# PREPARING THE PHYSICS QUALIFYING EXAM AT KNU 

# Electromagnetism and Quantum Mechanics 

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## Graduate School Education Committee

Department of Physics
Kyungpook National University

## [BK21 FOUR]

Education Research Center for Quantum Nature of Particles and Matter

## Have fun．．．

朱子曰，勿謂今日不學而有來日勿謂今年不學而有來年日月逝矣歲不我延嗚呼老矣是誰之衍少年易老學難成一寸光陰不可輕未覺池塘春草夢階前梧葉已秋聲陶淵明詩云，盛年 不重來 一日 難再晨

及時 當勉勵歲月不待人
筍子曰，不積步 無以至千里
不積小流 無以成江河
－明心寶鑑 勸學篇－

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ELECTROMAGNETISM

## CHAPTER 1

## EXERCISES

## EXERCISES

1 (Easy) Two infinite parallel planes carry equal but opposite uniform charge densities $\pm \sigma$. Find the field in each of the three regions: (i) to the left of both. (ii) between them, (iii) to the right of both.

2 (Medium) A toroidal coil with a very large number $N$ of turns is uniformly wound on a nonmagnetic core of square cross section with sides of length $a$. The inner and outer radii are $\rho_{1}$ and $\rho_{2}$ and the coil carries a current $I$.


Figure 1.1 Exercise 2
a) Show that the magnetic field $\vec{B}$ both inside and outside the toroid has only an azimuthal component, i.e., $B_{z}=B_{\rho} \equiv 0$ for two components of the field in cylindrical coordinates ( $\rho, \phi, z$ ) measured from the center of the toroid.
b) Derive an expressions for $B_{\phi}$ (both inside and outside the toroid) as a function of $\rho$.
c) Determine the self-inductance of the toroid.

3 (Medium) A long coaxial cable carries current $I$. The current flows down the surface of the inner cylinder (radius $a$ ), and back along the outer cylinder, radius $b$.


Figure 1.2 Exercise 3
a) Find the magnetic energy stored in a section of the cable of length $L$.
b) If the current now becomes an alternating current $I=I_{0} \cos (\omega t)$, explain why there will be an induced electric field in the longitudinal direction, pointing along the axis of the cylinders.
c) Assuming that the induced electric field goes to zero as $r \rightarrow \infty$, find $E(r, t)$, where $r$ is the perpendicular distance from the axis.
d) Find the magnetic field component that has an amplitude which increases in proportion to the square of the frequency $\omega$. Do not evaluate any nontrivial integrals.

4 (Medium) An infinite grounded conducting plate covers the $x y$ plane. A thin insulated rod of length $L$ made of non-conducting material lies on the $z$ axis. The rod extends from $z=D$ to $z=D+L$. A charge $Q$ is uniformly distributed along the rod.
a) Determine the surface charge density distribution induced on the conducting plate by the charge $Q$. Write your answer as a function of the variables $\rho=\sqrt{x^{2}+y^{2}}$ and $\phi=\arctan (y / x)$.
b) What is the total charge induced on the plate?
c) What is the electrostatic force on the rod due to the grounded plate?

5 (Medium) The vector potential $\vec{A}$ at point P due to an infinitesimal wire of length $d l$ carrying a current $I$ and located at the origin is

$$
\vec{A}_{\mathrm{inf}}=\frac{\mu_{0} I}{4 \pi} \frac{d \vec{l}}{r} .
$$

a) What is the vector potential $\vec{A}$ at point P due to a sinusoidally oscillating electric dipole of length $s$, oriented along the $z$ axis, and with end charge and alternating current given by


Figure 1.3 Figure of Exercise 4


Figure 1.4 Figure of Exercise 5

Assume that the distance from the origin to the point $P$ is much greater than the dipole size $(r \gg s)$, and remember to take into account the finite travel time of light signals.
b) Derive the magnetic field intensity $\vec{B}$. (Hint: use Cartesian coordinates.)
c) Assume that the average radiated power $S_{\text {av }}$ is proportional to $B^{2}$. How does it vary with $r, \theta$ at large distance?

6 (Medium) Two straight lines of length $L$ are separated by a distance $d(\ll L)$. They carry equal and opposite static charges of $Q$ and $-Q$.
a) Find the direction and magnitude of the electric field at a point midway between the two lines. Neglect edge effects.
b) If an observer $O$ is moving with velocity $\vec{v}(\ll c)$ parallel to the lines, what are the directions and magnitudes of the electric and magnetic fields observed in the rest frame of $O$ ?

7 (Medium) A charge $q$ is placed adjacent to two infinite grounded conducting planes as shown below.
a) Determine the electrostatic potential everywhere in the first quadrant.
b) Determine the work needed to bring $q$ to ( $a, a$ ) from infinity.


Figure 1.5 Figure of Exercise 6


Figure 1.6 Figure of Exercise 7
c) Determine the force on the charge $q$.

8 (Medium) Electrons undergoing cyclotron motion can be accelerated by increasing the magnetic field intensity with time - thus the induced electric field will impart a tangential acceleration. We require the radius of the electron's orbit to be kept constant during the process. This is the principle of the betatron accelerator.
Show that this can be achieved by designing a magnet such that the average $B$ over the area of the orbit is twice the field $B$ at the circumference.
Assume the electrons start from rest when $B=0$, and the apparatus is symmetric about the $z$-axis. Treat the problem non-relativistically. You may assume that the electron orbit is in the $x y$ plane of the magnet gap $(\theta=\pi / 2)$ oriented perpendicular to the applied magnetic field. Also, assume that the magnitude of the magnetic field in this plane has no $\phi$ dependence - only a radial dependence.
9 (Medium) In a perfect conductor, the conductivity is infinite, so $\vec{E}=0$ inside the conductor and any net charge resides on the surface.
a) What is the temporal behavior of the magnetic field inside a perfect conductor? What is the temporal behavior of the magnetic flux through a surface bound by a loop made of a perfectly conducting material?

A superconductor is a perfect conductor with the additional property that no magnetic field can exist inside the conductor.
b) Show that the current in a superconductor is confined to the surface.

The phenomenon of superconductor levitation, such as suspension of a small magnet in the air above a superconductor, can be analyzed using the image method by simplifying the system: (i) approximate the magnet by a perfect magnetic dipole moment $\vec{m}=m \frac{\vec{r}}{|\vec{r}|}$ located at $\vec{r}=(0,0, h)$. $(h>0)$ (ii) suppose that the space $z \leq 0$ is filled with superconducting material. Then replace the whole superconducting material by an image magnetic dipole $\vec{m}^{\prime}$ of the same magnitude as $\vec{m}$ but located at $(0,0,-h)$.
c) Which way should the image magnetic dipole point $+\hat{z}$ or $-\hat{z}$ ? Verify the superconducting boundary condition $\hat{z} \cdot \vec{B}(x, y, 0)$.
d) The force on the original magnetic dipole $\vec{m}$ by the superconducting material is just the force due to the magnetic field produced by the image dipole. Calculate this force.
e) Let $M$ be the mass of the magnet. Determine the height $h$ at which the magnet will float.

10 (Hard) A box is made up of six metal plates. The plates at $x=0, y=0, z=0, x=a$, and $y=a$ are grounded $(\Phi=0)$. The metal plate at $z=a$, insulated from the others, is held at a constant potential $\Phi_{0} \neq 0$. Find the potential $\Phi(x, y, z)$ inside the box.
11 (Hard) The potential on the surface of a spherical shell of radius $R$ is specified to be $V(\theta)=k\left(3 \cos ^{2} \theta-1\right)$, where $k$ is a constant. Assume that there are no charges anywhere except on the surface of the shell, and that $V \rightarrow 0$ as $r \rightarrow \infty$.
a) Find the potential inside the shell.
b) Find the potential outside the shell.
c) Find the surface charge density $\sigma(\theta)$ on the shell.

12 (Medium) In solids, the individual atomic dipoles are also able to contribute to the local field, and thus the local field may not necessarily the same as the external field. Let us assume that all the dipoles, which are distributed in a simple cubic lattice, are parallel to the external field.
(a) Show that the field at the center of an imaginary sphere is given by

$$
\begin{equation*}
E_{\text {sphere }}=\frac{1}{4 \pi \epsilon_{0}} \sum_{j} p_{j} \frac{3 z_{j}^{2}-r_{j}^{2}}{r_{j}^{5}} \tag{1.1}
\end{equation*}
$$

where $p_{j}$ is the dipole moment of atom $j$. The center of the dipole is excluded in the summation.
(b) Show that $E_{\text {sphere }}$ is 0 when the dipole moments are all identical.
(c) Show that the surface charge density is given as $-P \cos \theta$, where $\theta$ is an angle from the $z$ axis.
(d) Show that the field at the center of the sphere generated by the material outside the spherical surface equals to $-\mathbf{P} / 3 \epsilon_{0}$.
(e) Derive the Clausius-Mossotti relationship:

$$
\begin{equation*}
\frac{\epsilon_{r}-1}{\epsilon_{r}+2}=\frac{N \chi}{3}, \tag{1.2}
\end{equation*}
$$

where $\epsilon_{r}$ is the relative dielectric constant, $N$ is the number of atoms per unit volume, and $\chi$ is the electric susceptibility per atom. You may use the following equations.

$$
\begin{align*}
E_{\text {local }} & =E+\frac{\mathbf{P}}{3 \epsilon_{0}}  \tag{1.3}\\
\mathbf{P} & =N \epsilon_{0} \chi E_{\text {local }}  \tag{1.4}\\
\mathbf{P} & =\left(\epsilon_{r}-1\right) \epsilon_{0} E \tag{1.5}
\end{align*}
$$

13 (Medium) Forced oscillations of the atomic dipole are induced by the electric field of the light wave. If the motion of the nucleus is ignored, then the displacement $(x)$ of the electron is described by the following equation of motion:

$$
\begin{equation*}
m_{0} \frac{d^{2} x}{d t^{2}}+m_{0} \gamma \frac{d x}{d t}+m_{0} \omega_{0}^{2} x=-e E \tag{1.6}
\end{equation*}
$$

where $\gamma$ is the damping rate, $e$ is the electric charge of the electron, and $E$ is the electric field of the light wave.
(a) What is the physical meaning of each term?
(b) Assume that the light wave is monochromatic of angular frequency $\omega$.

$$
\begin{equation*}
E(t)=E_{0} \operatorname{Re}\left(\exp ^{-i(\omega t+\phi)}\right) \tag{1.7}
\end{equation*}
$$

where $E_{0}$ is the amplitude of the wave and $\phi$ is the phase. By letting that the position of electron also has a similar form, obtain the maximum displacement of an electron.
(c) If there are N number of atoms per unit volume, obtain the resonant polarizations per volume.
(d) The electric displacement $D$ of the isotropic medium is given as

$$
\begin{equation*}
D=\epsilon_{0} E+P=\epsilon_{0} E+\epsilon_{0} \chi E+P_{\text {resonant }}=\epsilon_{0} \epsilon_{r} E \tag{1.8}
\end{equation*}
$$

where $P$ is the polarization, $\chi$ is the electric susceptibility, $P_{\text {resonant }}$ is the resonant term of the polarization, and $\epsilon_{r}$ is the relative dielectric constant. Obtain $\epsilon_{r}$.
(e) Obtain the real part of $\epsilon_{r}$ at $\omega=0$ and $\omega=\infty$.
(f) Draw the real part and the imaginary part of $\epsilon_{r}$.

14 (Easy) The Kramers-Kronig relationships explains the relation between the real and imaginary part of the refractive index.

$$
\begin{align*}
n(\omega)-1 & =\frac{2}{\pi} \mathrm{P} \int_{0}^{\infty} \frac{\omega^{\prime} \kappa\left(\omega^{\prime}\right)}{\omega^{\prime 2}-\omega^{2}} d \omega^{\prime}  \tag{1.9}\\
\kappa(\omega) & =-\frac{2}{\pi \omega} \mathrm{P} \int_{0}^{\infty} \frac{\omega^{\prime 2}\left[n\left(\omega^{\prime}\right)-1\right]}{\omega^{\prime 2}-\omega^{2}} d \omega^{\prime}, \tag{1.10}
\end{align*}
$$

where $n$ and $\kappa$ are the real and the imaginary part of the refractive index, respectively. $\omega$ is the angular frequency.

Assume that $\kappa$ equals to a constant value $\kappa_{0}$ if $\omega$ is in between $\omega_{1}$ and $\omega_{2}$ and 0 for otherwise. $\omega_{2}$ is larger than $\omega_{1}$, and their difference is much smaller than $\omega_{1}$.
(a) Schematically draw $\kappa$. (b) Obtain $n(0)$. (c) $n(\infty)$.

15 (Medium) A parallel-plate capacitor of plate separation $d$ has the region between its plates filled by a block of solid dielectric of permittivity $\epsilon$. The dimensions of each plate


Figure 1.7 Dielectric slab partially withdrawn from between two charged plates
are length $l$, width $w$. The plates are maintained at the constant potential difference $\Delta \phi$. If the dielectric block is withdrawn along the $l$ dimension until only the length $x$ remains between the plates (Fig. 1.7), calculate the force tending to pull the block back into place.

16 (Hard) Consider a sphere of linear magnetic material of radius $a$ and permeability $\mu$ placed in a region of a space containing an initially uniform magnetic field, $\mathbf{B}_{0}$. We should like to determine how the magnetic is modified by the presence of the sphere and, in particular, to determine the magnetic field in the sphere itself.

