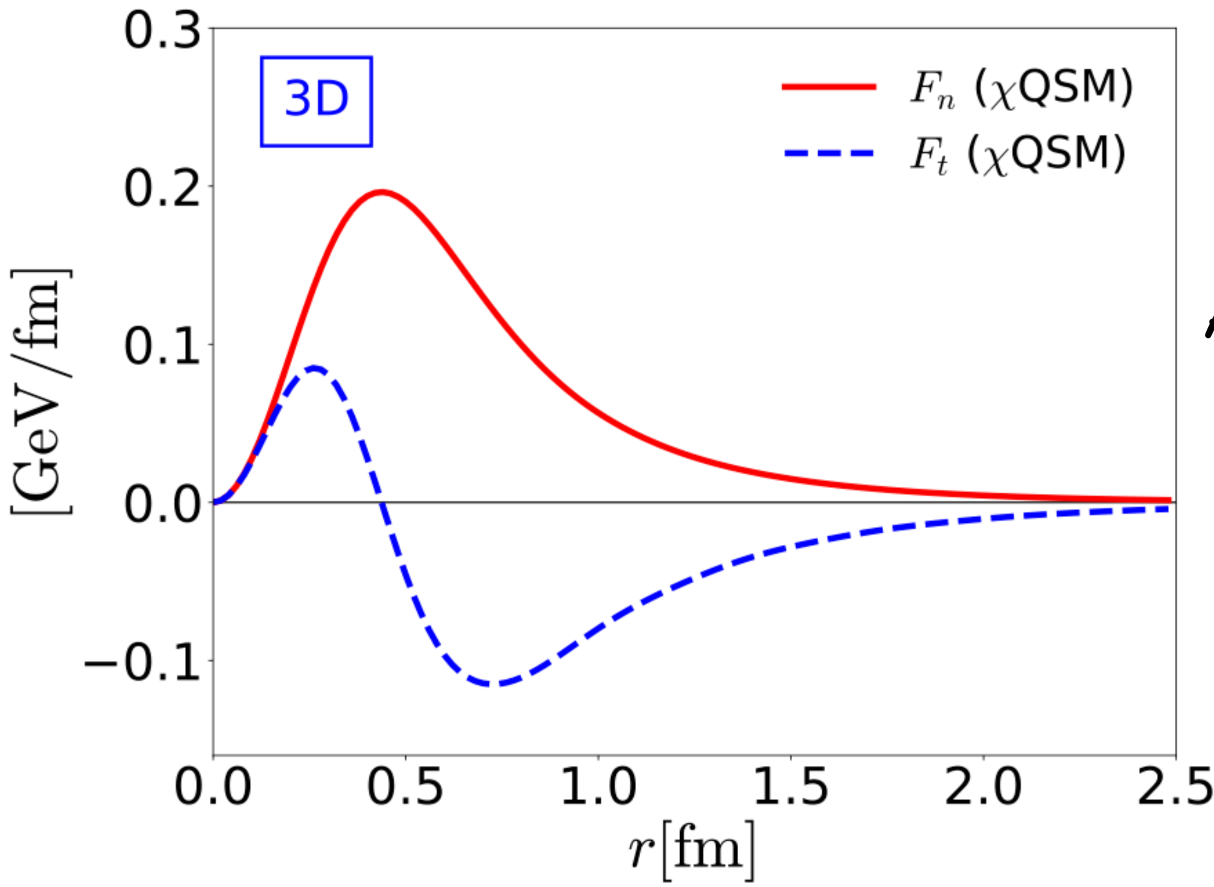
→ Inner part of the tangential force is opposite to its outer part.

$$\int_0^\infty dr\ r\ p_\phi = 0$$ (2D von Laue condition)

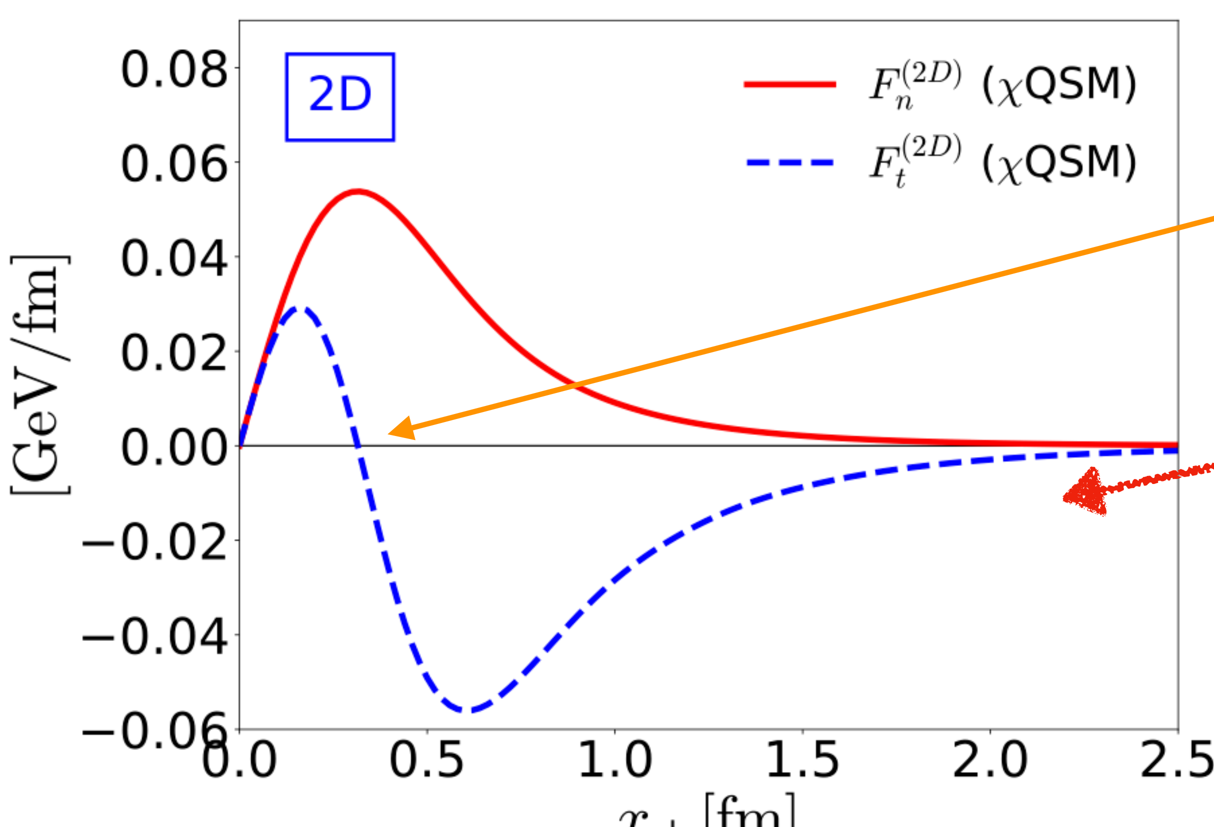
Kim, HChK, H. Son, M. Polyakov PRD 103 (2021)

