Toward a solution of the gap equation in Minkowski space

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In collaboration with Giovanni Salmè

Based on:
Eur.Phys.J.C 81 (2021) 1, 34

Disclaimer

- Rather technical talk as I was asked to speak specifically on our last paper
  $\rightarrow$ I try to make it as lively as possible
- For a review of application to observables, see *e.g.* P. Maris talk in December and topical review article on DSE-BSE
- I tried to come back on some points mentioned during the discussion following P. Maris talk (I could not attend to)

Introduction

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- Perturbation theory: a powerful tool to describe scattering in QFT
  - anomalous magnetic moment of the electron (multi-loop pQED)
  - structure functions scaling violations (major pQCD result)

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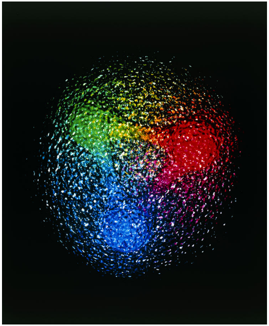

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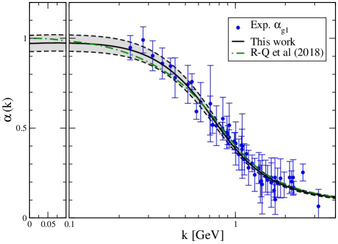

figure from Z.F. Cui *et al.*, Chin.Phys.C 44 (2020) 8, 083102

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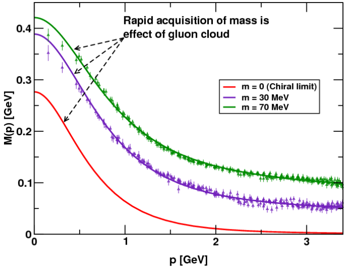

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- Different approaches to non-perturbative physics
  - Lightfront Hamiltonian
  - ADS/QCD
  - Lattice QCD
  - Dyson-Schwinger/Bether Salpeter equations

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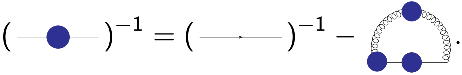
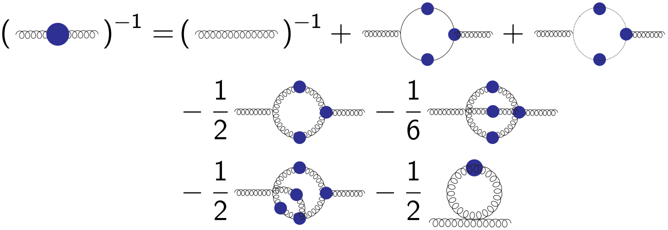
DSEs relate the N-point functions of a given QFT among each other.

Quark case:



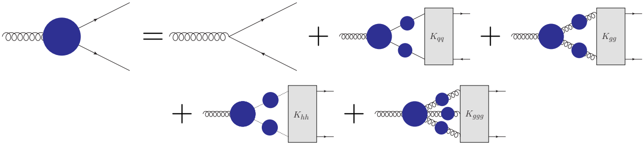
Ghost case:

Gluon case:



Coupled to higher N-point functions $\rightarrow$ infinite set of equations

- 3-point functions also obey their own DSEs:



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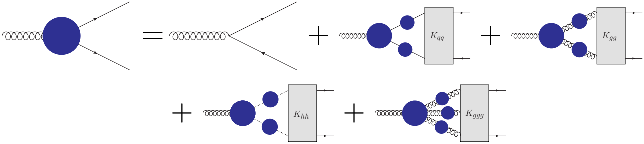
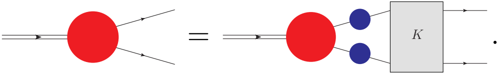


- Higher point functions present poles at bound-states masses
  $\rightarrow$ residues yield the Bethe-Salpeter wave functions