## CHAPTER 2

## PREVIOUS TEST PROBLEMS

This chapter collects the previous test problems since 2008. Some questions may be missing.

### 2.1 Fall 2020

## PROBLEMS

1 A point charge $q$ is located at the center of a grounded conducting cube defined by the surfaces $x=a, y=a, z=a$.
a) Find the potential in terms of rectangular coordinates.
b) Find the charge density at the center of top face
c) Instead of a point charge, if two charges $\pm q$, situated symmetrically with respect to the center of cube, are placed along the $z$ direction and are separated by a distance $d$, how would the result change $(d>a)$ ?

Hint: Use the result of subsection 3.12 (Eigenfunction Expansions for Greeen Functions) of Jackson's book and the expansion of Green function (of rectangular box defined by the six planes $x=0, y=0, z=0, x=a, y=b, z=c$ ) is

$$
G\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=\frac{32}{\pi a b c} \sum_{l, m, n=1}^{\infty} \frac{\sin \left(\frac{l \pi x}{a}\right) \sin \left(\frac{l \pi x^{\prime}}{a}\right) \sin \left(\frac{m \pi y}{b}\right) \sin \left(\frac{m \pi y^{\prime}}{b}\right) \sin \left(\frac{n \pi z}{c}\right) \sin \left(\frac{n \pi z^{\prime}}{c}\right)}{\frac{l^{2}}{a^{2}}+\frac{m^{2}}{b^{2}}+\frac{n^{2}}{c^{2}}}
$$

### 2.2 Spring 2020

## PROBLEMS

1 The expansion of the Green function for a spherical shell bounded by $r=a$ and $r=b(\geq a)$ is given by

$$
G\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=4 \pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{l m}^{*}\left(\theta^{\prime}, \phi^{\prime}\right) Y_{l m}(\theta, \phi)}{(2 l+1)\left[1-\left(\frac{a}{b}\right)^{2 l+1}\right]}\left(r_{<}^{l}-\frac{a^{2 l+1}}{r_{<}^{l+1}}\right)\left(\frac{1}{r_{>}^{l+1}}-\frac{r_{>}^{l}}{b^{2 l+1}}\right)
$$

The variable $\boldsymbol{x}^{\prime}$ refers to the location $P^{\prime}$ of the unit source, while the variable $\boldsymbol{x}$ is the point $P$ at which the potential is being evaluated. The general solution to the Poisson equation with speicified values of the potential on the boundary surface is

$$
\Phi(\boldsymbol{x})=\frac{1}{4 \pi \epsilon_{0}} \int_{V} \rho\left(\boldsymbol{x}^{\prime}\right) G_{D}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right) d^{3} x^{\prime}-\frac{1}{4 \pi} \oint_{S} \Phi\left(\boldsymbol{x}^{\prime}\right) \frac{\partial G_{D}}{\partial n^{\prime}} d a^{\prime}
$$

a) Let us consider an example of a hollow grounded sphere of radius $b$ with a concentric ring of charge of radius $a$ and total charge $Q$. The ring of charge is located in the $x y$ plane. Draw this ring (of charge of radius $a$ and total charge $Q$ inside a grounded conducting sphere of radius $b$ ) and the sphere in $x y z$ space.
b) Express the charge density of the ring with the help of delta function in angle and radius.
c) Find the electric scalar potential in terms of infinite sum.
d) In the limit $b \rightarrow \infty$, obtain the result of c) for a ring of charge.
e) Describe the alternative method for the c) with the help of the result of d).

### 2.3 Fall 2019

## PROBLEMS

1 For Dirichlet boundary conditions on the sphere of radius $a$, the Green function for a unit source and its image is given by

$$
G\left(x, x^{\prime}\right)=\frac{1}{\left|\boldsymbol{x}-\boldsymbol{x}^{\prime}\right|}-\frac{a}{x^{\prime}\left|\boldsymbol{x}-\frac{a^{2}}{x^{\prime 2}} \boldsymbol{x}^{\prime}\right|}
$$

The variable $x^{\prime}$ refers to the location $P^{\prime}$ of the unit source, while the variable $\boldsymbol{x}$ is the point $P$ at which the potential is being evaluated.
a) Express this Green function in terms of $x, x^{\prime}, \gamma$ and $a$. Note that $\gamma$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$. The symmetry in the variables $\boldsymbol{x}$ and $\boldsymbol{x}^{\prime}$ is obvious in this result, as is the condition that $G=0$ if either $\boldsymbol{x}$ or $\boldsymbol{x}^{\prime}$ is on the surface of the sphere.
b) For solution

$$
\Phi(\boldsymbol{x})=\frac{1}{4 \pi \epsilon_{0}} \int_{V} \rho\left(\boldsymbol{x}^{\prime}\right) G_{D}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right) d^{3} x^{\prime}-\frac{1}{4 \pi} \oint_{S} \Phi\left(\boldsymbol{x}^{\prime}\right) \frac{\partial G_{D}}{\partial n^{\prime}} d a^{\prime}
$$

of the Poisson equation, we need not only $G$, but also $\frac{\partial G}{\partial n^{\prime}}$. Express $\left.\frac{\partial G}{\partial n^{\prime}}\right|_{x^{\prime}=a}$ in terms of $x, a$ and $\gamma$.
c) Hence, the solution of the Laplace equation outside of a sphere with the potential specified on its surface can be written explicitly. Express $\Phi(x)$.
d) As an application of c ), we consider that the upper hemisphere of a conducting sphere (radius $a$ ) is set to have an uniform potential $V$ and the lower hemisphere is grounded. Find the potential on a point along the negative $z$-axis in terms of $z$, $a$ and $V$.

### 2.4 Spring 2019

## PROBLEMS

1 [60 pts] A metal sphere of radius a carries a charge $Q$. It is surrounded, out to radius $b$, by linear dielectric material of permittivity $\varepsilon$ and susceptibility $\chi_{e}$.


Figure 2.1 Problem 1 of the test at Fall 2019.
a) (20 pts) Calculate the $\vec{D}$ and $\vec{E}$ for regions of $r<a, a<r<b$, and $r>b$.
b) $(20 \mathrm{pts})$ Calculate the potential at the center.
c) $(20 \mathrm{pts})$ Find the energy of this configuration.

2 [20 pts] A steady current $I$ flows down a long cylindrical wire of radius $a$. Find the magnetic field, both inside and outside the wire, if the current is distributed in such a way that $J$ is proportional to $s$, the distance from the axis.


Figure 2.2 Problem 2 of the test at Fall 2019.

3 [20 pts] A metal bar of mass $m$ slides frictionlessly on two parallel conducting rails a distance $l$ apart. A resistor $R$ is connected across the rails and a uniform magnetic field $B$, pointing into the page, fills the entire region.


Figure 2.3 Problem 3 of the test at Fall 2019.
a) If the bar moves to the right at speed $v$, what is the current in the resistor?
b) If the bar starts out with speed $v_{0}$ at time $t=0$, and is left to slide, what is its speed at later time $t$ ?

### 2.5 Fall 2018

## PROBLEMS

1 [50 pts] A long coaxial cable carries a uniform volume charge density $\rho$ on the inner cylinder (radius $a$ ), and a uniform surface charge density on the outer cylindrical shell (radius $b$ ). This surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral.


Figure 2.4 Problem 1 of the test at Fall 2018.
a) (30 pts) Find the electric field in each of the three regions: (i) inside the inner cylinder $(s<a)$, (ii) between the cylinders $(a<s<b)$, (iii) outside the cable ( $s>b$ ).
b) ( 10 pts ) Plot $E$ as a function of $s$.
c) (10 pts) Find the potential difference between a point on the inner cylinder and a point on the outer cylinder.

2 [30 pts] Three charges are situated at the center of a rectangular shape as shown in the figure.


Figure 2.5 Problem 2 of the test at Fall 2018.
a) . (10 pts) Calculate the electric field in the point $P$ due to three charges.
b) ( 10 pts ) How much work does it take to bring in another charge $q_{4}=-q$ and place it in point $P$.
c) (10 pts) How much work does it take to assemble the whole configuration of four charges? The charges in the figure are $q_{1}=+q, q_{2}=+2 q$, and $q_{3}=-2 q$.

3 [20 pts] A metal sphere of radius $R$, carrying charge $q$, is surrounded by a thick concentric metal shell (inner radius $a$, outer radius $b$ ). The shell carries no net charge. The outer surface of the metal shell is connected to a grounding wire.


Figure 2.6 Problem 3 of the test at Fall 2018.
a) Find the surface charge density $\sigma$ at $R$, at $a$, and at $b$.
b) Find the potential at the center, using infinity as the reference point.

### 2.6 Spring 2018

- Solve 2 problems only out of 3 problems. (Don't solve 3 problems)
- Describe details of your solution as well as physical meaning.


## PROBLEMS

1 Consider circular metal plate with radius $R$ and surface charge density $\sigma$.


Figure 2.7 Problem 1 of the test at Spring 2018.
a) [20 pts] Determine electric field $E(z)$ as a function of $z$ using Coulomb's law.
b) [20 pts] Determine electrostatic potential $V(z)$ as a function of $z$ using Coulomb's law.
c) [10 pts] Calculate electric field $\vec{E}(z)$ through the resultant of b) again. Compare electric field with a).

2 Two spherical cavities, of radii $a$ and $b$, are hollowed out from the interior of a (neutral) conducting sphere of radius $R$. At the center of each cavity a point charge is placed - call these charges $q_{a}$ and $q_{b}$.


Figure 2.8 Problem 2 of the test at Spring 2018.
a).$[15 \mathrm{pts}]$ Find the surface charges $\sigma_{a}, \sigma_{b}$, and $\sigma_{R}$.
b) [10 pts] What is the electric field outside the conductor?
c) $[15 \mathrm{pts}]$ What is the electric field within each cavity?
d) [10 pts] What is the force on $q_{a}$ and $q_{b}$ ?

3 The volume between two concentric conducting spherical surfaces of radii $a$ and $b$ $(a<b)$ is filled with an inhomogeneous dielectric constant $\varepsilon_{r}$, where $K$ and $\varepsilon_{0}$ are constant and permittivity in air, respectively.

Thus, $\vec{D}(r)=\varepsilon \vec{E}(r)$. A charge $Q$ is placed on the inner surface, while the outer surface is grounded.


$$
\varepsilon_{r}=\frac{\varepsilon_{0}}{1+K r}
$$

Figure 2.9 Problem 3 of the test at Spring 2018.
a) [10 pts] Find the electrical displacement $(\vec{D}(r))$ using Coulomb's law in the region $a<r<b$.
b) [20 pts] Calculate the capacitance of the device.
c) $[10 \mathrm{pts}]$ Calculate the polarization (bounded) charge density $\left(\rho_{b}\right)$ in $a<r<b$.
d) $[10 \mathrm{pts}]$ Calculate the surface polarization (bounded) charge density $\left(\sigma_{b}\right)$ at $r=a$, $r=b$.

## Usefule expression:

$\overline{1 \text { Gauss's law: } \oint_{S} \vec{E}_{r}} \cdot d \vec{a}=\frac{1}{\epsilon_{0}} \int_{V} \rho d v$
2. Spherical coordinate:

$$
\begin{aligned}
& d \vec{l}=d r \hat{r}+r d \theta \hat{\theta}+r \sin \theta d \phi \hat{\phi} \\
& d a=r^{2} \sin \theta d \theta d \phi, \quad d v=r^{2} \sin \theta d r d \theta d \phi \\
& \nabla f=\frac{\partial f}{\partial r} \hat{r}+\frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta}+\frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi} \\
& \nabla^{2} f=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial f}{\partial r}\right)+\frac{1}{r^{2} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial f}{\partial \theta}\right)+\frac{1}{r^{2} \sin \theta} \frac{\partial^{2} f}{\partial \phi^{2}} \\
& \nabla \cdot \vec{v}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} v_{r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta v_{\theta}\right)+\frac{1}{r \sin \theta} \frac{\partial v_{\phi}}{\partial \phi}
\end{aligned}
$$

### 2.7 Fall 2017

Total 100 point $=80$ point required +20 point choice (choose 1 problem)

## PROBLEMS

1 A spherical charge distribution (dielectric material) has a volume charge density that is a function only of $r$, the distance from the center of the distribution. In other words, $\rho=\rho(r)$. If $\rho(r)$ is as given below, determine the electric field as a function of $r$. Integrate the result to obtain an expression for the electrostatic potential $V(r)$, subject to the restriction that $V(\infty)=0$.