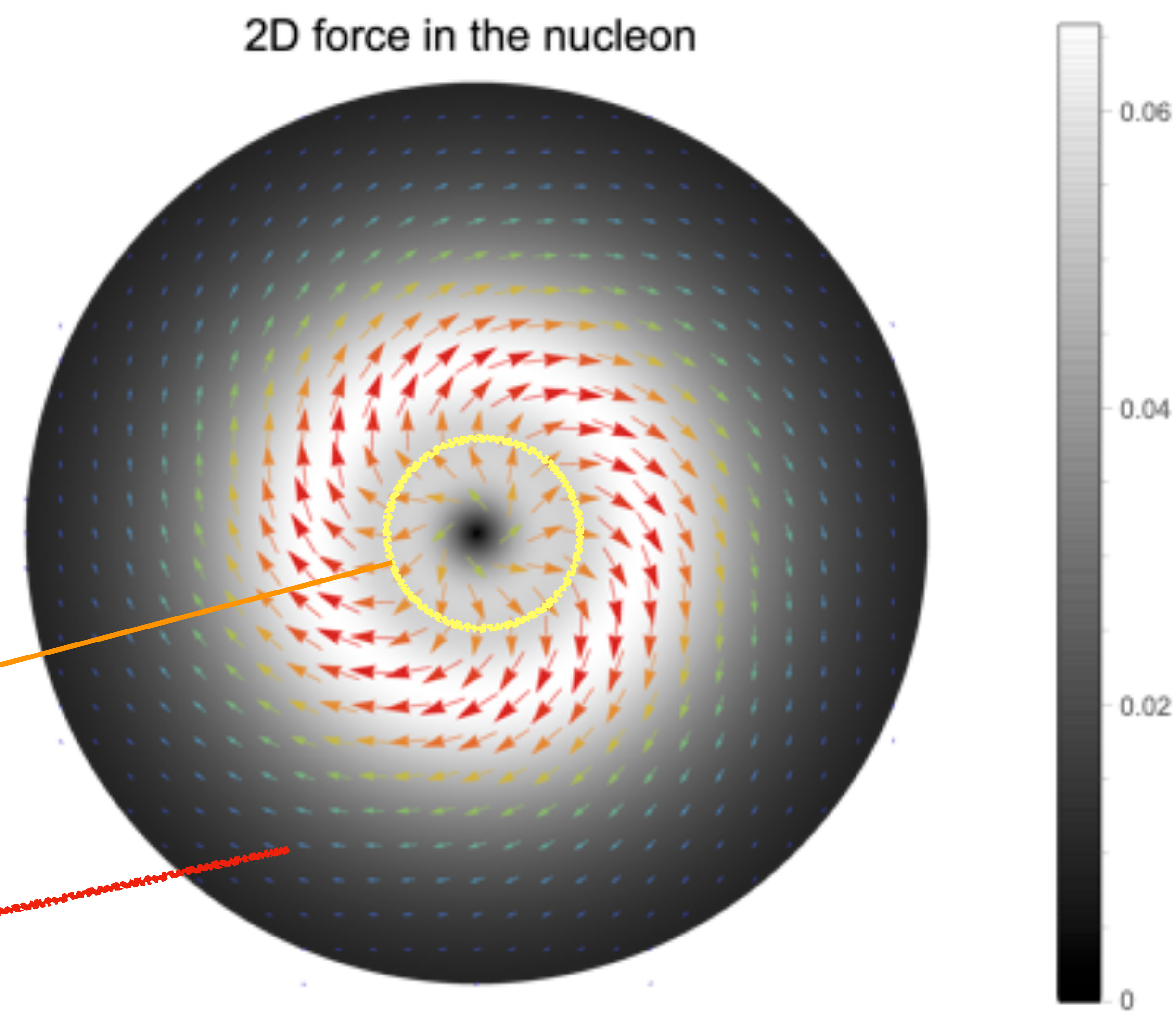
2D force fields & local stability

J. Y. Kim & HChK, ArXiv 2105.10279



Abel transformation





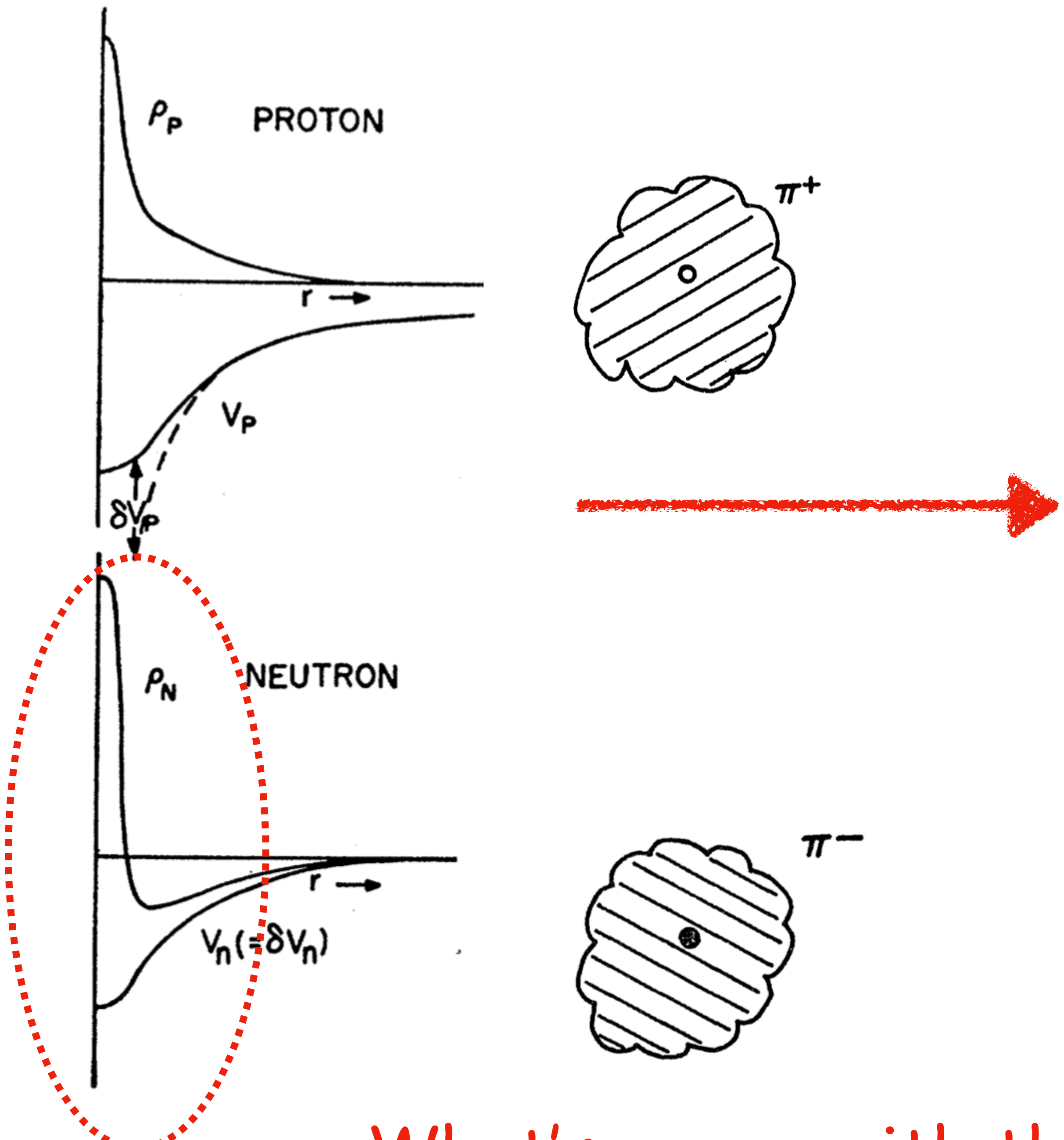
Outer part is governed by the tangential force. (Stability is acquired).