Vertex and Bethe-Salpeter equations

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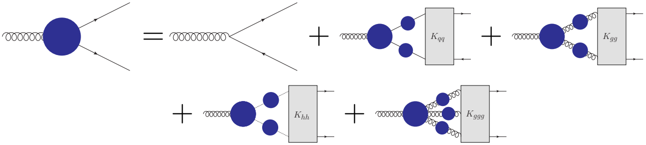

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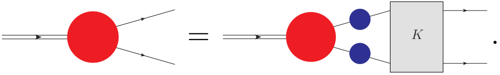

- DSE-BSE formalism provides a way to study bound states
- Approximations are required to close the system (truncations)

Euclidean vs. Minkowski Space

- Usually, DSE-BSE are solved numerically in Euclidean space
  - External momenta are continued in the complex plane
  - Can reveal itself numerically challenging
  - A direct access to lighfront quantities can be difficult

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K. Kusaka *et al.*, Phys. Rev. D56, 5071 (1997)
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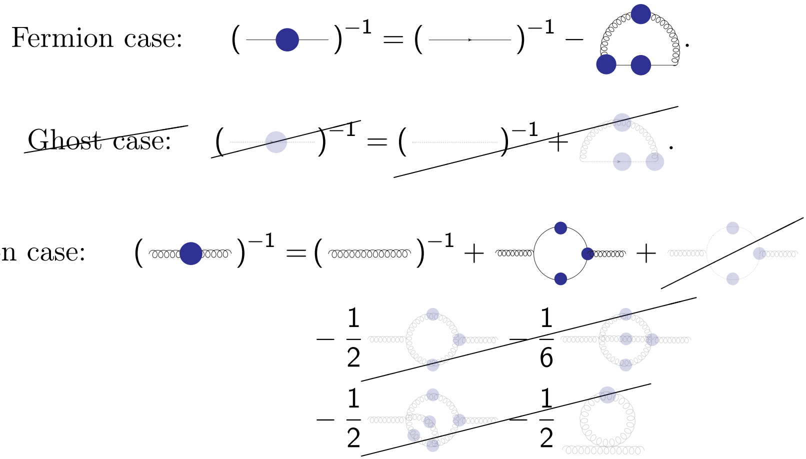
Our idea

Can we "export" to Minkowski space the theoretical progresses achieved in euclidean one?

Two-point functions in Minkowski space

- Abelian DSE are much simpler than QCD ones:



- Abelian theories presents interesting properties that can be exploited.

- Fermion Case:

$$iS_R^{-1}(p,\zeta^2) = \not{p} - m - \Sigma_R(p,\zeta^2)$$
$$\Sigma_R(p,\zeta^2) = \not{p}\mathcal{A}_R(p,\zeta^2) + \mathcal{B}_R(p,\zeta^2)$$

- Photon case in covariant gauges:

$$D_R^{\mu\nu}(q,\zeta^2) = -i\frac{T^{\mu\nu}(q)}{(q^2+i\epsilon)\left(1+\Pi_R(q^2,\zeta^2)\right)}$$
$$T^{\mu\nu}(q) = \eta^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}$$

- The Källen-Lehmann representation is a key property of the propagator:

$$S_R(p,\zeta^2) = i\mathcal{R}_S \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} + i\int_{s_{th}}^{\infty} \frac{\not{p}\sigma_v(s,\zeta) + \sigma_s(s,\zeta)}{p^2 - s + i\epsilon}$$

$$D_R^{\mu\nu}(q,\zeta) = -iT^{\mu\nu}(q)\left(\frac{\mathcal{R}_D}{q^2 + i\epsilon} + \int_{s_{th}^p}^{\infty} \frac{\sigma_\gamma(s,\zeta^2)}{q^2 - s + i\epsilon}\right)$$

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What about self energies?

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$$Z_2(\zeta)\Sigma(p,\Lambda) = \int_0^\infty [\mathrm{d}s]_\Lambda \frac{\not{p}\rho_A(s,\zeta^2) + \rho_B(s,\zeta^2)}{p^2 - s + i\epsilon}$$

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allowing the algebraic manipulation of the momenta.