Figure 2.10 Problem 1 of the test at Fall 2017.
a) $[20 \mathrm{pts}]$

$$
\begin{aligned}
& \rho=\rho_{0} \text { (i.e., constant) for } 0 \leq r \leq R, \\
& \rho=0 \text { for } r>R
\end{aligned}
$$

b) (Choice 1) [20 pts]

$$
\begin{array}{ll}
\rho=A / r(A \text { is a constant }) & \text { for } 0 \leq r \leq R, \\
\rho & =0 \quad \text { for } r>R
\end{array}
$$

c) [20 pts] Suppose that a sphere is metal, calculate electrical field and electrostatic potential as a function of $r$.

$$
\begin{aligned}
\sigma & =\sigma_{0} \text { (i.e., constant) } \quad \text { for } 0 \leq r \leq R, \\
\sigma & =0 \quad \text { for } r>R
\end{aligned}
$$

a) [20 pts] $\boldsymbol{H}$ is magnetic field and $\boldsymbol{B}=\mu_{0}(\boldsymbol{H}+\boldsymbol{M})$ is magnetic induction. $\boldsymbol{D}=$ $\epsilon_{0} \boldsymbol{E}+\boldsymbol{P}$ is electrical displacement. Write the Maxwell equations in differential form inside materials including $\boldsymbol{P}$ (polarization), $\boldsymbol{M}$ (magnetiztion), and $\boldsymbol{J}$ (current density).
b) [20 pts] Derive wave equation in free space using Maxwell equations.
c) $\left(\right.$ Choice 2) [20 pts] Derive wave equation in conducting media $\left(\boldsymbol{J}=\frac{\sigma}{1-i \omega \tau} \boldsymbol{E}\right)$ using Maxwell equations. Calculate the skin depth of metal when the light with long wavelength propagate to the metal.

## Usefule expression:

1 Gauss's law: $\oint_{S} \vec{E}_{r} \cdot d \vec{a}=\frac{1}{\epsilon_{0}} \int_{V} \rho d v$
2. Spherical coordinate: $d a=r^{2} \sin \theta d \theta d \phi, d v=r^{2} \sin \theta d r d \theta d \phi$
2.8 Spring 2017

## PROBLEMS

1 (50 pts) Electrons undergoing cyclotron motion can be accelerated by increasing the magnetic field intensity with time - thus the induced electric field will impart a tangential acceleration. We require the radius of the electron's orbit to be kept constant during the process. This is the principle of the betatron accelerator.

Show that this can be achieved by designing a magnet such that the average $B$ over the area of the orbit is twice the field $B$ at the circumference.

Assume the electrons start from rest when $B=0$, and the apparatus is symmetric about the $z$-axis. Treat the problem non-relativistically. You may assume that the electron orbit is in the $x-y$ plane of the magnet gap $(\theta=\pi / 2)$ oriented perpendicular to the applied magnetic field. Also, assume that the magnitude of the magnetic field in this plane has no $\phi$ dependence-only a radial dependence.

2 (50 pts) A box is made up of six metal plates. The plates at $x=0, y=0, z=0, x=a$, and $y=a$ are grounded $(\Phi=0)$. The metal plate at $z=a$, insulated from the others, is held at a constant potential $\Phi_{0} \neq 0$. Find the potential $\Phi(x, y, z)$ inside the box.

### 2.9 Fall 2016

## PROBLEMS

1 (30 pts) A cylindrical conductor of radius $a$ and current $I$ has a hole of radius $b$ parallel to, and centered a distance $d$ from the cylinder axis $(d+b<a)$. The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis.


Figure 2.11 Problem 1 of the test at Fall 2016.

Find the magnitude and direction of the magnetic field inside the hole.
2 (30 pts) In this problem we use a quantum mechanical setting to explore classical electrostatics. A neutral "hydrogen atom" in its ground state has a shell-electron for which
the charge density can be described by:

$$
\begin{aligned}
\rho_{e}(r) & =-\frac{e}{\pi a^{3}} \exp (-2 r / a) \\
e & =\text { electron charge, } \\
r & =\text { distance between electron and proton, } \\
a & =\text { Bohr radius. }
\end{aligned}
$$

The proton is taken to be a point charge situated at the origin of a coordinate system. Under the influence of a constant external electric field $\vec{E}_{0}$, the electron and its charge density distribution shift rigidly (without deformation) with respect to the proton by a vector $\vec{r}_{0}$. The proton can be assumed to remain fixed.
a) Write the expression for the total (i.e., positive and negative) charge density of the hydrogen atom without and with external field. (Help: a coordinate transformation from $\vec{r}=\vec{r}^{\prime}+\vec{r}_{0}$ will be useful.)
b) Calculate the dipole moment $\vec{p}$ of the hydrogen atom as a function of $\vec{r}_{0}$ in the presence of the external field $\vec{E}_{0}$.
c) Calculate the electrostatic force between the proton and the electron charge distribution under the influence of the external field $\vec{E}_{0}$.
3 (40 pts) A charge $q$ is placed adjacent to two infinite grounded conducting planes as shown.


Figure 2.12 Problem 3 of the test at Fall 2016.
a) Determine the electrostatic potential everywhere in the first quadrant.
b) Determine the work needed to bring $q$ to $(a, a)$ from infinity.
c) Determine the force on the charge $q$.

### 2.10 Spring 2016

## PROBLEMS

1 (40 pts) A long circular cylinder of radius $R$ carries a magnetization $\vec{M}=k s^{2} \hat{\phi}$, where $k$ is a constant, $s$ is the distance from the axis, and $\hat{\phi}$ is the usual azimuthal unit vector. Find the magnetic field due to $\vec{M}$, for points (a) inside and (b) outside the cylinder.

## Solve two of the following problems



Figure 2.13 Problem 1 of the test at Spring 2016.

2 (30 pts) Two concentric metal spherical shells, of radius $a$ and $b$, respectively, are separated by weakly conducting material of conductivity $\sigma$.
a) If they are maintained at a potential difference $V$, what current flows from one to the other?
b) What is the resistance between the shells?


Figure 2.14 Problem 2 of the test at Spring 2016.

3 (30 pts)
a) Calculate the magnetic dipole moments for the square loop and for the circular loop, respectively.
b) Calculate the torque exerted on the square loop due to the circular loop (assume $r$ is much larger than $a$ or $b$ ).
c) Calculate the energy of two magnetic dipoles separated by a displacement $r$.


Figure 2.15 Problem 3 of the test at Spring 2016.

4 (30 pts) A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic suscepptibility $\chi_{m}$. A current $I$ flows down the inner conductor and returns along the outer one; in each case the current distributes itself uniformly over the surface.
a) Find the magnetic field in the region between the tubes.
b) Calculate the magnetization in the region between the tubes.
c) Calculate the bound currents $\vec{J}_{b}$ and $\vec{K}_{b}$ in the region between the tubes.


Figure 2.16 Problem 4 of the test at Spring 2016.

### 2.11 Fall 2015

## PROBLEMS

1 (20 pts) A long circular cylinder of radius $R$ carries a magnetization $\vec{M}=k s^{2} \hat{\phi}$, where $k$ is a constant, $s$ is the distance from the axis, and $\hat{\phi}$ is the usual azimuthal unit vector. Find the magnetic field due to $\vec{M}$, for points (a) inside and (b) outside the cylinder.



Figure 2.17 Problem 1 of the test at Fall 2015.

## Solve two of the following problems

2 (40 pts) Suppose

$$
\vec{E}(r, \theta, \phi, t)=A \frac{\sin \theta}{r}\left[\cos (k r-\omega t)-\frac{1}{k r} \sin (k r-\omega t)\right] \hat{\phi},
$$

with $\frac{\omega}{k}=c$.
a) Find the associate magnetic field by using the Maxwell's equation.
b) Calculate the Poynting vector $\vec{S}$.
c) Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$.
d) Integrate $\vec{I} \cdot d \vec{a}$ over a spherical surface to determine the total power radiated.

3 (40 pts) Two concentric metal spherical shells, of radius $a$ and $b$, respectively, are separated by weakly conducting material of conductivity $\sigma$.
a) If they are maintained at a potential difference $V$, what current flows from one to the other?
b) What is the resistance between the shells?


Figure 2.18 Problem 3 of the test at Fall 2015.

4 (40 pts)
a) Calculate the magnetic dipole moments for the square loop and for the circular loop, respectively.
b) Calculate the torque exerted on the square loop due to the circular loop (assume $r$ is much larger than $a$ or $b$ ).
c) Calculate the energy of two magnetic dipoles separated by a displacement $r$.


Figure 2.19 Problem 4 of the test at Fall 2015.

5 (40 pts) A coaxial cable consists of two very long cylindrical tubes, separated by linear insulating material of magnetic suscepptibility $\chi_{m}$. A current $I$ flows down the inner conductor and returns along the outer one; in each case the current distributes itself uniformly over the surface.
a) Find the magnetic field in the region between the tubes.
b) Calculate the magnetization in the region between the tubes.
c) Calculate the bound currents $\vec{J}_{b}$ and $\vec{K}_{b}$ in the region between the tubes.


Figure 2.20 Problem 5 of the test at Fall 2015.

### 2.12 Spring 2015

## PROBLEMS

l l Choose two problems out of the 6 in section $A$ and choose one problem in section B.
[宫 Unless therwise stated, you will be graded on the quality of your explanation. Please be brief but clear!
[菅 If you need more room, use the back of the pages (or staple extra pages to the exam).

LT ued, (and where) if you need more room than the problem page itself.Section A: Choose two problems out of the $\mathbf{6}$ in section A.
1 (40 pts) You have a spherical shell (radius $R$ ) which has been covered by some surface charge distribution $\sigma(\theta)$. The potential everywhere is given by

$$
\begin{array}{ll}
\text { Outside shell : } & V_{\mathrm{out}}(r, \theta)=\left(\frac{c}{r^{2}}\right) \cos \theta \\
\text { Inside shell : } & V_{\mathrm{in}}(r, \theta)=\left(\frac{V_{0} r}{R}\right) \cos \theta
\end{array}
$$

a) (10 pts) Given these two formulas, I claim the constant " $c$ " in $V_{\text {out }}(r, \theta)$ can be immediately deduced. Tell me what $c$ is, and how you know.
b) ( 15 pts ) Find the surface charge density $\sigma(\theta)$ on the shell. (Briefly show/explain your work. Please be explicit about your reasoning here.)
NOTE: If you can't get part a), that's no problem. I'm mostly just looking for method here!
c) ( 15 pts ) Find the electric field inside. I just want a formula. (You do NOT need part b) for this at all!)

Useful dielectric relations I sometimes forget:
$\vec{D}=\varepsilon_{0} \vec{E}+\vec{P}$
Linear dielectrics: $\vec{P}=\chi_{e} \varepsilon_{0} \vec{E}$, or if you prefer, $\vec{D}=\varepsilon \vec{D} \varepsilon_{0} \varepsilon_{r} \vec{E}$,
where the dielectric constant and permittivity are related by $\varepsilon_{r}=1+\chi_{2}$.

2 (50 pts) A metal (conducting) sphere of radius " $a$ " has total negative charge $+Q$ placed to it. (The solid interior of this solid metal sphere is labeled region I.) Outside of it (region III) is a solid, neutral, linear dielectric shell. That shell is hollow, it he EMPTY region of space between the conductor and the dielectric is labeled region II ( $a<r<b$ ), The solid dielectric (region III) has dielectric constant $\varepsilon_{r}$, which extends from $b<r<c$.


Figure 2.21 Problem 2 of the test at Spring 2015.
a) (20 pts) Find the electric field " $E$ " and the electric displacement " $D$ " field everywhere in space. (be explicit, what are they in each of regions I, II, III, and IV in the figure?)
b) ( 10 pts ) If there were $\mathbf{N O}$ dielectric shell (or put another way, if $\varepsilon_{r}=1$ in region III), the voltage at the origin (with respect to infinity) would be $V_{0}$. With the dielectric shell present, how would the voltage at the origin compared with $V_{0}$ (clearly choose one!)
A) $V$ (origin, with dielectric present) $>V_{0}$
B) $V$ (origin, with dielectric present $)<V_{0}$
C) $V($ origin, with dielectric present $)=V_{0}$
D) Not enough information

Briefly, justify your answer mathematically:
c) (10 pts) Going back to part a), find a formula for the surface bound charge density $\sigma_{B}$ (magnitude and sign) on the inner surface of the plastic sphere, i.e., at $r=b$ ? (Give your answer only in terms of given constants: namely $Q, \varepsilon_{r}$ and the given radii.)
If you couldn't solve part a), just clearly explain in detail what procedure you would follow: What would you need, how would you proceed?

Let's do some explicit check of part c). (If you could not get an answer above, you can still get lots of partial credit by explaining clearly what you expect fir these check)
d) ( 2 pts ) Check the units of your symbolic answer to part c). (what should it be)
e) ( 3 pts ) Consider the limit $\varepsilon_{r} \rightarrow 1$ : what do you expect the answer for $\sigma_{B}$ should approach, and what does your result above give you?
f) ( 5 pts ) Consider the limit $\varepsilon_{r} \rightarrow \infty$ : what do you expect the answer for $\sigma_{B}$ should approach, and what does your result above give you?
3 (40 pts) A charge $+q$ is located midway between 2 infinitely wide, grounded conducting planes. We want to find the voltage $V(r)$ everywhere in space. A friend has proposed using the method of images, their idea is to put two images charges into the problem:

- a negative charge $(-q)$ located a distance $d$ above the base of the upper plate,
- and, a second (negative) charge $-q$ a distance $d$ below the top of the lower plate.


Figure 2.22 Problem 3 of the test at Spring 2015.
a) (10 pts) ASSUMING for the moment their proposed method will work, write down the simple resulting formula for the voltage at a point $\vec{r}=(x, y, z)$ imply. (Note where the origin is located in the figure!)
b) ( 10 pts ) Comment on whether this solution method is correct or incorrect. (If correct, justify the method briefly. If incorrect, what's the problem?)
c) ( 10 pts ) Where in all of space (if anywhere) is $V$ discontinuous in this problem? (Note: you do not need to have solved either of the previous parts to answer this or the next question!)
d) (10 pts) Where in all of space (if anywhere) is the divergence of $\vec{E}$ zero in this problem?
4 (50 pts) You have a cubical box (sides all of length $a$ ) made of 6 metal plates which are insulated from each other.