Transverse charge distribution of the polarized Neutron

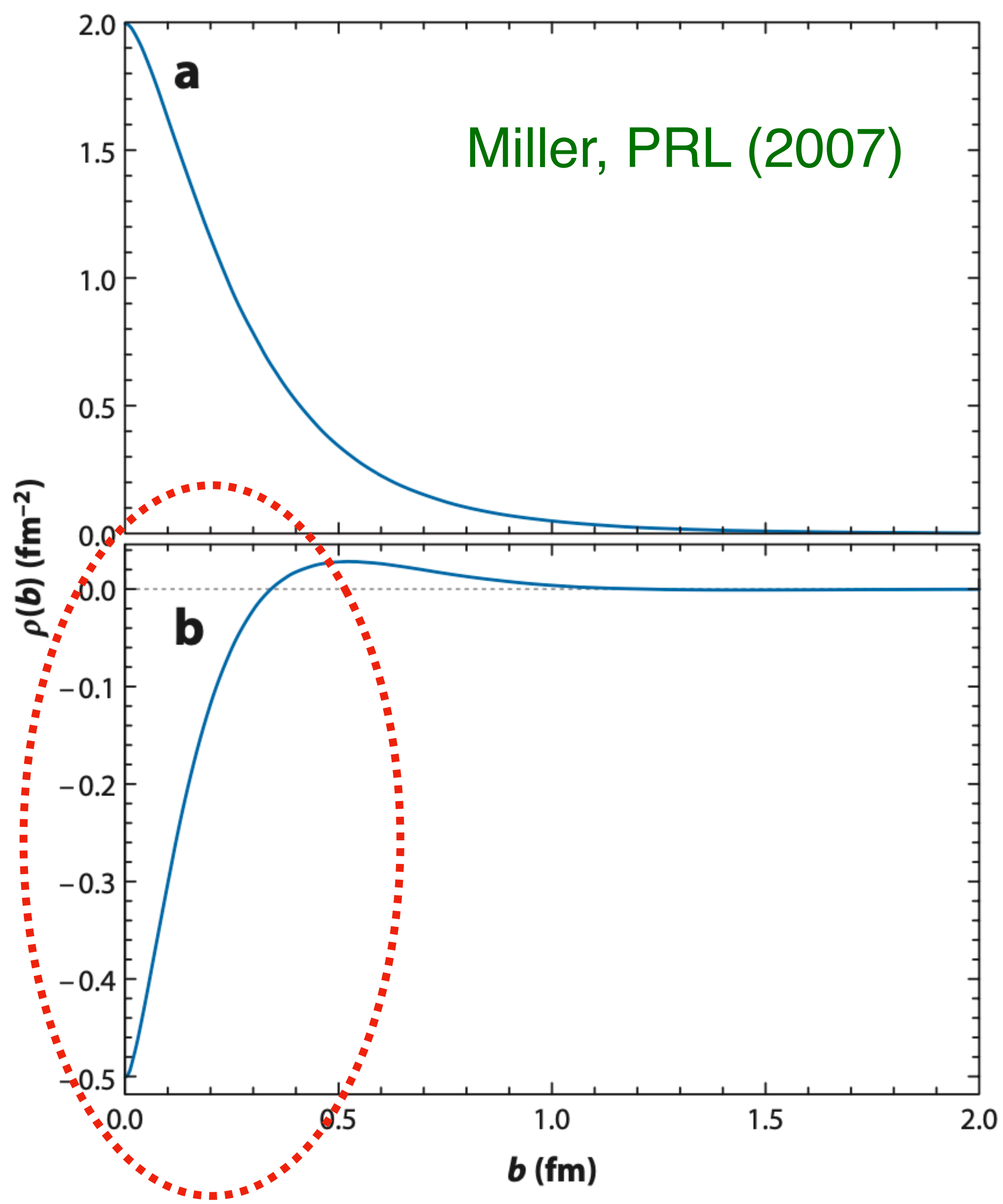
Charge distributions of the nucleon

3D charge densities of the nucleon in the BF

Yennie et al., RMP (1957)



2D Transverse charge densities of the nucleon in the IMF

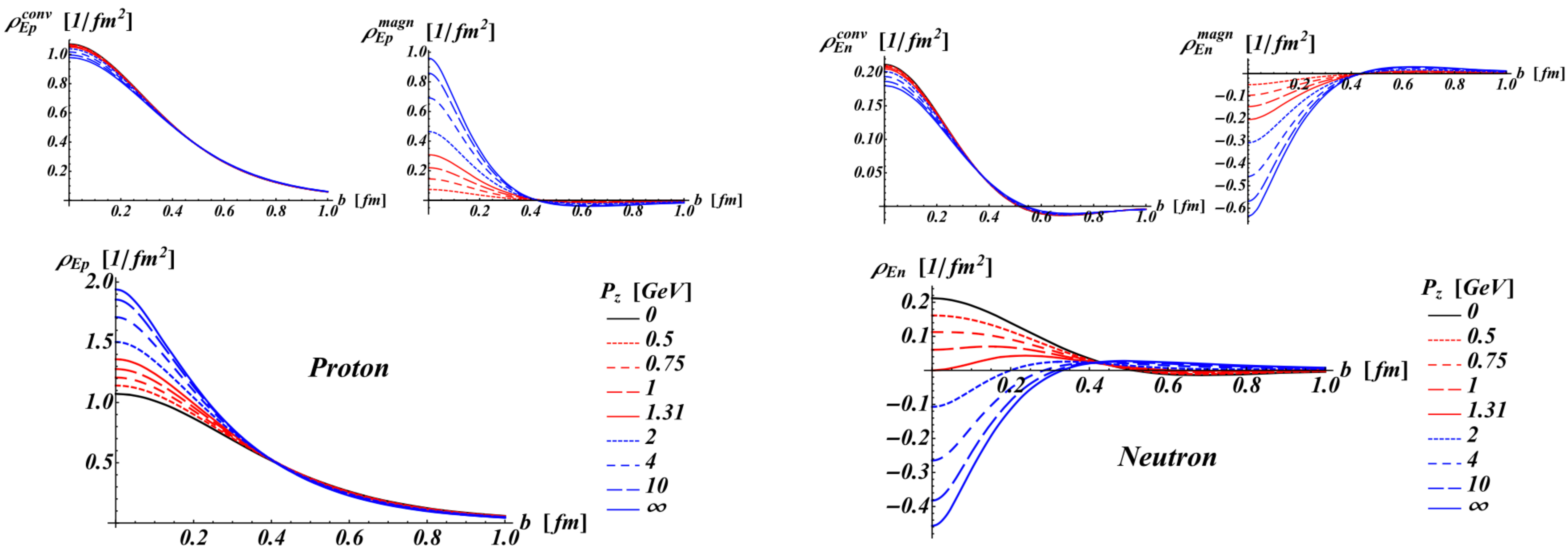


What's wrong with this?