- Such a representation is proven to hold at **all order** of perturbation theory

Advantages

- Crucially, the ρ are **unique** and **independent** of the momenta.
- Such relations can be generalised to higher N-point functions, such as the vertex function (N=3), the scattering amplitude (N=4)...

see seminar by P. Maris in December

$$\Gamma(k, P, \Lambda) = \int_{-1}^{1} [\mathrm{d}z]_\Lambda \int_0^\infty [\mathrm{d}\beta]_\Lambda \frac{\rho(z,\beta)}{\beta - \left(k + \frac{z}{2}P\right)^2}$$

Nakanishi representation II

Advantages and drawback

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Drawback

- All order of perturbation theory $\neq$ non-perturbatively
  $\rightarrow$ weaker than the "Källen-Lehmann" proof (at least in Abelian case)
  However:
  - assume that the Nakanishi representation hold non-perturbatively
  - price to pay might be that ρ are not smooth functions

Nakanishi representation III

Previous studies

- Previously used for the self-energy through direct computations

  V. Sauli, J. Phys., 2004, G30, 739-758
  ...

- And also in the case of the vertex function
  - Using simple algebraic ρ functions

    C. Mezrag *et al.*, PLB 741 (2015) 190-196
    N. Chouika *et al.*, PLB 780 (2018) 287-293
    ...

  - attempts of direct calculations in Minkowski space

    J. Carbonell *et al.*, Eur. Phys. J., 2017, C77, 58
    J. H. Alvarenga Nogueira *et al.*, PRD 100, 2019, 016021
    ...

  - or trying to solve the inverse "Nakanishi problem" through Bayesian techniques in euclidean space

    F. Gao *et al.*, PLB770 551-555 (2017)

We will look for a direct computation through the DSE

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  - Renormalisation constant given in terms of Nakanishi weights:

$$Z_2(\zeta = m, \Lambda) = 1 + \int_{m^2}^{\infty} [\mathrm{d}s]_\Lambda \frac{\rho_A(s, \zeta)}{m^2 - s + i\epsilon}$$

- Exploit our various expressions for S_R (and $D_R^{\mu\nu}$)

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- Using the Sokhotski–Plemelj formula for $p^2 > m^2$

$$\sigma_v(p^2,\zeta^2) = \frac{-1}{\pi}\Im\left(\frac{\left(1-\mathcal{A}_R(p^2,\zeta^2)\right)}{p^2\left(1-\mathcal{A}_R(p^2,\zeta^2)\right)^2-\left(m+\mathcal{B}_R(p^2,\zeta^2)\right)^2}\right)$$

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Divide & rule strategy

First set of equations

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- This yields 3 equations in our system of 6 unknown $(\sigma_v, \sigma_s, \sigma_\gamma, \rho_A, \rho_B, \rho_\gamma)$

- Exploit our various expressions for S_R (and $D_R^{\mu\nu}$)
- Using the Sokhotski–Plemelj formula for $p^2 > m^2$
- It yields the σ as (non-linear) functions of the ρ

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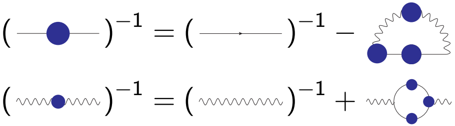
- 3 more are provided by the gap equations.