The left wall at $y=0$ is held at constant potential $V=-V_{0}$. The right wall at $y=a$ is held at constant potential $V=V_{0}$. All four other walls are grounded, $V=0$. We might want to find the voltage $V(r)$ everywhere inside the cube, but you do NOT have to solve for voltage in this problem!! Just read the questions and answer only what $I$ ask.
a) (20 pts) Use the method of separation of variables to separate Laplace's equation into three ordinary differential equations, one that depends only on $x$, one that depends only on $y$, and one only on $z$. Show and briefly/clearly explain all your work. You do NOT need to SOLVE these equations!


Figure 2.23 Problem 4 of the test at Spring 2015.
b) ( 15 pts ) The 3 equations above should contain some not-yet-determined constants. Given the boundary conditions above:

- what if anything, can you tell me about the SIGNS of all three constants?
- what, if anything, can you tell about any relationship among the three constants?
(or are they all completely unrelated to one another?) Briefly, explain.
c) ( 15 pts ) Suppose I change the previous problem so that ALL SIX sides have the same constant voltage V0 (none are grounded) What is the potential at the very center of the cube? Briefly but clearly justify/explain your reasoning!


Figure 2.24 Problem 4c) of the test at Spring 2015.

5 (45 pts) Consider a long straight coaxial cable consisting of an inner cylindrical conductor of radius $a$ carrying a uniform current $I$ in the $+z$ direction and an outer very thin cylindrical conductor of inner radius $b$ carrying a uniform return current of the same magnitude $I$ in the ${ }^{\breve{ }} z$ direction. The space between the conductors $(a<s<b)$ is filled with a linear paramagnetic material with magnetic susceptibility $\chi_{m}$.


Figure 2.25 Problem 5 of the test at Spring 2015.
a) ( 5 pts ) What are the SI metric units of the $\vec{H}$ field?
b) (10 pts) Solve for the $\vec{H}$-field for $a<s<b$, i.e. between the inner and outer conductors (direction and magnitude).
c) ( 10 pts ) Solve for the magnetic field $\vec{B}$ between the inner and outer conductors (direction and magnitude).
d) (10 pts) Solve for any bound currents (surface, or volume). Indicate clearly the direction of any bound currents, keeping in mind that the material is paramagnetic.
e) (10 pts) Briefly, comment on why the direction of the bound currents in part d) makes sense physically, given that the material is paramagnetic.

6 Consider an infinitesimally thin charged disk of radius $R$ and uniform surface charge density $\sigma$ that is in the $x y$ plane centered on the origin.
a) ( 15 pts ) Calculate the potential along the z axis from Coulomb's law.
b) ( 25 pts ) Calculate the monopole moment, dipole moment and quadrupole moment of the disk. These are defined as

$$
\begin{aligned}
Q & =\int d^{3} r \rho(r), \\
\vec{P} & =\int d^{3} r \rho(r) \vec{r}, \\
Q_{i j} & =\int d^{3} r \rho(r)\left[3 r_{i} r_{j}-r^{2} \delta_{i j}\right] .
\end{aligned}
$$

## Section B: Choose one problem out of the $\mathbf{4}$ in section $B$.

1 ( 10 pts ) A rectangular copper strip 1.5 cm wide and 0.10 cm thick carries a current of 5.0 A. A 1.2 T magnetic field is applied perpendicular to the strip. Find the resulting Hall voltage. The molar mass of copper is 63.5 g , and the density of copper is $8.95 \mathrm{~g} / \mathrm{cm}^{3}$. Assume each copper atom contributes on free electron to the body of the material.

2 ( 10 pts ) Consider a set of 12 identical capacitors, each of capacitance $C$. As shown in the figure below, they are connected together such that they form the geometry of a cube. Find the equivalent total capacitance of this arrangement, as measured between points diagonally opposite one another (e.g., measured between the lower left point on the figure and the upper right point).


Figure 2.26 Problem 2 of the test (Section B) at Spring 2015.

3 (10 pts) Starting from the basic definition of magnetic vector potential $\vec{A}$, show that $\oint \vec{A} \cdot d \vec{\ell}=\Phi_{M}$ (in words: that the line integral of the magnetic vector potential $\vec{A}$ around any closed loop is always given by the total magnetic flux through the loop enclosed) Explain your steps, briefly but clearly.
4 (10 pts) You have a sheet with a uniform (positive) surface charge density. The sheet lies in the $x y$ plane (perpendicular to the plane of this page, it is shown in perspective). There are lots of OTHER charges just out of the picture (not shown!) contributing to the $\vec{E}$ field. At a point $\mathrm{P}_{1}$ just (infinitesimally) below this sheet, the electric field is $\vec{E}=E_{0} \hat{x}$. The numerical values of $\sigma$ and $E_{0}$ are given by $\sigma / \varepsilon_{0}=3 \mathrm{~N} / \mathrm{C}$ and $E_{0}=4 \mathrm{~N} / \mathrm{C}$. What is the electric field at point $\mathrm{P}_{2}$ just above the sheet (infinitesimally above point $\mathrm{P}_{1}$ ?) (Note: $\vec{E}$ is a vector, so I either need components, or a magnitude and direction!)


Figure 2.27 Problem 4 of the test (Section B) at Spring 2015.

### 2.13 Fall 2014

## PROBLEMS

## A. Solve one of the following problems

1 (50 pts) A long coaxial cable carries a uniform volume charge density $\rho$ on the inner cylinder (radius $a$ ), and a uniform surface charge density on the outer cylindrical shell (radius $b$ ). This surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral.


Figure 2.28 Problem 1 of the test (Section A) at Fall 2014.
a) (15 pts) Find the electric field in each of the three regions: (i) inside the inner cylinder $(s<a)$, (ii) between the cylinders $(a<s<b)$, (iii) outside the cable $(s>b)$.
b) ( 5 pts$)$ Plot $E$ as a function of $s$.
c) (10 pts) Find the potential difference between a point on the inner cylinder and a point on the outer cylinder.
d) ( 10 pts ) Find the capacitance per unit length of two coaxial cylinders.
e) (10 pts) Find the energy stored in the gap between the inner and the outer cylinder.

2 (40 pts) A metal sphere of radius $R$, carrying charge $q$, is surrounded by a thick concentric metal shell (inner radius $a$, outer radius $b$ ). The shell carries no net charge.
a) (10 pts) Find the surface charge density $\sigma$ at $R$, at $a$, and at $b$.
b) (20 pts) Find the electric field (i) $r>b$, (ii) $a<r<b$, (iii) $R<r<a$, and (iv) $r<R$.
c) $(5 \mathrm{pts})$ Find the potential at the center, using infinity as the reference point.
d) $(5 \mathrm{pts})$ Find the capacitance for this capacitor.


Figure 2.29 Problem 2 of the test (Section A) at Fall 2014.

3 (60 pts) A metal sphere of radius $a$ carries a charge $Q$. It is surrounded by linear dielectric material of susceptibility $\chi_{2}$, out to radius $b$.


Figure 2.30 Problem 3 of the test (Section A) at Fall 2014.
a) Find the electric field at $r$ for $a<r<b$.
b) Find the electric field at $r$ for $r>b$.
c) Find the potential at the center (relative to infinity).
d) Find the polarization in the dielectric.
e) Find the bound charge density $\left(\rho_{b}, \sigma_{b}\right)$ in the dielectric.
f) Find the energy of this configuration.

4 (50 pts) The space between the plates of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness $a$, so the total distance between the plates is $2 a$, Slab 1 has a dielectric constant of 0.1 , and slab 2 has a dielectric constant of 1 . The free charge density on the top plate is $\sigma$ and on the bottom plate $-\sigma$.


Figure 2.31 Problem 4 of the test (Section A) at Fall 2014.
(a) Find the electric displacement D in each slab. (b) Find the electric field E in each slab. (c) Find the polarization P in each slab. (d) Find the potential difference between the plates. (e) Find the capacitance of this capacitor and compare the capacitance without any dielectric material in between.

## B. Solve one of the following problems

1 (60 pts) A point charge $q$ is situated a distance $a$ from the center of a ground conducting sphere of radius $R$.


Figure 2.32 Problem 1 of the test (Section B) at Fall 2014.
a) Find the potential at $P$ situated a distance $r$ from the center by using the image method.
b) Find the electric field at $P$ using the result of a).
c) Find the induced surface charge on the sphere, as a function of $\theta$.
d) Find the total induced charge in the sphere.
e) Find the force between the charge and the sphere.
f) Calculate the energy of this configuration.

2 (50 pts) An uncharged metal sphere of radius $R$ is placed in an otherwise uniform electric field $\vec{E}=E_{0} \hat{z}$.


Figure 2.33 Problem 2 of the test (Section B) at Fall 2014.
a) ( 30 pts ) Find the potential in the region outside the sphere.
b) ( 10 pts ) Find the induced charged density and the total induced charge in this metal sphere.
c) $(10 \mathrm{pts})$ Find the electric field in the region outside the sphere.

### 2.14 Spring 2014

## PROBLEMS

l
l宴 Describe details of your solution as well as physical meaning.
1 Draw triangle relation of three fundamental quantity of volume charge density $\rho$, electric potential $V$ and electric field $\vec{E}$. Write down relation of them with 6 different equations and explain each equation what it means. (For example, you can obtain $\vec{E}$ when you know $\rho$.)
2 For the point charge $Q$, solve the above 6 equations. You need to use three dimensional delta function since it is a point charge.

3 A long straight shielded cable length $L$ (Ex: video or signal cable), carrying a line charge density $\lambda$ with radius $r_{1}$ is surrounded by PMMA (Acryl) with dielectric constant of $\varepsilon$ and shielded with radius $r_{2}\left(r_{2}>r_{1}\right)$


Figure 2.34 Problem 3 of the test at Spring 2014.
a) Write down the Gauss's law in dielectric.
b) What is the relation between electric displacement $\vec{D}$ and electric field $\vec{E}$ inside dielectric material?
c) Solve the equation to obtain the electric displacement ( $\vec{D}$ ) inside PMMA ( $r_{1}<$ $r<r_{2}$ )
d) Derive the electric potential $(V)$ inside PMMA.
e) What is the capacitance $(C)$ of this shielded cable?

### 2.15 Fall 2013

## PROBLEMS

Volume charge density $\rho$ is given by

$$
\rho= \begin{cases}\rho_{0}, & 0 \leq r<a \\ 0, & a<r\end{cases}
$$

Here $r$ is the distance of a given point from the origin.
1 Find the electric field $\vec{E}(\vec{r})$ due to this charge distribution for all $\vec{r}$.
a) when $|\vec{r}|=r \geq a$.
b) when $|\vec{r}|=r<a$.

2 Find the electric potential $V(\vec{r})$ due to this charge distribution for all $\vec{r}$.
(Let $V(|\vec{r}| \rightarrow \infty)=0$ )
a) when $|\vec{r}|=r \geq a$.
b) when $|\vec{r}|=r<a$.

### 2.16 Spring 2013

PROBLEMS Consider a positive point change $Q$ fixed at the origin and assume that no other charges are around.
1 Express the electric field $\vec{E}$ due to the charge $Q$ as a vector function of $\vec{r}$.
2 Calculate $\vec{\nabla} \cdot \vec{E}$ (the divergence of this field $\vec{E}$ ) and express the result as a function of $\vec{r}$.
3 Now we introduce a positive test charge $q$ with mass $m$. We put this test charge at a point whose coordinates are $(a, 0,0)$. The initial speed of $q$ is zero. What will be the speed of $q$ when it passes the point $(2 a, 0,0)$ ? (Note that $Q$ is assumed to be fixed all the time and neglect all forces other than the Coulomb force.)

### 2.17 Fall 2012

## PROBLEMS

A point charge $Q(Q>0)$ is fixed at $\vec{r}_{Q}$.
1 Express the electric field $\vec{E}\left(\vec{r}_{p}\right)$ at point $\vec{r}_{p}$ (due to the point charge $Q$ at $\vec{r}_{Q}$ ).
2 Express the electrostatic potential $V\left(\vec{r}_{p}\right)$ at a point $\vec{r}_{p}$ (due to the point charge $Q$ at $\vec{r}_{Q}$ ). $V(\vec{r})$ is zero when $|\vec{r}|$ is infinite.

3 Consider a test charge $q(q>0)$ with mass $m$. We put this charge $q$ at $\vec{r}_{p}$ and let it move due to the repulsive Coulomb force (by the point charge $Q$ at $\vec{r}_{Q}$ (fixed)). Express the final speed of the test charge in terms of $q, m$ and $V\left(\vec{r}_{p}\right)$.

### 2.18 Spring 2012

## PROBLEMS

1 Describe the Gauss's law.
2 Consider a uniformly charged sphere with radius $R$. The sphere is centered at the origin and the charge density is $\rho$. Calculate the magnitude of electric field at position $P$ for the following two cases. (Here, $r$ is the distance between the point $P$ and the origin. There is no other charge around the sphere.)
a) what is $|\vec{E}|$ at $P$ when $0 \leq r<R$ ?
b) what is $|\vec{E}|$ at $P$ when $r>R$ ?