(Actually, nothing wrong.)

2D density exhibits correctly QM probabilistic meaning.

Charge distributions of the nucleon



$$\langle p', s'|\hat{j}^\mu(0)|p, s\rangle = \sum_{s'_B, s_B} D^{*(j)}_{s'_B s'}(p'_B, \Lambda) D^{(j)}_{s_B s}(p_B, \Lambda) \times \Lambda^\mu{}_\nu \langle p'_B, s'_B|\hat{j}^\nu(0)|p_B, s_B\rangle,$$

$$\tilde{\rho}_E = \tilde{\rho}_E^{\text{conv}} + \tilde{\rho}_E^{\text{magn}}$$

$$\rho_E^X(b; P_z) = \int_0^\infty \frac{dQ}{2\pi} Q J_0(Qb) \tilde{\rho}_E^X(Q; P_z)$$

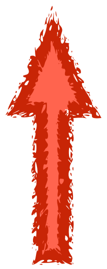
$$\tilde{\rho}_E^{\text{conv}}(Q; P_z) = \frac{P^0 + M(1+\tau)}{(P^0 + M)(1+\tau)} G_E(Q^2),$$

$$\tilde{\rho}_E^{\text{magn}}(Q; P_z) = \frac{\tau P_z^2}{P^0(P^0 + M)(1+\tau)} G_M(Q^2)$$

C. Lorce, PRL 125 (2020)

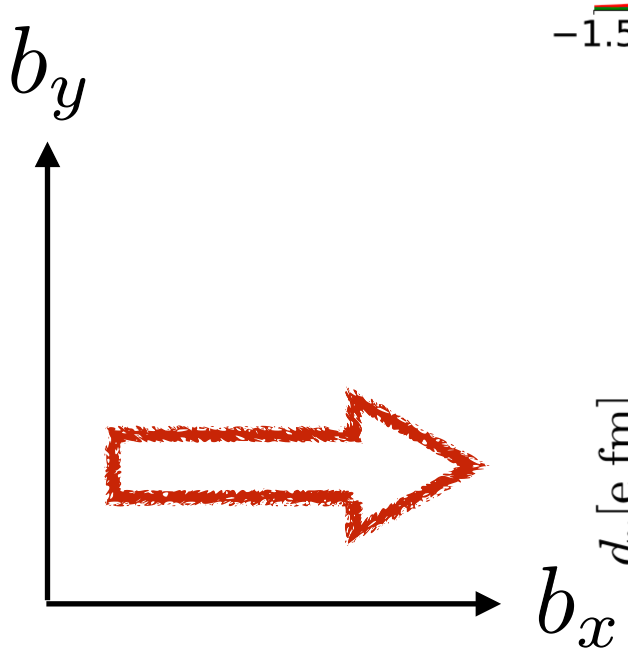
Charge distributions of the tr. polarized nucleon