Abelian DSEs in Minkowski space

In Search of Lost Vertex



$$\Sigma_R(\zeta;p) = -\,iZ_1(\zeta,\Lambda)\ e_R^2 \int_\Lambda \frac{\mathrm{d}^4k}{(2\pi)^4}\ \gamma^\beta\ S_R(\zeta,k)$$
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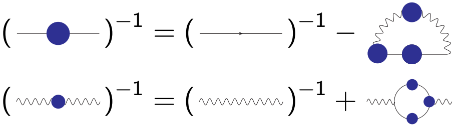
$$\times\left\{\frac{1}{q^2}\Gamma^\nu_R(\zeta,k,q)\ S_R(\zeta,k-q) - \frac{1}{\zeta_p^2}\left[\Gamma^\nu_R(\zeta,k,q)\ S_R(\zeta,k-q)\right]_{q^2=\zeta_p^2}\right\}\Big]$$

Abelian Gap equations



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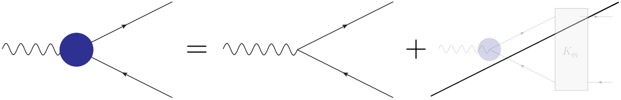
The last thing to get under control is the vertex

- First thing one could look at: neglecting higher point functions:



- Independent of the momenta $\rightarrow$ all momenta degrees of freedom of the self-energy can be algebraically manipulated
- It works well for the fermion self-energy

Tree-level vertex approximation

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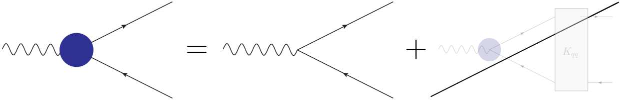

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Major issues

- The vacuum polarisation tensor $\Pi^{\mu\nu}$ is *not* transverse anymore
- Quadratic divergences (proportional to $\eta^{\mu\nu}$) do not vanish as they should

Ward-Takahashi Identities

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&(k_2 - k_1)_\mu \Gamma^\mu_R(k_2, k_1, \zeta) = iS_R^{-1}(k_2, \zeta) - iS_R^{-1}(k_1, \zeta) \\
= &\underbrace{(\not{k}_2 - \not{k}_1)}_{\text{tree-level vertex}} - (\not{k}_2 \mathcal{A}_R(k_2) - \not{k}_1 \mathcal{A}_R(k_1)) - (\mathcal{B}_R(k_2) - \mathcal{B}_R(k_1))
\end{aligned}
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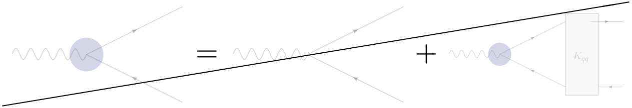
Need to build a vertex fulfilling all the required symmetry properties



Ball-Chiu vertex I
Exploiting WTI

- Instead of using the DSE to build the vertex, use directly the WTI:

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- The idea behind the use of the Ball-Chiu vertex:

$$\begin{aligned} \Gamma^\mu_R(k_2, k_1, \zeta) =& \Gamma^\mu_{BC}(k_2, k_1, \zeta) + \Gamma^\mu_T(k_2, k_1, \zeta) \\ =& \underbrace{\sum_{i=1}^{4} \lambda^\mu_i(k_2, k_1) F_i(k_2, k_1, \zeta)}_{\text{Fully determined by the WTI}} + \sum_{j=1}^{8} \tau^\mu_j(k_2, k_1) F^T_j(k_2, k_1, \zeta) \end{aligned}$$

J. Ball and T.-W. Chiu, PRD 22 (1980) 2550

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J. Ball and T.-W. Chiu, PRD 22 (1980) 2550

Ball-Chiu approximation: $\Gamma_R^\mu(k_2, k_1, \zeta) = \Gamma_{BC}^\mu(k_2, k_1, \zeta)$

Ball-Chiu Vertex II

Obtaining a closed system

- Detailed structure of the BC vertex:

$$\lambda_1^\mu = \frac{\gamma^\mu}{2} \quad \rightarrow \quad F_1(k_2,k_1,\zeta) = 2 - \mathcal{A}_R(k_2^2,\zeta^2) - \mathcal{A}_R(k_1^2,\zeta^2)$$

$$\lambda_2^\mu = -\frac{\not{k}_1 + \not{k}_2}{2}(k_1+k_2)^\mu \quad \rightarrow \quad F_2(k_2,k_1,\zeta) = \frac{\mathcal{A}_R(k_2^2,\zeta^2) - \mathcal{A}_R(k_1^2,\zeta^2)}{k_2^2 - k_1^2}$$