### 2.19 Fall 2011

## PROBLEMS

1 Derive the value of $\vec{E}$ (electric field) at $\left(x_{0}, y_{0}, z_{0}\right)$ due to a uniform charge density $\sigma>0$ on the $x y$ plane (i.e., $z=0$ plane). Assume $z_{0}>0$ )
2 For the situation of the above problem, What is the difference of potential between two points $\left(x_{1}, y_{1}, z_{1}\right)$ and $\left(x_{2}, y_{2}, z_{2}\right)$ (with $\left.z_{1}>0, z_{2}>0\right)$ ?

### 2.20 Spring 2011

## PROBLEMS

Consider an infinite wire with a uniform electric charge density $\lambda>0$.
1 Find the magnitude of the electric field at point whose distance from the wire is $r$.
2 Find the difference of the electrostatic potential (scalar potential) between two points, one at $r_{1}$ and the other at $r_{2}$ apart from the wire.

3 A point charge $q>0$ begins to accelerate from zero velocity. The distance between the starting point and the wire is $d$. What is the speed of the particle when this distance is doubled?

### 2.21 Fall 2010

## PROBLEMS

Consider an infinite conducting plane and a charged particle with charge $q$. They are separated by a distance $\ell$ as in the figure.


Figure 2.35 Problems of the test at Fall 2010.

1 Using the image charge nethod, find the magnitude of the electric field at point $A$ (see the figure: just out of the surface of the plane to the direction of the charged particle.)
2 Find the electrostatic force on $q$ due to the conducting plane.
a) magnitude?
b) is the force repulsive or attractive?

### 2.22 Spring 2010

## PROBLEMS

## 1 T

1 A nonconducting sphere of radius $a$ has a spherical cavity of radius $b$ located at $c$ as shown in the figure. The sphere of radius $a$ contains a uniform charge density $\rho$ and the cavity of radius $b$ is empty.


Figure 2.36 Problem 1 of the test at Spring 2010.
a) Find the electric field at $r>a$.
b) Find the electric field at any point in the cavity.

## 2 Suppose

$$
\vec{E}(r, \theta, \phi)=A \frac{\sin \theta}{r}\left[\cos (k r-\omega t)-\frac{1}{k r} \sin (k r-\omega t)\right] \hat{\phi},
$$

with $\omega / k=c$.
a) Find the associate magnetic field by using the Maxwell's equation.
b) Calculate the Poynting vector $\vec{S}$.
c) Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$.
d) Integrate $\int \vec{I} \cdot d \vec{a}$ over a spherical surface to determine the total power radiated.

3 The potential at the surface of a sphere (radius $R$ ) is given by

$$
V_{0}=k \cos 3 \theta,
$$

where $k$ is a constant.
a) Find the potential inside and outside the sphere, as well as
b) the surface charge density $\sigma(\theta)$ on the sphere. (Assume that there is no charge density inside or outside the sphere.)

4 A metal sphere of radius $a$ carries a charge $Q$. It is surrounded, out to radius $b$, by linear dielectric material of permittivity $\varepsilon$. Find the potential at the center.
5 A long coaxial cable consists of two concentric cylindrical conducting sheets of radii $R_{1}$ and $R_{2}$, respectively ( $R_{2}>R_{1}$ ). The two conductors are connected through a battery which maintains a voltage $V_{0}$ between them.
a) Calculate the electric field and
b) the potential between the cylinder ( $R_{1}<r<R_{2}$ ).
c) What is the surface charge density on each of the two sheets?
d) How much charge per unit length is there on each sheet? (You may assume that the electric field is zero outside the cable.)


Figure 2.37 Problem 4 of the test at Spring 2010.

### 2.23 Spring 2009

## PROBLEMS

I官 Solve two problems out of three.
IT 宣 Describe the problem solving procedure and physical meaning of the results.

1 The field of a uniform spherical charge. Consider a spherical charge of uniform volume charge density $\rho$, total charge $Q$ and radius $R$.
a) Write down the Laplacian of $V$ in (a) rectangular coordinates, (b) cylindrical coordinates and (c) spherical coordinates.
b) Derive electric field and potential inside of sphere using the Poisson or Laplace equation.
c) Derive electric field and potential outside of sphere using the Poisson or Laplace equation.

2
a) Derive Green's first identity and
b) Green's second identity.
c) Using Green's second identoty with $\phi=1 / r$ and Poisson's equation, derive electric potential $\Phi$.
d) Explain about first, second and third terms in final equation in case of c).

3 A steady current $I$ flows down a long cylindrical wire of radius $a$. The current is distributed in such a way that $J$ is proportional to $s$ (the distance from the axis).
a) Find the magnetic field inside the wire.
b) Find the magnetic field outside the wire.
c) If alternative current $I(t)$ flows instead of steady current $I$, which Maxwell equation needs to be applied outside of the wire? Explain why.

### 2.24 Fall 2008

## PROBLEMS

1 Uniformly magnetized sphere: Let us consider a sphere of radius $a$ with a uniform permanent magnetization $\vec{M}$ of magnitude $M$ and parallel to the $z$ axis, embedded in a nonpermeable medium. Via the magnetic scalar potential in spherical coordinates and surface magnetic charge density, the potential is found to be

$$
\Phi_{M}(r, \theta)=\frac{1}{3} M a^{2} \frac{r_{<}}{r_{>}^{2}} \cos \theta,
$$

where $\left(r_{<}, r_{>}\right)$are smaller and larger of $(r, a)$. You don't have to prove this.
a) What is the magnetic field $\vec{H}$ inside the sphere?
b) What is the magnetic induction $\vec{B}$ inside the sphere?
c) From the expression for the potential outside the sphere, can you identify what is the dipole moment $\vec{m}$ ?

2 The field of a uniform spherical charge: Consider a spherical charge of uniform volume charge density $\rho$, total charge $Q$ and radius $R$.
a) Derive electric field $(\vec{E})$ inside of sphere using Gauss's law.
b) Derive electric field $(\vec{E})$ inside of sphere (1) using Poisson equation.
c) Derive electric field $(\vec{E})$ outside of sphere using Gauss's law.
d) Derive electric field $(\vec{E})$ outside of sphere (1) using Poisson equation.
e) Derive electric potential $(V)$ from $\vec{E}$ inside of sphere.
f) Derive electric potential $(V)$ from $\vec{E}$ outside of sphere.

## QUANTUM MECHANICS

## CHAPTER 3

## EXERCISES

## EXERCISES

1 (Easy) Obtain the expansion coefficients of a wavefunction $\psi$ when the Hilbert space is spanned by
a) the sequence of functions $\left\{\varphi_{n}\right\}$, and
b) the continuous set of momentum eigenfunctions.

2 (Medium) Translation operators and the fundamental commutation relations.
a) By considering an infinitesimal translation by $d \mathbf{x}^{\prime}$, prove the fundamental commutation relation $\left[\hat{x}_{j}, \hat{p}_{k}\right]=i \hbar \delta_{j k}$. State all the properties of the translation operator required in deriving the result.
b) Using the above result, derive the translation operator for a finite translation and use the commutativity of translations in different directions to derive $\left[\hat{p}_{j}, \hat{p}_{k}\right]=0$

3 Momentum operator and momentum space.
a) (Medium) Using the fact that the momentum operator is the generator of translation (in position), prove that the momentum operator acting on the wavefunction (in the position representation) obeys $\hat{p}_{x} \psi(x)=-i \hbar \frac{\partial}{\partial x^{\prime}} \psi(x)$.
b) (Easy) Use the above result to derive the momentum eigenfunction. That is, the momentum eigenket in the position space.
c) (Easy) Show that the position- and momentum-space wavefunctions are related through Fourier transformation.

4 Evaluate the following commutators.
(Easy)
a) $[\hat{x}, \hat{p}]$.
b) $\left[\hat{x}^{2}, \hat{p}\right]$.
c) $\left[\hat{x}, \hat{p}^{2}\right]$.
(Medium)
d) $[\hat{p}, g(x)]$.
e) $[\hat{p}, \hat{x} f(\hat{p})]$.
f) $[\hat{p}, \hat{x} g(\hat{x})]$.

5 (Medium) Uncertainty relations.
a) Prove the Schwarz inequality $\langle\alpha \mid \alpha\rangle\langle\beta \mid \beta\rangle \geq|\langle\alpha \mid \beta\rangle|^{2}$.
b) Using the above inequality prove that $\left\langle(\Delta A)^{2}\right\rangle\left\langle(\Delta B)^{2}\right\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^{2}$ for $\Delta A \equiv$ $A-\langle A\rangle$ and similarly for $\Delta B$.
6 (Easy) The time derivative of the expectation value of an observable $\langle\hat{A}\rangle$ is expressed in terms of the Hamiltonian as

$$
\frac{d}{d t}\langle\hat{A}\rangle=\left\langle\frac{i}{\hbar}[\hat{H}, \hat{A}]+\frac{\partial \hat{A}}{\partial t}\right\rangle
$$

a) Derive the above expression.
b) If $\hat{A} \neq \hat{A}(t)$, and $\hat{H}$ and $\hat{A}$ commute with each other, show that $\langle\hat{A}\rangle,\left\langle\hat{A}^{2}\right\rangle$, and $\Delta A$ are constant in time.
7 (Medium) Answer the following questions for the matrix $A=\left(\begin{array}{ccc}1 & i & 1 \\ -i & 0 & 0 \\ 1 & 0 & 0\end{array}\right)$.
a) Is $A$ Hermitian?
b) Find the eigenvalues of $A$.
c) Find the matrix $U$ that diagonalizes $A$.
d) Show that the eigenvectors comprising $U$ are orthonormal.
e) Carry out the matrix multiplication to verify that $U^{-1} A U$ is diagonal.

8 (Easy) Normalize the wavefunction, $\psi(x)=\frac{1}{\sqrt{a}} \cos \frac{\pi x}{2 a}+\frac{2}{\sqrt{a}} \sin \frac{\pi x}{a}$ for $-a \leq x \leq a$ and plot its probability density function.
9 (Easy) Prove the following angular momentum commutator relations by using the basic commutator relations $[\hat{x}, \hat{p}]=i \hbar$ and others related to it.
a) $\left[\hat{L}_{i}, \hat{L}_{j}\right]=i \hbar \varepsilon_{i j k} \hat{L}_{k}$ (or in other words $\vec{L} \times \vec{L}=i \hbar \vec{L}$ ).
b) $\left[L_{i}, L^{2}\right]=0$.
c) $\left[L_{i}, x_{k}\right]=i \hbar \varepsilon_{i k l} x_{l}, \quad\left[L_{i}, x_{j}\right]=\left[x_{i}, L_{j}\right]$.
d) $\left[L_{i}, p_{k}\right]=i \hbar \varepsilon_{i k l} p_{l}, \quad\left[L_{i}, p_{j}\right]=\left[p_{i}, L_{j}\right]$.

10 (Easy-Medium ) Establish the following properties of the Pauli spin matrices.
a) Show that the Pauli spin matrices are Hermitian.
b) $\sigma_{x}^{2}=\sigma_{y}^{2}=\sigma_{z}^{2}=I$.
c) $\sigma_{i} \sigma_{j}=i \varepsilon_{i j k} \sigma_{k}$.
d) $\left[\sigma_{i}, \sigma_{j}\right]=2 i \varepsilon_{i j k} \sigma_{k}$.
e) $\left\{\sigma_{i}, \sigma_{j}\right\}=0$ if $i \neq j$.
f) $(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b})=\vec{a} \cdot \vec{b} I+i \vec{\sigma} \cdot(\vec{a} \times \vec{b})$.
g) Show that $\exp (i \vec{e} \cdot \vec{\sigma} \phi)=(\cos \phi) I+i(\sin \phi) \vec{e} \cdot \vec{\sigma}$.

11 (Easy) A spin $1 / 2$ particle with a magnetic moment $\mu$ is placed in a magnetic field. The Hamiltonian is $H=-\frac{2 \mu}{\hbar} \mathbf{S} \cdot \mathbf{B}$. Find the eigenvalues and eigenstates for $\mathbf{B}=B_{x} \hat{x}+B_{z} \hat{z}$.
12 Rotations in the two-component formalism
a) (Medium) Write $\exp \left(\frac{-i}{\hbar} \mathbf{S} \cdot \hat{\mathbf{n}} \phi\right)$ in the 2 by 2 matrix form. $\hat{\mathbf{n}}$ is a unit vector.
b) (Easy) Use the above result to find the ket resulting from rotating the twocomponent spinor $\chi=[\alpha ; \beta]$ (a column vector) by an angle $\theta$ around the $x$ axis.
c) (Easy) Now find the probability that the resulting state is found in $|\uparrow\rangle$, i.e. the spin up state in the $z$-direction.

13 (Medium) Prove $\left[J_{x}, J_{y}\right]=i \hbar J_{z}$ by using the infinitesimal forms of rotation operators

$$
R_{x}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{array}\right), R_{y}=\left(\begin{array}{ccc}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{array}\right), R_{z}=\left(\begin{array}{ccc}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right) .
$$

14 Angular momentum eigenvalues and eigenstates
a) (Easy) Prove $\left[J_{+}, J_{-}\right]=2 \hbar J_{z}$ and $\left[J_{z}, J_{ \pm}\right]= \pm \hbar J_{ \pm}$.
b) (Easy) For the simultaneous eigenstates $|a, b\rangle$ such that $\mathbf{J}^{2}|a, b\rangle=a|a, b\rangle$ and $J_{z}|a, b\rangle=b|a, b\rangle$, show that $J_{ \pm}$act as raising/lowering operators.
c) (Medium) Prove that $a$ can be written as $\hbar^{2} j(j+1)$ in which $j$ can take integer and half-integer values, and $b$ can be written as $m \hbar$. What are the possible values of $b$ for a given $j$ ?
d) (Medium) Find the matrix elements of $\mathbf{J}^{2}, J_{z}$ and $J_{ \pm}$in the eigen basis $|j, m\rangle$.