Carlson & Vanderhaghen, PRL

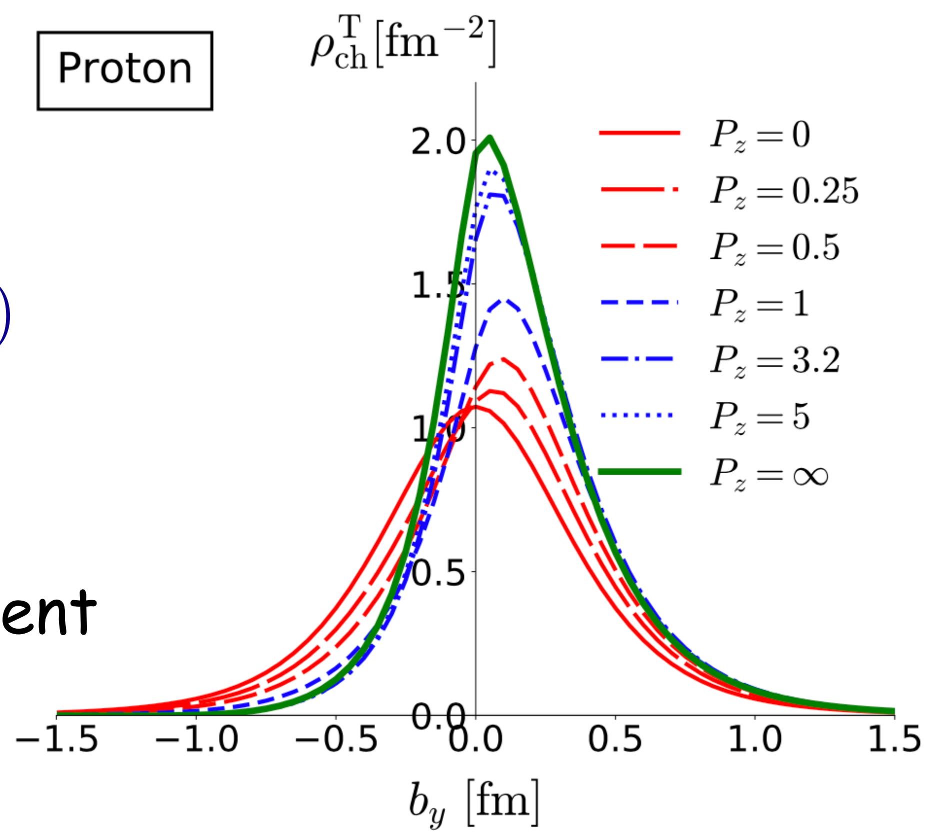


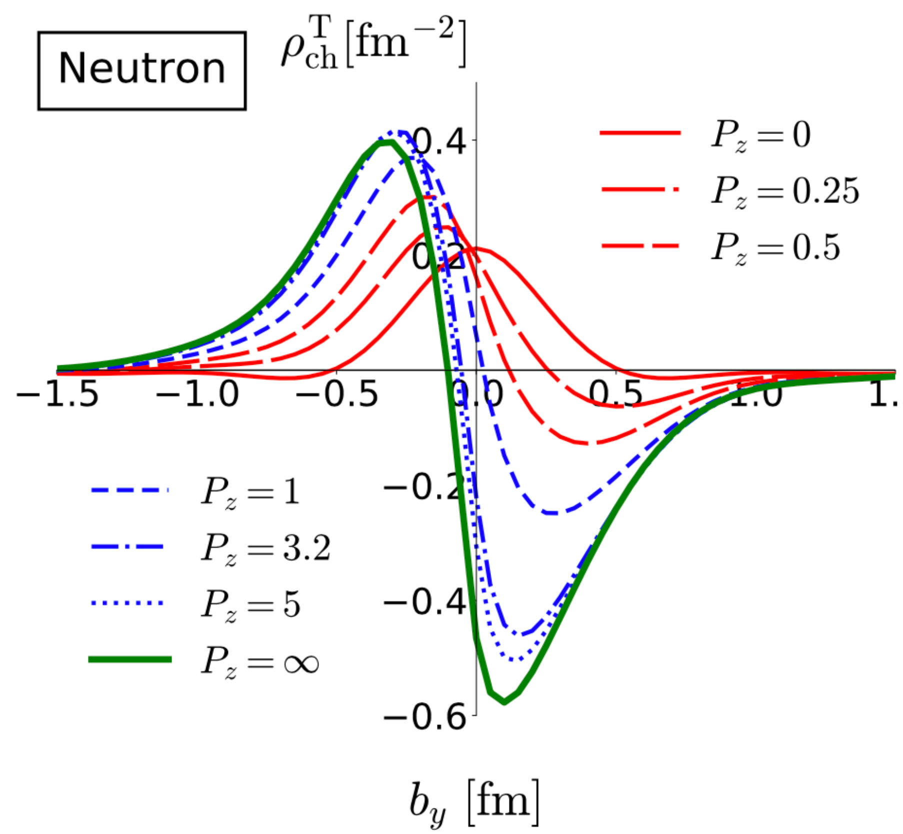
$$\boldsymbol{E}' = \gamma(\boldsymbol{v} \times \boldsymbol{B})$$

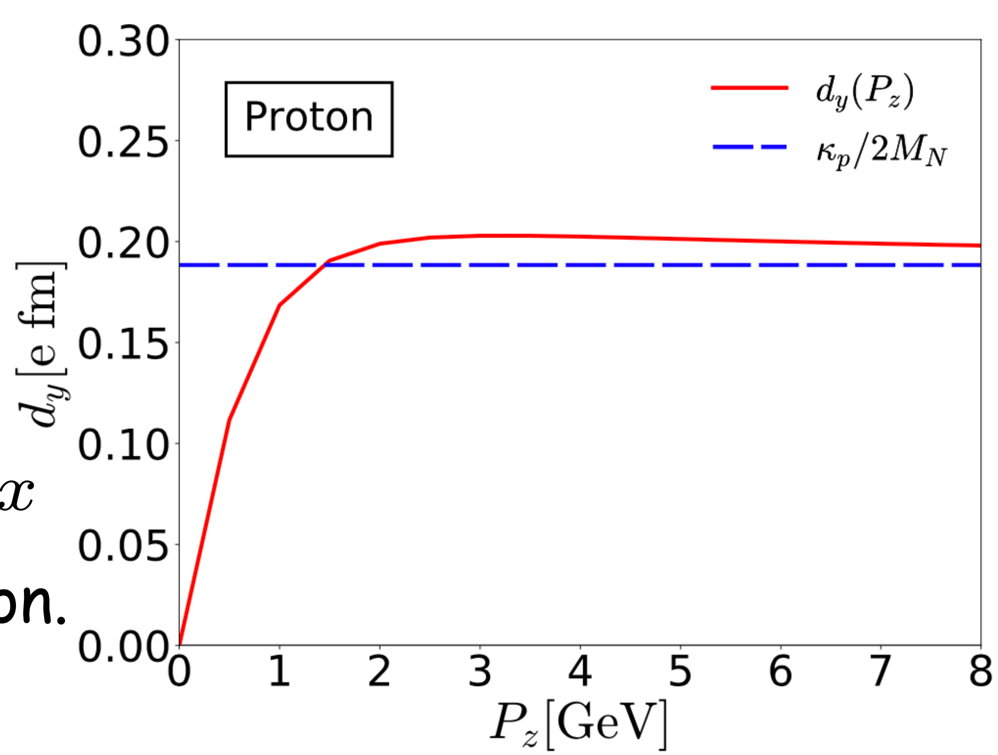
Induced electric dipole moment

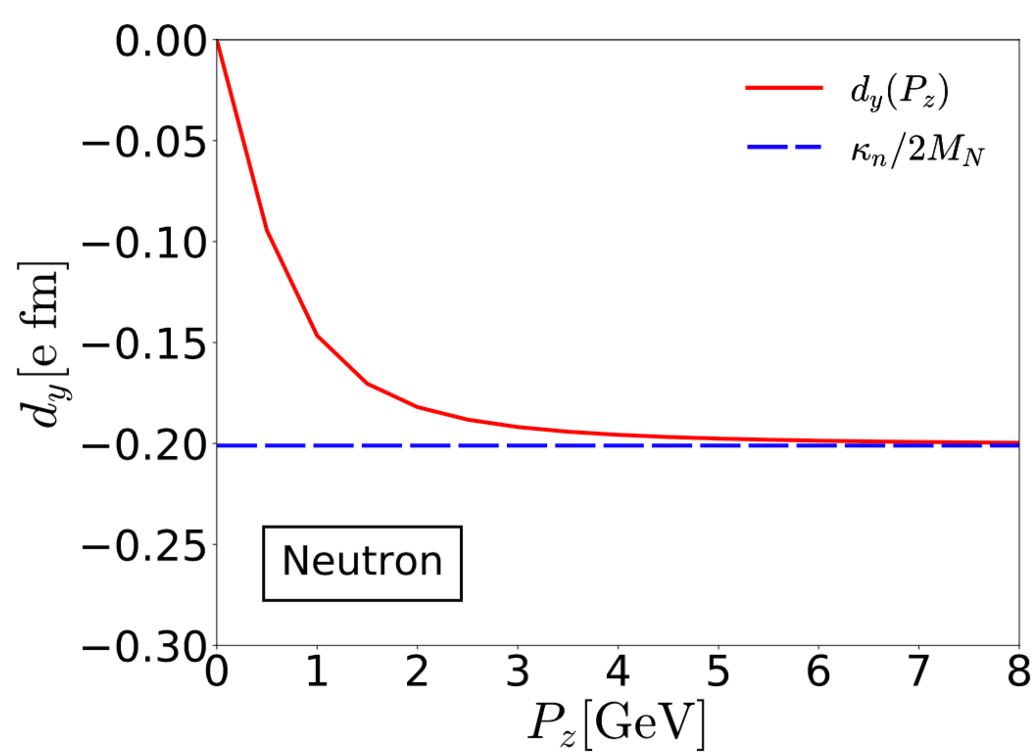


Polarized in x direction.