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$$\mathcal{A}_R(p^2, \zeta^2) = (\zeta^2 - p^2) \int_0^\infty \mathrm{d}s \frac{\rho_A(s, \zeta^2)}{(p^2 - s + i\epsilon)(\zeta^2 - s)}$$

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we obtain a vertex which

- **by construction** fulfil the WTI $\rightarrow \Pi_R^{\mu\nu}$ is transverse and finite
- depends **only** on the fermion self energy $\rightarrow$ the system is closed
- allow algebraic manipulation of the momenta degrees of freedom

Ball-Chiu Vertex III

Renormalisation

$$\Pi_R(\zeta,\zeta_p;q) = -\, iZ_1(\zeta,\Lambda)\ \frac{4}{3}\ e_R^2 \int_\Lambda \frac{d^4k}{(2\pi)^4}\ \mathcal{P}_{\mu\nu}\frac{1}{4}\mathrm{Tr}\Big[\gamma^\mu S_R(\zeta,k) \times \left\{\frac{1}{q^2}\Gamma^\nu_{BC}(\zeta,k,q)\ S_R(\zeta,k-q) - \frac{1}{\zeta_p^2}\left[\Gamma^\nu_{BC}(\zeta,k,q)\ S_R(\zeta,k-q)\right]_{q^2=\zeta_p^2}\right\}\Big]$$

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$\Sigma_R(\zeta;p)$ becomes logarithmically divergent !

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$\Sigma_R(\zeta;p)$ becomes logarithmically divergent !

Where do these new singularities come from?

$$F_1(k,p,\zeta) = 2 - \mathcal{A}_R(k^2,\zeta^2) - \underbrace{\mathcal{A}_R(p^2,\zeta^2)}_{\to 0 \text{ when } p^2\to\zeta^2}$$

→ some logarithmic singularities are not subtracted by our renormalisation procedure

Renormalisation and BC vertex

- Is this a problem with our renormalisation condition $\mathcal{A}_R(\zeta^2, \zeta^2) = 0$?
  $\rightarrow$ no, if $\mathcal{A}_R(\zeta^2, \zeta^2) \neq 0$ the singularities do not compensate

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D. Curtis and M. Pennington, PRD 42, 1990, 4165-4169

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- Solution to the issue is not unique $\rightarrow$ constraints on the purely transverse components to fulfil multiplicative renormalisation

A. Bashir *et al.*, PRC 85, 045205 (2012)

- Lesser known transverse WTIs ($q = k - p, t = k + p$):

$$
\begin{aligned}
q_\mu \Gamma_\nu(k,p) - q_\nu \Gamma_\mu(k,p) =& S^{-1}(p)\sigma_{\mu\nu} + \sigma_{\mu\nu} S^{-1}(k) \\
& + 2im\Gamma_{\mu\nu}(k,p) + t_\lambda \epsilon_{\lambda\mu\nu\rho}\Gamma^A_\rho(k,p) + A^V_{\mu\nu}(k,p) \\
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$$

Y. Takahashi, 1985, Print-85-0421 (Alberta)
K.-I. Kondo, Int. J. Mod. Phys.A12, 5651 (1997)
H.-X. He, arXiv:hep-th/0202013
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S.-X. Qin *et al.*, Phys.Lett.B 722 (2013) 384-388

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  - Take advantage of the curl of the vertex ($\nabla \times \Gamma$)

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- Some comments:
  - Take advantage of the curl of the vertex ($\nabla \times \Gamma$)
  - Coupled equations between vector and axial-vector vertices
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q_\mu \Gamma^A_\nu(k,p) - q_\nu \Gamma^A_\mu(k,p) =& S^{-1}(p)\gamma_5\sigma_{\mu\nu} + \gamma_5\sigma_{\mu\nu} S^{-1}(k) \\
&+ t^\lambda \epsilon_{\lambda\mu\nu\rho}\Gamma^\rho(k,p) + V^A_{\mu\nu}(k,p)
\end{aligned}
$$