15 (Medium) A system is in the superposition state $\varphi(\theta, \phi)=\sqrt{\frac{3}{4 \pi}} \sin \theta \sin \phi$ at a given time (Use a table for spherical harmonics which will be given in the qualifying exam).
a) Discuss the possible values when $L_{z}$ is measured.
b) Obtain $\left\langle L_{x}\right\rangle,\left\langle L_{y}\right\rangle,\left\langle L_{z}\right\rangle$, and $\left\langle L^{2}\right\rangle$ in this state.

16 (Medium) Addition of Angular momentum.
a) Consider the total angular momentum $\mathbf{J}=\mathbf{J}_{1}+\mathbf{J}_{2}$. Derive the recurrence relations for the Clebsch-Gordan coefficients $\left\langle j_{1} j_{2} ; m_{1} m_{2} \mid j_{1} j_{2} ; j m\right\rangle$.
b) There are two spin $1 / 2$ particles. By computing the Clebsch-Gordan coefficients (using the above recurrence relations or otherwise), write all the total angular momentum states $\left|\frac{1}{2} \frac{1}{2} ; j m\right\rangle$ in terms of the single particle eigenstates $\left.\left|\frac{1}{2} \frac{1}{2}\right\rangle \equiv \right\rvert\, \uparrow$ $\rangle,\left|\frac{1}{2},-\frac{1}{2}\right\rangle \equiv|\downarrow\rangle$. Remember to correctly label the states with appropriate $j$ and $m$.

## CHAPTER 4

## PREVIOUS TEST PROBLEMS

This chapter collects the previous test problems since 2008. Some questions may be missing.

### 4.1 Fall 2020

## PROBLEMS

1 At a certain time, the simultaneous measurement of two components of the linear momentum and one component of the position (all three in different directions) yields the following values: $p_{x}=\hbar k_{1}, y=b, p_{z}=\hbar k_{2}$. Write down the wavefunction (in the position representation) which describes the quantum state of the system right after the measurement. You do not need to explicitly specify the normalisation prefactor.
2 In a certain quantum state, the spin of an electron has average value of its $z$-component $\left\langle s_{z}\right\rangle=\hbar / 4$. Calculate the uncertainty of the $z$-component of the spin, $\Delta s_{z}=\sqrt{\left\langle s_{z}^{2}\right\rangle-\left\langle s_{z}\right\rangle^{2}}$, for that particular state.
3 A particle is in the ground state of a one-dimensional harmonic oscillator and the uncertainty of its position is known to be $\Delta x$. Calculate the uncertainty of its momentum.
4 The state of a particle in one dimension is given by the wavefunction $\Psi(x)=\mathcal{N} x \exp \left(-x^{2} / \lambda\right)$. $\mathcal{N}$ is a normalisation prefactor and $\lambda>0$ a constant. Where (at which position) is the probability to find the particle highest?

### 4.2 Fall 2019

You choose type A or type B

## PROBLEMS Type A

1 Consider a 3-dimensional Hilbert space on which the Hamiltonian is given by

$$
H=\left(\begin{array}{ccc}
0 & a & a \\
a & 0 & 0 \\
0 & 0 & 2 a
\end{array}\right)
$$

with $a>0$.
a) Find all the eigenvalues and the corresponding eigenstates of the Hamiltonian.
b) At $t=0$, a state is given as $|\Psi(t=0)\rangle=\left(\begin{array}{c}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0\end{array}\right)$. What is $|\Psi(t>0)\rangle$ ?
c) Calculate the expectation value of a physical quantity $L=\left(\begin{array}{lll}b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d\end{array}\right)$ for the state $|\Psi(t>0)\rangle$ in the previous problem.

## PROBLEMS Type B

1 [30 pts] Consider a particle in a quantum state which is described by a real wavefunction $\Psi(x, y, z)$. Calculate the average value of each one of the three components of the orbital angular momentum of that particle. Your answer should be a number!

2 [35 pts] Which of the following pairs of observables are compatible (i.e. they can be measured simultaneously with zero uncertainty) and which are not?

$$
\left(x, l_{x}\right), \quad\left(x, l_{y}\right), \quad .\left(p_{x}, l_{x}\right), \quad .\left(l_{x}, l_{y}\right)
$$

where $l_{i}$ is the $i$-th component of the orbital angular momentum. Explain your answer.
3 [35 pts] Repeated measurements of the energy of a quantum harmonic oscillator (onedimensional) in the same quantum state gave only two values: $E_{0}=(1 / 2) \hbar \omega$ and $E_{1}=$ (3/2) $\hbar \omega$ with probabilities $P_{0}=1 / 3$ and $P_{1}=2 / 3$. In that quantum state we also have $\langle x\rangle=0$. Write down the quantum state as a linear combination of two energy eigenstates. Which are the two energy eigenstates? What are coefficients in the linear combination? (Hint: The coefficients are, in general, complex numbers!)

### 4.3 Spring 2019

## PROBLEMS

1 Consider a quantum state with the Hamiltonian $H=\left(\begin{array}{cc}h_{1} & 0 \\ 0 & h_{2}\end{array}\right)$. Here $h_{1}<h_{2}$. A state of this system is described by two complex numbers. We consider a normalized state $|\Psi\rangle=\binom{a}{b}$ where $|a|^{2}+|b|^{2}=1$. We now make a measurement of $H$ on this state $|\Psi\rangle=\binom{a}{b}$.
a) The possible values of the measurement are $h_{1}$ and $h_{2}$. What is the probability for the measured value to be $h_{2}$ ?
b) What is the expectation value of $H$ for the state $|\Psi\rangle=\binom{a}{b}$ ?
c) $S=\left(\begin{array}{ll}0 & s \\ s & 0\end{array}\right)$ is another physical observable. Here $s$ is real and positive (not complex). When we make a measurement of $S$, what are the possible measured values? List all of them.
d) Now we make a measurement of $S=\left(\begin{array}{ll}0 & s \\ s & 0\end{array}\right)$ on the same state $|\Psi\rangle=\binom{a}{b}$. What is the probability for the measured value to be the biggest one among the possible values?

### 4.4 Fall 2018

## PROBLEMS

1 Consider a quantum system with the Hamiltonian $H=\left(\begin{array}{ll}0 & h \\ h & 0\end{array}\right)$. Here we use the bases $\left\{\binom{1}{0},\binom{0}{1}\right\}$. It is easy to see that the two basis states are not eigenstates of $H$.
a) Calculate the expectation value $\langle\Psi| H|\Psi\rangle$ of a state $|\Psi\rangle=\binom{1}{0}$.
b) Find the ground state eigenvalue of $H$ and the corresponding normalized eigenstate.
c) At time $t=0$, a physical state is given by $|\Psi(t=0)\rangle=\binom{1}{0}$. Later when $t=T$, how the physical state $|\Psi(t=T)\rangle$ is described?

### 4.5 Spring 2018

## PROBLEMS

1 The Hamiltonian of an $N$-dimensional quantum system $H$ is given with its eigenvalues and eigenstates as follows.

$$
H\left|\psi_{n}\right\rangle=E_{n}\left|\psi_{n}\right\rangle, \quad n=1,2, \cdots N
$$

Assume that a state at $t=0,|\psi(t=0)\rangle$ can be written as

$$
|\psi(t=0)\rangle=\sum_{n=1}^{N} a_{n}\left|\psi_{n}\right\rangle \quad \text { with } \sum_{n=1}^{N}\left|a_{n}\right|^{2}=1
$$

a) Express $|\psi(t \neq 0)\rangle$, the state given above at $t \neq 0$, as a linear combination of $\left|\psi_{n}\right\rangle$ 's.
b) What is the expectation value of $H$ of this state?
c) Consider an operator $Q=\left|\psi_{1}\right\rangle\left\langle\psi_{2}\right|+\left|\psi_{2}\right\rangle\left\langle\psi_{1}\right|$ in Schrödinger picture. Here, $\left|\psi_{1}\right\rangle$, $\left|\psi_{2}\right\rangle$ are two eigenstate of $H$ given above. Express this operator in Heisenberg picture.

### 4.6 Fall 2017

## PROBLEMS

1 [50 pts] Let's consider a potential of the form

$$
V(x)=-\alpha \delta(x)
$$

where $\alpha$ is a positive constant.
a) (10 pts) Write the time-independent Schrödinger equation for the delta-function well.
b) (10 pts) Find the solutions of Schrödinger equation of (a) for the region $x<0$ and $x>0$ at the bound state $(E<0)$. (Leave the coefficient as unknown)
c) (10 pts) Write two boundary conditions.
d) (10 pts) Using the boundary conditions in (c) and the normalization of $\psi$, find the wave function and allowed energy. (You have to find the coefficient of the wave function.)
e) (10 pts) Find the general solutions of (a) for both regions $x<0$ and $x>0$ at the scattering state $(E>0)$.

2 [50 pts] Addition of Angular Momenta. Suppose that there are two spin-1/2 particles for questions (a)-(c).
a) (10 pts) What are the allowed values for total spin?
b) ( 10 pts ) What are the allowed values of $s_{z}$ ?
c) ( 10 pts ) Write the corresponding eigenstates in the notation $\left|s m, s_{1} s_{2}\right\rangle$.
d) ( 20 pts ) Let's consider the hyperfine interaction between the electron and proton in the hydrogen atom. The Hamiltonian describing this interaction, which is due to the magnetic moments of the two particles is

$$
H_{h f}=A \vec{A}_{1} \cdot \vec{S}_{2}, \quad(A>0)
$$

This formula assumes the orbital state of the electron is $|1,0,0\rangle$. The total Hamiltonian is the Coulomb Hamiltonian plus $H_{h f}$. $H_{h f}$ splits the ground state into two levels. Find these two energy levels.

### 4.7 Spring 2017

## PROBLEMS

1 [25 pts] The Pauli matrices $\sigma_{i}$ satisfy the following commutation and anti-commutation relations:

$$
\left[\sigma_{i}, \sigma_{j}\right]=2 i \varepsilon_{i j k} \sigma_{k}, \quad\left\{\sigma_{i}, \sigma_{j}\right\}=2 \delta_{i j}
$$

a) ( 15 pts ) Using the two relations show that

$$
(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b})=\vec{a} \cdot \vec{b} I+i \vec{\sigma} \cdot(\vec{a} \times \vec{b}),
$$

where $\vec{a}$ and $\vec{b}$ are three-dimensional vectors and $I$ is the $2 \times 2$ unit matrix.
b) ( 10 pts ) Evaluate $(\vec{\sigma} \cdot \hat{n})^{2}$, where $\hat{n}$ is an arbitrary unit vector, $|\hat{n}|=1$.

2 [25 pts] You are given the following information:

$$
S_{x}| \pm\rangle_{x}= \pm \frac{\hbar}{2}| \pm\rangle_{x}, \quad S_{z}| \pm\rangle_{z}= \pm \frac{\hbar}{2}| \pm\rangle_{z}
$$

The two bases $| \pm\rangle_{x}$ and $| \pm\rangle_{z}$ are related to each other by

Find the matrix representation of $S_{x}$ in the $S_{z}$ basis.
3 [25 pts] Consider a 1-dimensional wave function given by

$$
\psi(x)=N \exp (-a|x-5|),
$$

where $N$ is the normalization constant and $a$ is a positive real number.
a) ( 15 pts ) Find the expectation value of the position.
b）（10 pts）Find the probability that the particle is observed in the range $-1<x<1$ ． Hint：You may use the following identities：

$$
\int e^{\alpha x}=\frac{e^{\alpha x}}{\alpha}, \quad \int x e^{\alpha x}=\frac{e^{\alpha x}}{\alpha^{2}}(\alpha x-1), \quad \int x^{2} e^{\alpha x}=\frac{e^{\alpha x}}{\alpha^{3}}\left(\alpha^{2} x^{2}-2 \alpha x+2\right)
$$

4 ［25 pts］Consider a system of three states whose Hamiltonian is given by the matrix of

$$
H=V_{0}\left(\begin{array}{ccc}
1 & \varepsilon & 0 \\
\varepsilon & 2 & 2 \varepsilon \\
0 & 2 \varepsilon & 3
\end{array}\right)
$$

where $V_{0}$ is a constant energy and $\varepsilon$ is a small number $(\varepsilon \ll 1)$ so that perturbation theory can be applied．
a）（ 5 pts ）Find the eigenvectors and eigenvalues of the unperturbed Hamiltonian， where $\varepsilon=0$ ．
b）（20 pts）Obtain the leading correction to the energy of the highest－energy state in the zeroth－order Hamiltonian．What is the corrected total energy of this state？

Hint：The first－order and second－order energy corrections to the state $\left|n^{(0)}\right\rangle$ are given by

$$
\begin{aligned}
E_{n}^{(1)} & =\left\langle n^{(0)}\right| H^{\prime}\left|n^{(0)}\right\rangle \\
E_{n}^{(2)} & =\sum_{m \neq n} \frac{\left.\left|\left\langle n^{(0)}\right| H^{\prime}\right| m^{(0)}\right\rangle\left.\right|^{2}}{E_{n}^{(0)}-E_{m}^{(0)}}
\end{aligned}
$$

where $H^{\prime}$ is the perturbing small Hamiltonian．

## 4．8 Fall 2016

## PROBLEMS

> I皃 Justify your answers. Please be brief but clear!
> t写 Put each final answer in a box.
> t菅 Solve all problems.