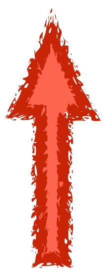






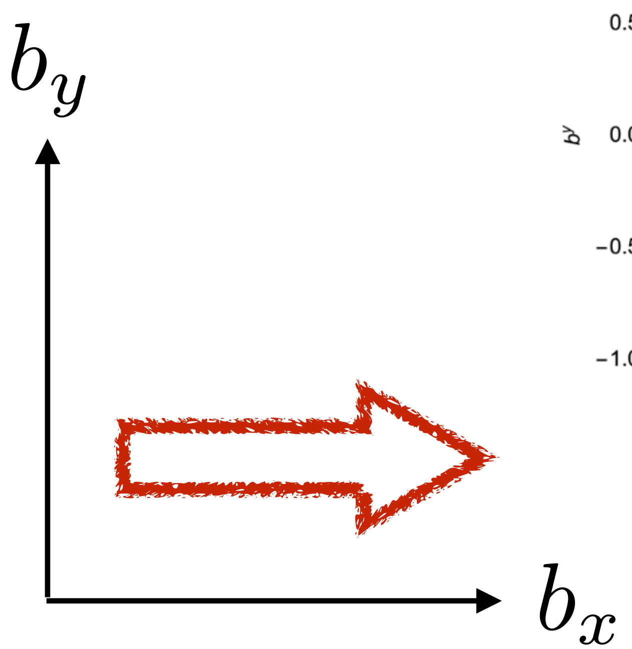
J. Y. Kim & HChK, ArXiv 2106.10986

Charge distributions of the tr. polarized proton

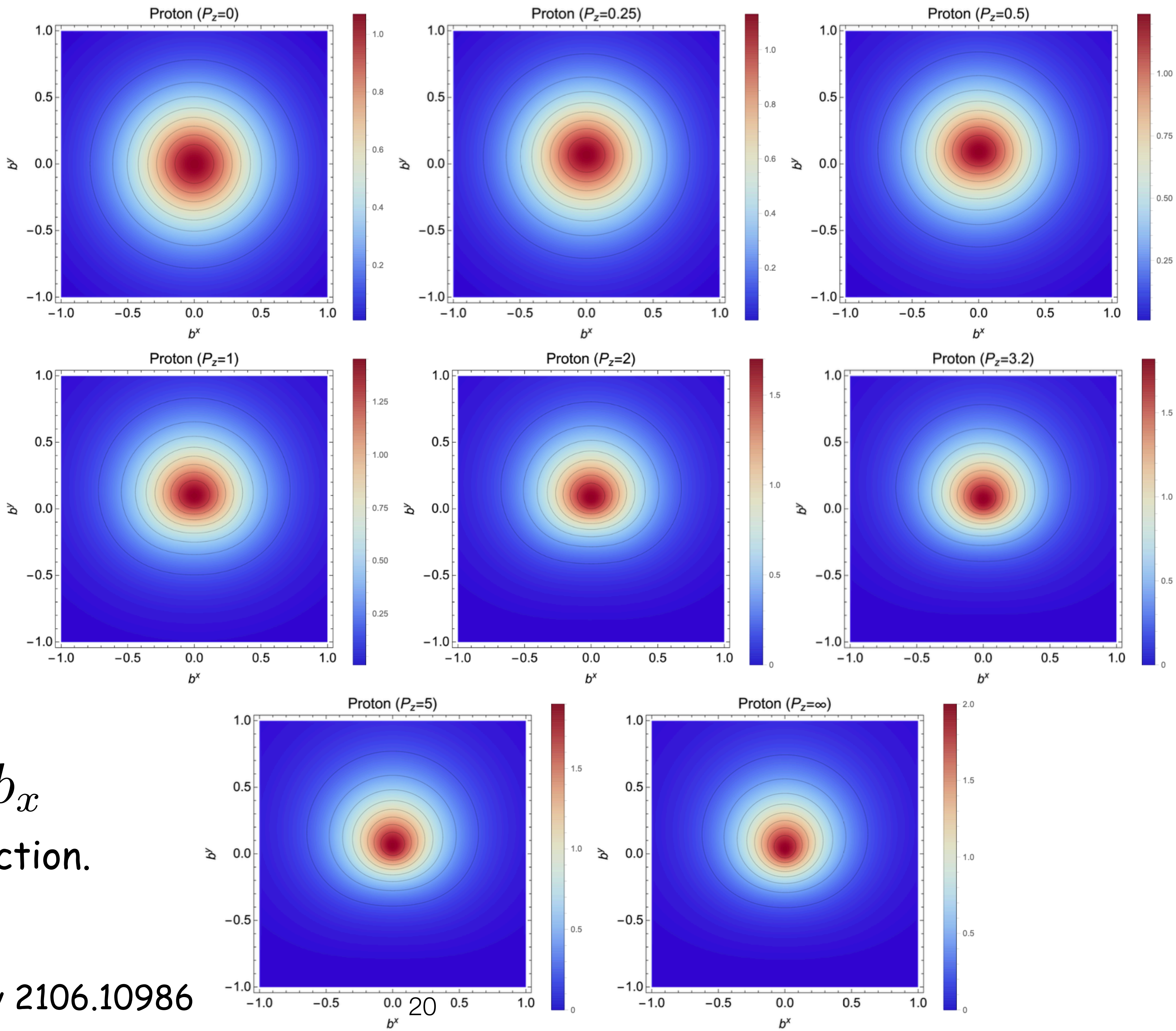


$$\boldsymbol{E}' = \gamma(\boldsymbol{v} \times \boldsymbol{B})$$

Induced electric dipole moment

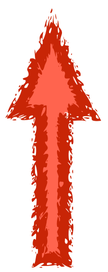


Polarized in x direction.