Y. Takahashi, 1985, Print-85-0421 (Alberta)
K.-I. Kondo, Int. J. Mod. Phys.A12, 5651 (1997)
H.-X. He, arXiv:hep-th/0202013
H.-X. He, Commun. Theor. Phys.46, 109 (2006)
H.-X. He, Int. J. Mod. Phys.A22, 2119 (2007)
S.-X. Qin *et al.*, Phys.Lett.B 722 (2013) 384-388

- Some comments:
  - Take advantage of the curl of the vertex ($\nabla \times \Gamma$)
  - Coupled equations between vector and axial-vector vertices
  - Involve higher-point functions $\Gamma_{\mu\nu}$, $A^V_{\mu\nu}$ and $V^A_{\mu\nu}$
  - Fully constrain the vertex in terms of the self energy for 1+1 QED

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The Qin vertex I
Definition

- In QED 3+1 the tWTI fully constrain the transverse vertex Γ_μ^T relating it to higher N-point function
  - for $j \in (1, 2, 4, 6, 7)$, F_j^T solely depends on higher N-point functions
  - for $j \in (3, 5, 8)$, F_j^T depends also on the fermion self-energy

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$$F_5^T(k,p) = \frac{\mathcal{B}_R(k^2) - \mathcal{B}_R(p^2)}{(k^2 - p^2)}$$

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This is *not* a standard vertex truncation:

- it does *not* involve any "graph" discussion (ladder, cross-ladder, . . .)
- it purely relies on *symmetry* considerations

- Impact of the Qin vertex :
  - $F_3^T(k,p)$ and $F_8^T(k,p)$ **together** cure the BC vertex
    $\rightarrow$ we get both Σ_R and Π_R finite !
  - $F_5^T(k,p)$ is not considered here $\rightarrow$ it might create troubles

A. Bashir *et al.*, PRC 85, 045205 (2012)

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- The Qin vertex is a long way from the tree-level one
- The symmetries are merciless $\rightarrow$ they determine the truncation

Abelian DSEs in Minkowski space

Coupled equations for Nakanishi weights

Back to the Gap equations

$$\Pi_R(\zeta,\zeta_p;q) = -\,iZ_1(\zeta,\Lambda)\ \frac{4}{3}\ e_R^2 \int_\Lambda \frac{d^4k}{(2\pi)^4}\ \mathcal{P}_{\mu\nu}\frac{1}{4}\mathrm{Tr}\Big[\gamma^\mu S_R(\zeta,k)$$

$$\times\left\{\frac{1}{q^2}\Gamma_Q^\nu(\zeta,k,q)\ S_R(\zeta,k-q) - \frac{1}{\zeta_p^2}\left[\Gamma_Q^\nu(\zeta,k,q)\ S_R(\zeta,k-q)\right]_{q^2=\zeta_p^2}\right\}\Big]$$

$$\Sigma_R(\zeta;p) = -\,iZ_1(\zeta,\Lambda)\ e_R^2\ \int_\Lambda \frac{\mathrm{d}^4k}{(2\pi)^4}\ \gamma^\beta\ S_R(\zeta,k)$$

$$\times\left\{D^R_{\beta\alpha}(\zeta,p-k)\ \Gamma_Q^\alpha(\zeta;k,p) - \Big[D^R_{\beta\alpha}(\zeta,p-k)\ \Gamma_Q^\alpha(\zeta;k,p)\Big]_{p^2=\zeta^2}\right\}$$

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- Straightforward, albeit tedious, steps:
  - replace Π_R, $\mathcal{A}_R$, $\mathcal{B}_R$ with their Nakanishi representations
  - replace S_R and D_R with their Källen-Lehmann representations
  - reduce the rhs to the same denominator through the Feynman trick
  - integrate over k for p and q *spacelike*