1 （30 points）You are given a quantum state

$$
|\Psi\rangle=N\left(\left|\Psi_{1}\right\rangle+2\left|\Psi_{2}\right\rangle+|\Psi\rangle_{3}\right)
$$

where $|\Psi\rangle_{1},|\Psi\rangle_{2},|\Psi\rangle_{3}$ are normalized eigenkets of an observable $\hat{A}$ with eigenvalues $a_{1}=-2, a_{2}=0, a_{3}=2$ ．Calculate the normalization constant $N$ and then the average value $\langle\hat{A}\rangle$ and the uncertainty $\Delta \hat{A}$ of the observable $\hat{A}$ ．
2 （50 points）Consider a two－particle system in which the momenta and masses of the particles are $\left(\vec{p}_{1}, m_{1}\right)$ and $\left(\vec{p}_{2}, m_{2}\right)$ ，respectively．The potential of interaction is a function of the radial distance between particles，called a central potential．We want to express physical quantities in terms of the center of mass and relative coordinates．The center of mass position and momentum are

$$
\vec{R}=\frac{m_{1} \vec{r}_{1}+m_{2} \vec{r}_{2}}{m_{1}+m_{2}}, \quad \vec{P}=\vec{p}_{1}+\vec{p}_{2}
$$

and the relative position and momentum are

$$
\vec{r}=\vec{r}_{2}-\vec{r}_{1}, \quad \vec{p}=\frac{m_{1}}{m_{1}+m_{2}} \vec{p}_{2}-\frac{m_{2}}{m_{1}+m_{2}} \vec{p}_{1}
$$

a) (20 pts) Show that

$$
\frac{\vec{p}_{1}^{2}}{2 m_{1}}+\frac{\vec{p}_{2}^{2}}{2 m_{2}}=\frac{\vec{p}^{2}}{2 \mu}+\frac{\vec{P}^{2}}{2 M},
$$

where $\mu$ is the reduced mass and $M$ is the total mass. Give the detailed derivation and find the expression of $\mu$.
b) ( 15 pts ) Show that $m_{1} \vec{r}_{1}^{2}+m_{2} \vec{r}_{2}^{2}=\mu r^{2}+M \vec{R}^{2}$. Give the detailed derivation.
c) ( 15 pts ) Show that $\vec{p}_{1} \cdot \vec{r}_{1}+\vec{p}_{2} \cdot \vec{r}_{2}=\vec{p} \cdot \vec{r}+\vec{P} \cdot \vec{R}$. Give the detailed derivation.

3 (20 points) Given the matrix

$$
A=\left(\begin{array}{ccc}
1 & i & 1 \\
-i & 0 & 0 \\
1 & 0 & 0
\end{array}\right)
$$

a) (10 pts) Is $A$ Hermitian?
b) (10 pts) Find the eigenvalues of $A$.

## 4．9 Spring 2016

## PROBLEMS

L 宫 Justify your answers．
I客 Put each final answer in a box．
l容 All problems carry equal weight．
1 You are given a quantum state

$$
|\Psi\rangle=N\left(\left|\Psi_{1}\right\rangle+2\left|\Psi_{2}\right\rangle+|\Psi\rangle_{3}\right),
$$

where $|\Psi\rangle_{1},|\Psi\rangle_{2},|\Psi\rangle_{3}$ are eigenkets of an observable $\hat{A}$ with eigenvalues $a_{1}=-1, a_{2}=0$ ， $a_{3}=1$ ．Calculate $N$ and then the average value $\langle\hat{A}\rangle$ and the uncertainty $\Delta \hat{A}$ of the observable $\hat{A}$ ．

2 A particle of mass $m$ ，in one dimension，experiences a force $F=-k x(k>0)$ ，and its quantum state，at a given instant，is represented by the wavefunction

$$
\Psi(x)=N \exp \left(-\lambda x^{2} / 2\right)
$$

where $N$ is a normalization prefactor．Calculate the value of the parameter $\lambda$ for which the energy of the particle becomes sharply defined（in other words，its uncertainty vanishes）．

3 The wavefunction of a particle in one dimension，at a given instant，has the form

$$
\Psi(x)=\frac{1}{\sqrt{2}}\left[\Psi_{1}(x)+\Psi_{2}(x)\right]
$$

where $\Psi_{1}(x)$ and $\Psi_{2}(x)$ are normalised eigenfunctions of the energy with eigenvalues $E_{1}$ and $E_{2}$ ，respectively．Assuming that $\Psi_{1}(x)$ is an even function and $\Psi_{2}(x)$ is an odd function （both real），calculate the average position of the particle at time $t:\langle x\rangle_{t}$ ．Express your answer in terms of $E_{2}-E_{1}$ and $\langle x\rangle_{t=0}$ ．

## 4．10 Fall 2015

## PROBLEMS

I宫 Justify your answers．
I客 Put each final answer in a box．
IT 军 All problems carry equal weight．
1 Calculate the average value of the momentum for a state described by a wavefunction of the form

$$
\Psi(x)=\psi(x) \exp (i k x)
$$

where $k$ is a real number and $\psi(x)$ is a real，square integrable function with $\int_{-\infty}^{+\infty} \psi^{2}(x) d x=$ 1.

2 In a certain quantum state，the spin of an electron has average value of its $z$－projection equal to $\hbar / 4$ ．Calculate the probabilities $\left(P_{+}, P_{-}\right)$to find the spin up or down in the $z$－direction for that particular state．

3 A particle is trapped in an infinitely deep square potential well（one－dimensional）．The energy of the ground state is 3 eV ．Calculate its average energy in a state described by the wavefunction

$$
\Psi=\frac{1}{\sqrt{3}} \Psi_{1}+i \sqrt{\frac{2}{3}} \Psi_{2},
$$

where $\Psi_{1}$ and $\Psi_{2}$ are the ground and the first excited state，respectively．

## 4．11 Spring 2015

## PROBLEMS

1 When 1－dimensional potential is given by $V(x)=V_{0} \delta(x)$ ．Here，$\delta(x)$ is the Dirac delta function．Calculate the coefficients of transmission and reflection．

2 Show the following relations on $\delta(x)$ function．
a）$\delta(-x)=\delta(x)$
b）$x \delta^{\prime}(x)=-\delta^{\prime}(x) \quad$（Prime denotes differentiation．）
c）$\delta(a x)=\frac{\delta(x)}{a} \quad(a>0)$

### 4.12 Fall 2014

## PROBLEMS

Consider a 3-dimensional Hilbert space on which the Hamiltonian is given by

$$
H=\left(\begin{array}{ccc}
\epsilon & 0 & 0 \\
0 & 2 \epsilon & 0 \\
0 & 0 & 3 \epsilon
\end{array}\right)
$$

with $\epsilon>0$.
1 Find all the eigenstates and the corresponding eigenvalues of the Hamiltonian.
2 At $t=0$, a state is given as

$$
|\Psi(t=0)\rangle=\left(\begin{array}{c}
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}}
\end{array}\right) .
$$

What is $|\Psi(t>0)\rangle$ ?
3 A physical quantity $L$ is defined as

$$
L=\left(\begin{array}{ccc}
0 & 0 & i \\
0 & 0 & 0 \\
-i & 0 & 0
\end{array}\right)
$$

with $i=\sqrt{-1}$.
a) When we measure this quantity $L$, what are the possible outcomes? In other words, list all the possible values of the measurement.
b) Consider the state $|\Psi(t>0)\rangle$ in question 2 . We want to make a measurement of $L$ for this state $|\Psi(t>0)\rangle$. Calculate the probability for the outcome of the measurement to be the maximum value among the list in a).

### 4.13 Spring 2014

## PROBLEMS

Consider the one-dimensional harmonic oscillator with the Hamiltonian,

$$
H=\frac{p^{2}}{2 m}+\frac{1}{2} m \omega^{2} x^{2}
$$

1 Write the Heisenberg equations of motion for $p$ and $x$.
2 Write the Heisenberg equations of motion for

$$
a=\frac{1}{\sqrt{2}}(Q+i P) \quad \text { and } \quad a^{\dagger}=\frac{1}{\sqrt{2}}(Q-i P) .
$$

Here, we define new operators

$$
P=\frac{p}{\sqrt{m \omega \hbar}} \quad \text { and } \quad Q=x \sqrt{\frac{m \omega}{\hbar}}
$$

3 Find the solutions of question 2. That is, determine $a(t)$ and $a^{\dagger}(t)$.
4 Check the time dependence or independence of the number operator $N=a^{\dagger} a$.
5 Find $x(t)$ and $p(t)$.
6 Express $\exp \left(\frac{i H t}{\hbar}\right) x(0) \exp \left(-\frac{i H t}{\hbar}\right)$ in terms of $x(0), p(0), m, \omega$ and $t$ and compare with $x(t)$ found in question 5.

### 4.14 Fall 2013

## PROBLEMS

Consider the one-dimensional harmonic oscillator with the Hamiltonian

$$
H=\frac{p^{2}}{2 m}+\frac{1}{2} m \omega^{2} x^{2}
$$

We define new operators

$$
P=\frac{p}{\sqrt{m \omega \hbar}} \quad \text { and } \quad Q=x \sqrt{\frac{m \omega}{\hbar}}
$$

1 Write the Hamiltonian in terms of $P$ and $Q$.
2 Compute the commutator relation $[P, Q]$.
3 For the operators

$$
a=\frac{1}{\sqrt{2}}(Q+i P) \quad \text { and } \quad a^{\dagger}=\frac{1}{\sqrt{2}}(Q-i P)
$$

write the Hamiltonian in terms of $a$ and $a^{\dagger}$.
4 Compute $[a, H]$ and $\left[a^{\dagger}, H\right]$ in terms of $a$ or $a^{\dagger}$.
5 Using the eigenvalue equation of the energy $H|n\rangle=\hbar \omega\left(n+\frac{1}{2}\right)|n\rangle$, what are the eigenvalues of $a a^{\dagger}$ and $a^{\dagger} a$ acting on the state $|n\rangle$ ?

6 Find the eigenvalue of $H$ acting on the state $a^{\dagger}|n\rangle$. Similarly, find the eigenvalue of $H$ acting on the state $a|n\rangle$. From this observation, find the $n$ dependent coefficients $A$ and $B$ in $a|n\rangle=A|n-1\rangle$ and $a^{\dagger}|n\rangle=B|n+1\rangle$.

### 4.15 Spring 2013

## PROBLEMS

1 An electron in the Coulomb field of a proton is in a state described by the wave function

$$
\frac{1}{6}\left[4 \psi_{100}(r)+3 \psi_{211}(r)-\psi_{210}(r)+\sqrt{10} \psi_{21-1}(r)\right]
$$

a) What is the expectation value of the energy?
b) What is the expectation value of $\vec{L}^{2}$ ?
c) What is the expectation value of $L_{z}$ ?

2 Hamiltonian of 1-dimensional harmonic oscillator

$$
H=\frac{p^{2}}{2 m}+\frac{1}{2} m \omega^{2} x^{2}
$$

can be written in terms of the non-Hermitian operator $a$ and its adjoint $a^{\dagger}$ as $H=\hbar \omega\left(a^{\dagger} a+c\right)$.
a) Determine $a$ and $a^{\dagger}$, and the constant $c$ if $\left[a, a^{\dagger}\right]=1$.
b) If $|n\rangle$ is an eigenstate of $N=a^{\dagger} a$ with eigenvalue $n$, show that

$$
a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle
$$

### 4.16 Fall 2012

## PROBLEMS

1 Use the uncertainty relation to estimate the ground state energy of a harmonic oscillator. The energy is given by

$$
E=\frac{p^{2}}{2 m}+\frac{1}{2} m \omega^{2} x^{2}
$$

2 Consider for the potential of the form

$$
V(x)= \begin{cases}\infty, & x<0 \\ 0 & 0<x<a \\ \infty & a<x\end{cases}
$$

a) find the normalized eigenfunction and eigenvalue.
b) find the expectation value of the kinetic energy.

3 Consider a particle in a box as in the above problem. Its wave function is $\psi(x)=\sqrt{\frac{2}{a}}$. Calculate the probability that an energy measurement yields the ground state energy.

### 4.17 Spring 2012

## PROBLEMS

1 Use the uncertainty relation to estimate the ground state energy of a harmonic oscillator. The energy is given by

$$
E=\frac{p^{2}}{2 m}+\frac{1}{2} m \omega^{2} x^{2}
$$

2 Calculate the following commutators.
a) $\left[p, p^{2}\right]$, where $p$ is the momentum operator.
b) $[x, p]$
c) $[x p, p]$
d) $[\mathbb{P}, H]$, where $H$ is general energy operator and $\mathbb{P}$ is the parity operator.

3 Consider for the potential of the form

$$
V(x)= \begin{cases}\infty, & x<0 \\ 0 & 0<x<a \\ \infty & a<x\end{cases}
$$

a) find the normalized eigenfunction and eigenvalue.
b) find the expectation value of the kinetic energy.