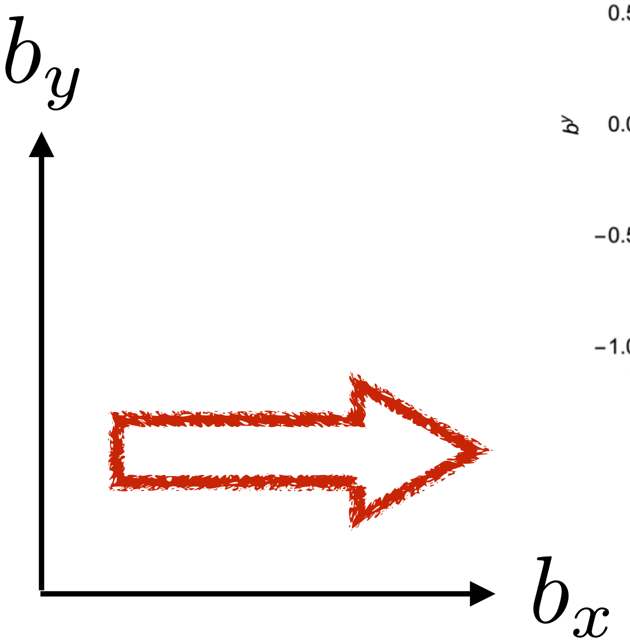
J. Y. Kim & HChK, ArXiv 2106.10986

Charge distributions of the tr. polarized neutron

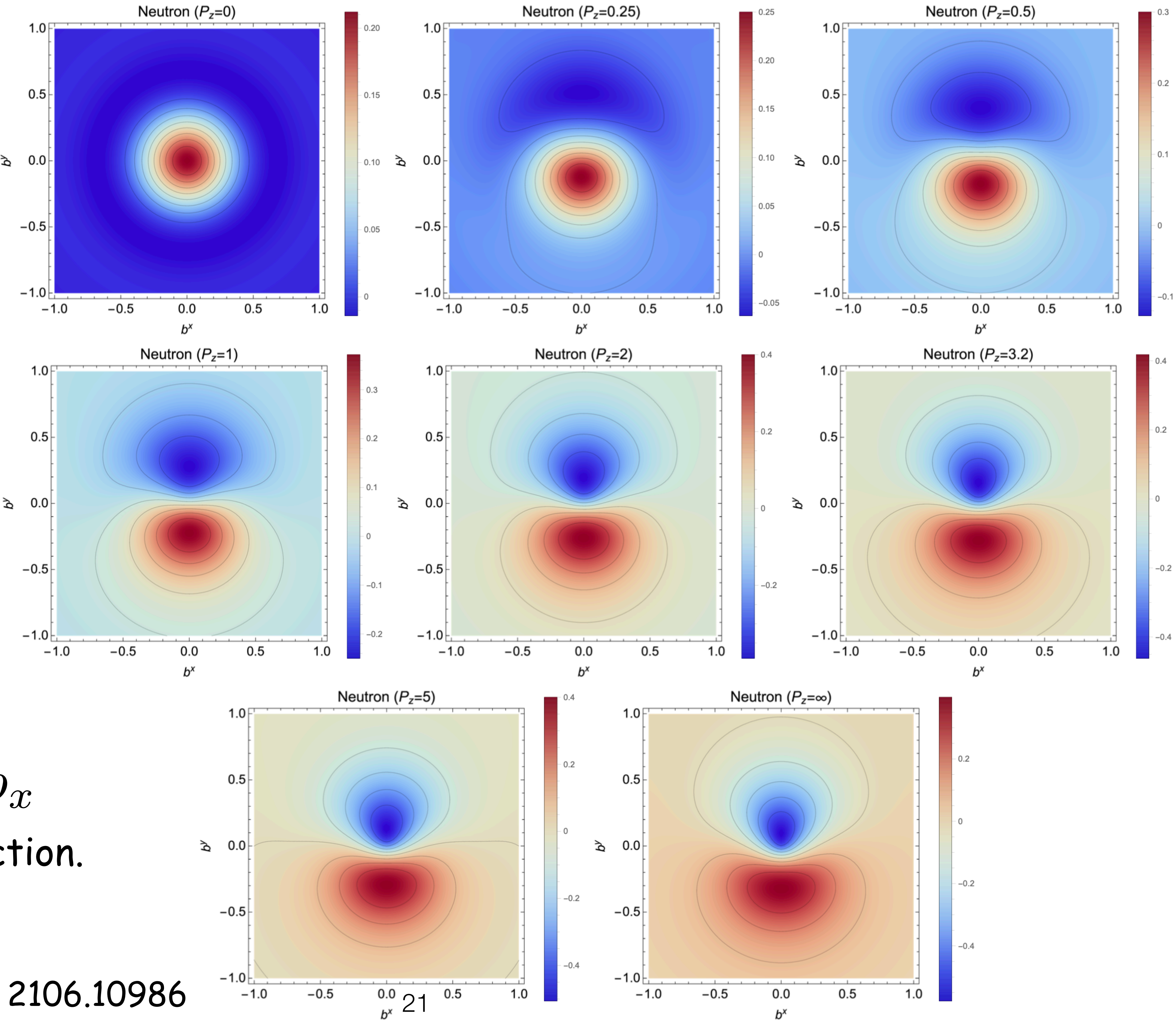


$$E' = \gamma(v \times B)$$

Induced electric dipole moment



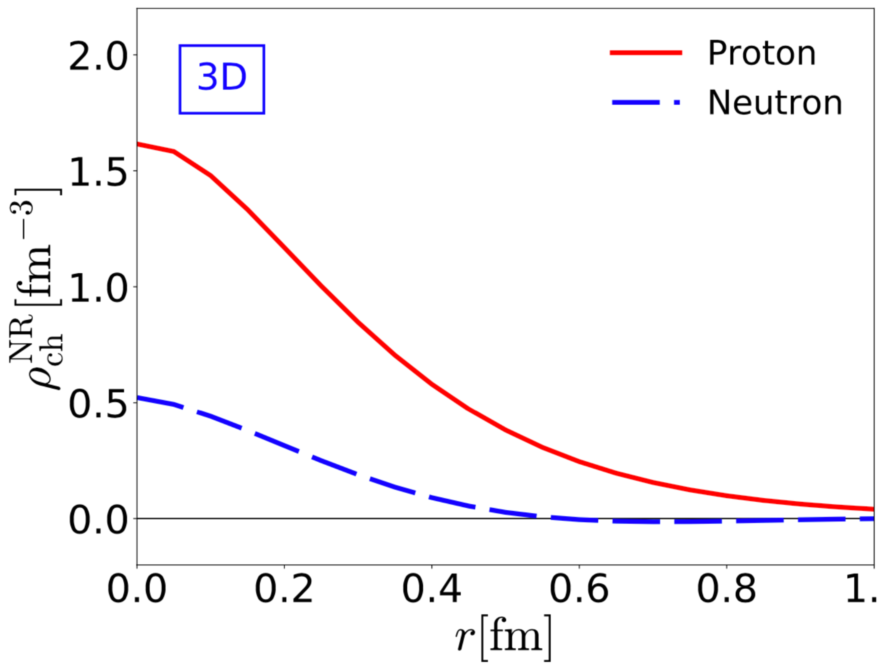
Polarized in x direction.