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  - reduce the rhs to the same denominator through the Feynman trick
  - integrate over k for p and q *spacelike*
- Less straightforward steps:
  - rearrange the rhs and perform the proper change of variable to obtain the same structure of external momentum than the lhs
  - finally use the unicity of the Nakanishi representation to identify the gap equation fulfilled by the weight

Change of Variable

- The Nakanishi representations yield:

$$\Pi_R(\zeta,\zeta_p;q) = (\zeta_p^2 - q^2)\int_{s_{th}^p}^{\infty} \mathrm{d}s \frac{\rho_\gamma(s,\zeta^2)}{(q^2 - s + i\epsilon)(\zeta_p^2 - s)}$$

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  - potentially 6 unbounded integration variables (ρ and σ)
  - various number of Feynman parameters

$\rightarrow$ many integration parameters needs to be rearranged

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- one needs to get the *same* denominator power
  - achieve through approriate change of variable and integration on specific variables
  - or obtained through integration by parts

An example of result

$$\Theta\left(y - s_{th}\right)\ \rho_A(y,\zeta) = \frac{3}{(4\pi)^2}\ e_R^2 \lim_{\Lambda\to\infty}\ Z_1(\zeta,\Lambda)\int_0^\infty\ d\omega\ \bar{\sigma}_\gamma(\omega,\zeta,\zeta_p,\Lambda)\ \int_0^1 d\xi\ \int_0^\infty\ ds'$$

$$\Bigg\{\bar{\sigma}_V(s',\zeta,s'_{th},\Lambda)\ \Bigg[\xi\ \Theta\Big(y\xi(1-\xi) - \xi\omega - (1-\xi)s'\Big)$$

$$- \int_0^{1-\xi} dt \Theta\Big(yt(1-t) - \xi\omega - ts'\Big)\Bigg] + \bar{\sigma}_V(s',\zeta,s'_{th},\Lambda)$$

$$\times\Bigg[\ \int_{s_{th}}^\infty\ ds\ \rho_A(s,\zeta,\Lambda)\ \mathcal{C}_{AV}^{(0)}(\zeta,\omega,s,s',\xi,y)$$

$$+ y\ \int_{s_{th}}^\infty\ ds\ \rho_A(s,\zeta,\Lambda)\ \mathcal{C}_{AV}^{(1)}(\zeta,\omega,s,s',\xi,y)\Bigg]$$

$$- y\bar{\sigma}_S(s',\zeta,s'_{th},\Lambda)\int_0^{1-\xi} dt \int_0^{1-\xi-t} dw$$

$$\times\ \int_{s_{th}}^\infty ds\ \rho_B(s,\zeta,\Lambda)\Delta'\left[y - s + \frac{s\mathcal{A}_4(t,w) - \xi\omega - ts' - ws}{\mathcal{A}_4(t,w)}\right]\Bigg\}$$

$$\Theta\Big(y-s_{th}\Big)\rho_A^{(1)}(y,\zeta) = -\frac{e_R^2}{2(4\pi)^2}\frac{1}{\zeta_p^2 y^2}\Theta(y-m^2)\Bigg\{\Theta\Big[[m+\zeta_p]^2-y\Big]\Big(y-m^2\Big)^3$$

$$+\Theta\Big[y-[m+\zeta_p]^2\Big]\ \Big(y-m^2\Big)^3\Big[1-f(y,\zeta,\zeta_p^2)\Big]\Bigg\},$$