4 Consider a particle in a box as in the above problem. Its wave function is $\psi(x)=\sqrt{\frac{2}{a}}$. Calculate the probability that an energy measurement yields the ground state energy.

### 4.18 Fall 2011

## PROBLEMS

1 Consider a paricle whose normalized wave function is

$$
\Psi(x)= \begin{cases}2 \alpha \sqrt{\alpha} x e^{-\alpha x}, & x>0 \\ 0, & x<0\end{cases}
$$

a) Calculate $\langle x\rangle$ and $\left\langle x^{2}\right\rangle$.
b) Calculate $\Phi(p)$ and use this to calculate $\langle p\rangle$ and $\left\langle p^{2}\right\rangle$.

2 An electron in the Coulomb field of a proton is in a state described by the wave function

$$
\frac{1}{6}\left[4 \Psi_{100}(r)+3 \Psi_{211}(r)-\Psi_{210}(r)+\sqrt{10} \Psi_{21-1}(r)\right]
$$

a) What is expectation value of the energy?
b) What is expectation value of $\vec{L}^{2}$ ?
c) What is expectation value of $L_{z}$ ?

### 4.19 Spring 2011

## PROBLEMS

1 Obtain $\langle r\rangle$ and $\left\langle r^{2}\right\rangle$ for an electron in the ground state of hydrogen which is given by

$$
\psi_{100}(r, \theta, \phi)=\frac{1}{\sqrt{\pi a^{3}}} e^{-r / a}
$$

Here, $a$ is the Bohr radius.
2 Hamiltonian of 1-dimensional harmonic oscillator is given by

$$
H=\frac{p^{2}}{2 m}+\frac{1}{2} m \omega^{2} x^{2}
$$

and it can be also given by

$$
H=\hbar \omega\left(a^{\dagger} a+c\right)
$$

Here $a^{\dagger}$ is the adjoint of non-Hermitian operator $a$.
a) Show $a$ and $a^{\dagger}$, and the constant $c$ if $\left[a, a^{\dagger}\right]=1$.
b) If $|n\rangle$ is an eigenstate of $N=a^{\dagger} a$ with eigenvalue $n$, show that

$$
\begin{aligned}
a^{\dagger}|n\rangle & =\sqrt{n+1}|n+1\rangle \\
a|n\rangle & =\sqrt{n}|n-1\rangle .
\end{aligned}
$$

( $n$ is a non-negative integer.)

### 4.20 Fall 2010

## PROBLEMS

1 Consider a particle incident from left on a barrier of height $V$ an width $a$ as shown in the figure.

$$
V(x)= \begin{cases}0 & (x<0) \\ V_{0} & (0<x<a) \\ 0 & (x>a)\end{cases}
$$



Figure 4.1 Problem 1 of the test at Fall 2010.
a) Using a time-dependent Schrödinger equation, show that the following current conservation law is satisfied.

$$
\frac{\partial}{\partial t} \rho(x, t)+\frac{\partial}{\partial x} J(x, t)=0
$$

Here, $\rho(x, t)$ is the probability density and $J(x, t)$ is the probability current density.
b) Write Schrödinger equations and obtain their solutions in three regions of $x<0$, $0<x<a$ and $x>a$.
c) Obtain the probability that an incident particle penetrates beyond the barrier.

2 Consider a harmonic oscillator with Hamiltonian of

$$
\hat{H}=\frac{\hat{p}^{2}}{2 m}+\frac{1}{2} m \omega^{2} \hat{x}^{2}
$$

a) Using annihilation operator $(A)$ and creation operator $\left(A^{\dagger}\right)$, show that the commutator of $A$ and $A^{\dagger}$ is $\left[A, A^{\dagger}\right]=1$.
b) Obtain the commutators of $A$ and $A^{\dagger}$ with the Hamiltonian.
c) Show that

$$
U_{n}=\frac{\left(A^{\dagger}\right)^{n}}{\sqrt{n!}} U_{0}
$$

Here, $U_{n}$ and $U_{0}$ are the $n$-th and lowest eigenfunctions, respectively.

### 4.21 Spring 2010

## PROBLEMS

1 Calculate the expectation values of the potential and kinetic energies in any stationary state of the harmonic oscillator.

2 Find the transmission coefficient of a particle through a rectangular barrier for $E<V_{0}$.


Figure 4.2 Problem 2 of the test at Spring 2010.

### 4.22 Spring 2009

## PROBLEMS

1 A particle is confined in an infinite well of width $L$.
a) What is the probability that the particle in the ground state is found between $\frac{L}{4}$ and $\frac{3 L}{4}$ ?
b) What is the probability that the particle in the first excited state is found between $\frac{L}{4}$ and $\frac{3 L}{4}$ ?

2 Consider a system which is described by the state

$$
\psi(\theta, \phi)=\sqrt{\frac{3}{8}} Y_{11}(\theta, \phi)+\sqrt{\frac{1}{8}} Y_{10}(\theta, \phi)+A Y_{1-1}(\theta, \phi),
$$

where $A$ is a real constant.
a) Calculate $A$ so that $|\psi\rangle$ is normalized.
b) Find $L_{+} \psi(\theta, \phi)$.
c) Calculate the expectation values of $L_{x}$ and $\vec{L}^{2}$ in the state $|\psi\rangle$.
d) Find the probability associated with a measurement that gives zero for the $z$ component of the angular momentum.
e) Calculate $\langle\chi| L_{z}|\phi\rangle$ and $\langle\chi| L_{-}|\phi\rangle$, where

$$
\begin{gathered}
\chi(\theta, \phi)=\sqrt{\frac{8}{15}} Y_{21}(\theta, \phi)+\sqrt{\frac{4}{15}} Y_{10}(\theta, \phi)+\sqrt{\frac{3}{15}} Y_{2-1}(\theta, \phi), \\
\text { (Hint) } \quad L_{ \pm}|l, m\rangle=\hbar \sqrt{l(l+1)-m(m \pm 1)}|l, m \pm 1\rangle \\
L_{x}=\frac{1}{2}\left(L_{+}+L_{-}\right), \quad \vec{L}^{2}=L_{+} L_{-}+L_{z}^{2}-\hbar L_{z}
\end{gathered}
$$

3
a) Show that the operators $O_{1} \psi(x)=x^{3} \psi(x)$ and $O_{2} \psi(x)=\left(x \frac{d}{d x}\right) \psi(x)$ are linear operators.
b) Calculate the commutator of $\left[O_{1}, O_{2}\right]$.
c) Show that the following operators are not linear operators.

$$
O_{4} \psi(x)=e^{\psi(x)}, \quad O_{5} \psi(x)=\frac{d \psi(x)}{d x}+a
$$

(Hint) Definition of linear operator $L: L\left[\psi_{1}(x)+\psi_{2}(x)\right]=L \psi_{1}(x)+L \psi_{2}(x)$ and $L c \psi(x)=c L \psi(x)$.

### 4.23 Fall 2008

## PROBLEMS

LT T Y Yu can solve only one problem among two problems.
1 One dimensional harmonic oscillator: For the Hamiltonian

$$
H=\frac{p^{2}}{2 m}+\frac{m \omega^{2} x^{2}}{2} .
$$

one can introduce

$$
a=\sqrt{\frac{m \omega}{2 \hbar}}\left(x+\frac{i p}{m \omega}\right), \quad a^{\dagger}=\sqrt{\frac{m \omega}{2 \hbar}}\left(x-\frac{i p}{m \omega}\right) .
$$

a) Express $x^{2}$ in terms of $a$ and $a^{\dagger}$.
b) Find the expectation value of $x^{2}$ for the ground state $|0\rangle$.
c) Express $p^{2}$ in terms of $a$ and $a^{\dagger}$.
d) Find the expectation value of $p^{2}$ for the ground state $|0\rangle$.
e) Find the expectation value of $H$ for the ground state $|0\rangle$.

2 Addition of spin angular momentum: Let $\vec{S}=\vec{S}_{1}+\vec{S}_{2}$ be the total spin angular momentum of two spin $1 / 2$ particles. Note that

$$
S_{ \pm}\left|S, m_{S}\right\rangle=\hbar \sqrt{S(S+1)-m_{S}\left(m_{S} \pm 1\right)}\left|S, m_{S} \pm 1\right\rangle .
$$

One can compute C-G coefficients $\left\langle m_{1}, m_{2} \mid S, m_{S}\right\rangle$ by successive applications of $S_{ \pm}=$ $S_{x} \pm i S_{y}$ on the vectors $\left|S, m_{S}\right\rangle$. Work out in the two subspaces $S=1$ and $S=0$. In other words, find out $|1,1\rangle,|1,0\rangle,|1,-1\rangle,|0,0\rangle$ in terms of $|+,+\rangle,|+,-\rangle,|-,+\rangle,|-,-\rangle$ explicitly.

## APPENDIX A

## OLDER TEST PROBLEMS

## A. 1 Fall 2006 (QT: Quantum Mechanics)

## PROBLEMS

1 The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the $z$-direction can be written as

$$
H=A \vec{S}^{\left(e^{-}\right)} \cdot \vec{S}^{\left(e^{+}\right)}+\left(\frac{e B}{m c}\right)\left(S_{z}^{\left(e^{-}\right)}-S_{z}^{\left(e^{+}\right)}\right)
$$

Suppose the spin function of the system is given by $\chi_{+}^{\left(e^{-}\right)} \chi_{-}^{\left(e^{+}\right)}$.
a) Is this an eigenfunction of $H$ in the limit $A \rightarrow 0, e B / m c \neq 0$ ?
b) If it is, what is the energy eigenvalue? If it is not, what is the expectation value of $H$ ?
2 Show that the substitution $\vec{\nabla} \rightarrow \vec{\nabla}-\frac{i e}{\hbar c} \vec{A}$ in $\vec{j}=\frac{\hbar}{2 m i}\left[\psi^{*} \vec{\nabla} \psi-\left(\vec{\nabla} \psi^{*}\right) \psi\right]$ produces a gauge-invariant current density and that this new $\vec{j}$ satisfies the continuity equation

$$
\frac{\partial \rho}{\partial t}+\vec{\nabla} \cdot \vec{j}=0
$$

for the Schrödinger equation

$$
i \hbar \frac{\partial \psi(\vec{x}, t)}{\partial t}=-\frac{\hbar^{2}}{2 m}\left(\vec{\nabla}-\frac{i e}{\hbar c} \vec{A}\right)^{2} \psi(\vec{x}, t)+e \Phi \psi(\vec{x}, t)
$$

in the presence of an electromagnetic field.

## A. 2 Spring 1992 (QT: Quantum Mechanics)

## PROBLEMS

1 Let's assume that the nuclear charge $Z e$ is uniformly distributed inside the sphere of radius $R$ for the hydrogen-like atom.
a) Express the potential energy difference due to the finite size of the nucleus.
b) Using the above results, calculate the first order correction to the ground state energy of a hydrogen-like atom.

$$
\text { (Note): } R_{10}(r)=2\left(a_{0}\right)^{-3 / 2} \exp \left(-r / a_{0}\right)
$$

2 Consider a sysytem of two spin- $1 / 2$ particles which are fixed at different positions. Let the only interaction between two particles is given by

$$
W=A\left(\vec{S}_{1} \cdot \vec{S}_{2}+B S_{1 z} s_{2 z}\right)
$$

where $\vec{S}_{1}$ and $\vec{S}_{2}$ are the spins of the two particles.
a) Represent all possible eigenstates $\left|S, m_{s}\right\rangle$ of the operators $\vec{S}\left(=\vec{S}_{1}+\vec{S}_{2}\right)$ and $S_{z}$.
b) Show that the state $\left|S, m_{s}\right\rangle$ is the eigenstate of the Hamiltonian $W$ and find the corresponding eigenvalue.
c) If each particle has the spin value $S_{1 z}=\frac{1}{2}$ and $S_{2 z}=-\frac{1}{2}$, respectively, at $t=0$, find the state of the system as a function of time. Find the time required to recover the initial configuration.
3
a) Write down the Heisenberg picture equations of motion for the harmonic oscillator and solve them.
b) If $N=a^{\dagger} a$, estimate $\langle N\rangle$ and $\Delta N$ for an oscillator with mass $m=1 \mathrm{gm}$ and frequency $\omega=1 \mathrm{~Hz}$, if the wave packet has a speed of $10 \mathrm{~cm} / \mathrm{s}$ at the minimum of the potential.

## A. 3 June 1991 (Master degree QT: Quantum Mechanics)

## PROBLEMS

1 A particle of mass $m$ is trapped in a one-dimensional potential such as

$$
V= \begin{cases}\frac{1}{2} k x^{2}, & \text { for } x>0 \\ \infty, & \text { for } x<0\end{cases}
$$

a) What are the ground state enegy of the first excited state?
b) What is the expectation value $\left\langle x^{2}\right\rangle$ for the ground state?

2 Consider a system with $j=1$. Obtain the explicit form of

$$
\left\langle j=1, m^{\prime}\right| J_{y}|j=1, m\rangle
$$

in a $3 \times 3$ matrix form.
3 At $t=0$ the system is known to be in the first state, represented by $\binom{1}{0}$. Using the time-dependent perturbation theory and assuming that $E_{1}^{0}-E_{2}^{0}$ is not close to $\pm \hbar \omega$, derive an expression for the probability that the system be found in the second state represented by $\binom{0}{1}$ as a function of $t(t>0)$.