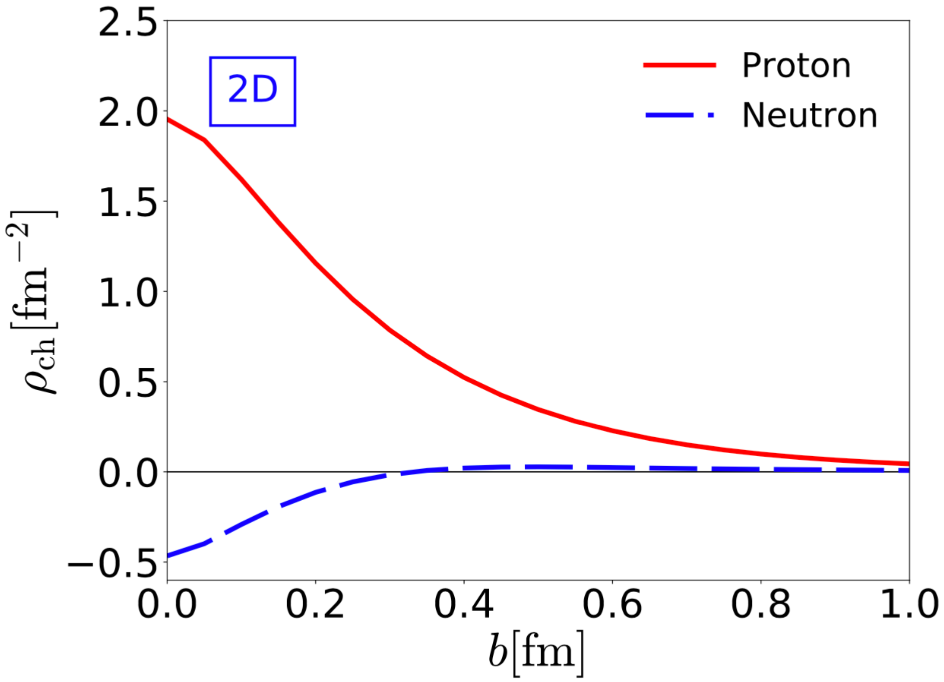


J. Y. Kim & HChK, ArXiv 2106.10986

Abel transforms of charge & magnetization distributions

$$\rho_{\rm ch}(b) + \frac{1}{4M_N^2}\partial_\perp^2 \rho_{\rm M}(b) = \int_b^\infty \frac{2r\,dr}{\sqrt{r^2-b^2}}\rho_{\rm ch}^{\rm NR}(r), \quad \rho_{\rm ch}(b) + \rho_M(b) = \int_b^\infty \frac{2r\,dr}{\sqrt{r^2-b^2}}\rho_{\rm M}^{\rm NR}(r)$$

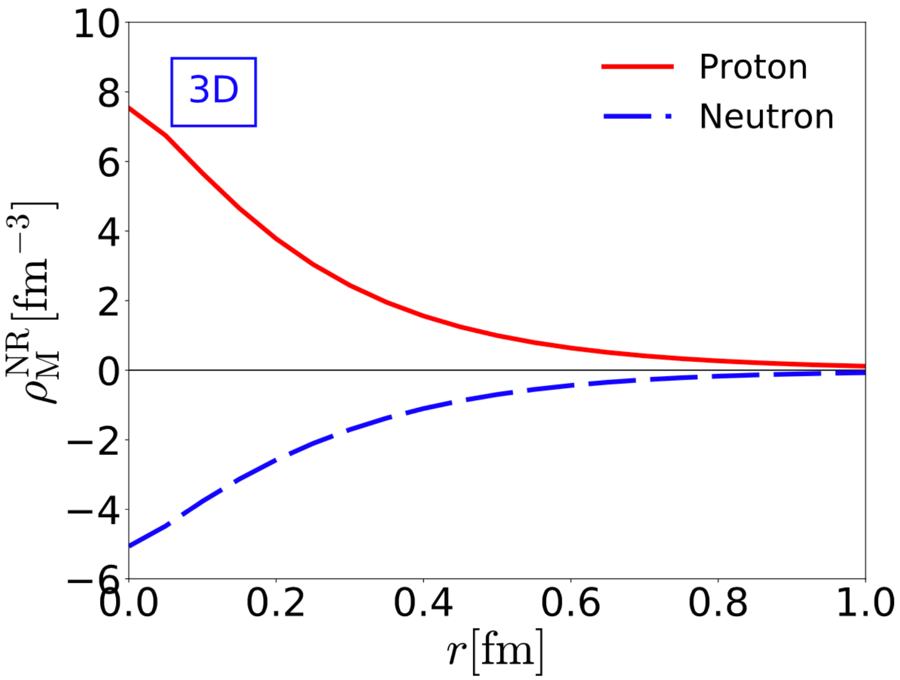


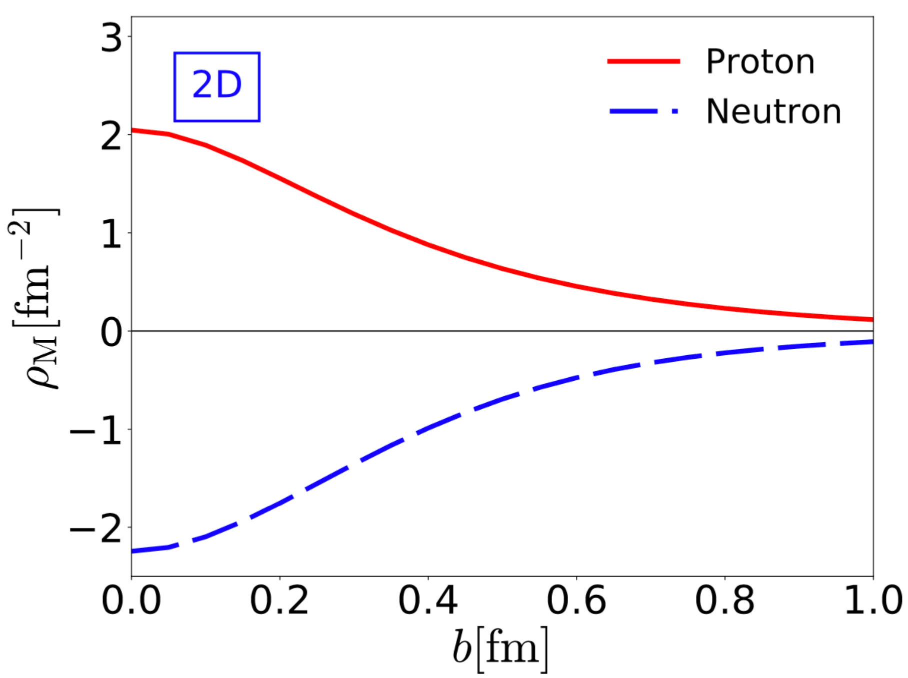


From 3D at rest

to 2D in IMF

Abel transforms





J. Y. Kim & HChK, ArXiv 2106.10986

Summary & Conclusions

2D transverse structure of the Nucleon

- The nucleon is *per se* a relativistic particle.
- The 3D BF distributions have only a quasi-probabilistic meaning in a Wigner sense.
- Abel transform makes 3D BF densities equivalent to 2D IMF ones.

  In 2D, we restore quantum mechanically probabilistic meaning of the densities.
- **The 3D global & local stability conditions are all conveyed to the 2D ones!**
- **3D distributions in BF still provide physical intuitions, even though they have only a quasi-probabilistic meaning.**
- Higher-spin baryons are under investigation by using the Radon transform.

Though this be madness, yet there is method in it.

Hamlet Act 2, Scene 2

by Shakespeare

Thank you very much for the attention!