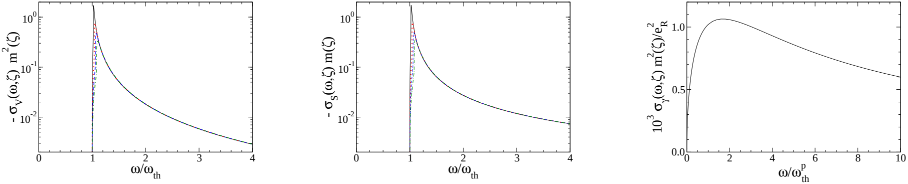
$$f(y,\zeta^2,\zeta_p^2) = \sqrt{1-\zeta_p^2\,\frac{2y+2m^2-\zeta_p^2}{(y-m^2)^2}}\left[1+\zeta_p^2\,\frac{y+m^2-2\zeta_p^2}{\Big(y-m^2\Big)^2}\right],$$

$$\Theta\Big(y-s_{th}\Big)\ \rho_B^{(1)}(y,\zeta) = -\frac{3e_R^2}{(4\pi)^2}\Theta\Big[y-[m+\zeta_p]^2\Big]\frac{m}{y}\sqrt{[y-m^2-\zeta_p^2]^2-4m^2\zeta_p^2}$$

$$\Theta(y-s^p_{th})\ \rho_\gamma^{(1)}(y,\zeta) = -\frac{e_R^2}{3(2\pi)^2}\Theta(y)\Theta(y-4m^2)\Big(1+2\frac{m^2}{y}\Big)\ \sqrt{1-4\frac{m^2}{y}}$$

- Expected behaviour for $\zeta_p \to 0$
- Expected behaviour for $y \to \infty$

Recovering the 1-loop results II
Källen-Lehmann weights



- The Källen-Lehmann weights behave as expected:
  - rapid increase from threshold, reach maximum and slowly go to zero at infinity
  - for fermions, IR divergences noticeable

Summary

Truncation of the gap equations

- Getting a workable and consistent truncation is **not** easy
- In particular, the bare vertex cannot be used in the photon case
- We learn a great deal on the impact of the symmetries on the interaction
- In the end, the symmetries leave us **no choice** but working with the Qin vertex as a "minimal" vertex

Minkowski space computation

- From the Qin vertex, Källen-Lehmann and Nakanishi representations allows us to handle the momenta algebraically
- We obtained 6 coupled and non-linear equations for six unknown functions
- We checked that we recover expected one-loop results

Perspectives

Scheme dependence

- Check whether things hold in the standard on-shell schemes
- Modification should be of a finite amount despite mixing $\mathcal{A}$ and $\mathcal{B}$

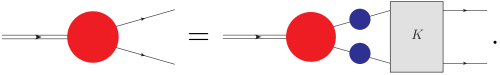
Gauge dependence

- Open question: does the framework hold in the lightcone gauge ?
- Algebraic momentum dependence: we can expect that yes
  $\rightarrow$ good news to compute lightcone quantities (PDFs, GPDs, . . .)

Numerical effort

- Is the framework workable numerically speaking?
- All final integral are finite, but it does not mean the system will converge toward a solution

- Naive idea: plug the results of our equations in the BSE:



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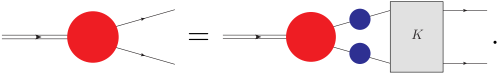

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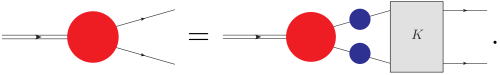

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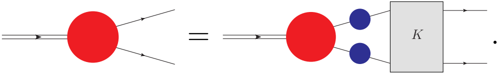

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Once again, the symmetries will dictate the structure of the kernel
$\rightarrow$ this needs to be worked out in our case

Extension to QCD

A speculative slide

Modification in QCD

- KL representation not proved but compatible with lattice results

  D. Binosi and R.-A. Tripolt, PLB 801 (2020) 135171

- Nakanishi → pQCD working at large p, so representation valid with modification of the singularities (*e.g.* complex conjugate poles)?
- WTI are replaced by STI. Non-abelian BC vertex available (quarks-gluon and 3-gluons cases).

  A.C. Aguilar *et al.*, PRD 98 (2018) 1, 014002
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Extension to QCD relies on progresses on the gauge constraints on the 3-point and 4-point functions entering the gap equations.

Thank you for your attention