Non-triviality of the vacuum in light-front quantization; and implications for correctly formulating light-front quantization

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Two parts of seminar:

- Non-triviality of vacuum in light-front quantization:
  - It is often said that the vacuum in light-front quantization (LFQ) is trivial, and that correspondingly vacuum bubbles are zero.
  - Why is that not true?
- Implications and related problems:
  - There are serious unresolved conceptual and mathematical difficulties in formulating LFQ and even equal-time quantization (ETQ).
  - How to go about solving them fully?

Vacuum non-triviality

- A commonly stated advantage of light-front quantization is triviality of the vacuum, and that vacuum bubbles vanish.
- Hence, it appears that LFQ and ETQ quantization are inequivalent, and that LFQ solves cosmological constant problem (Brodsky & Shrock).
- But the LF method leads to a paradox in self-energy graphs (Gross quoted by Yan (1973)).
- So something is wrong, and vacuum triviality fails.
- Locate error in derivation of x^+-ordered perturbation theory from LF quantization, without invoking any equivalence to Feynman perturbation theory.

(See JCC, arXiv:1801.003960)

Background

- Light-front coordinates $(x^+, x^-, \boldsymbol{x}_T) = ((t+z)/\sqrt{2}, (t-z)/\sqrt{2}, \boldsymbol{x}_T)$.
- Use Heisenberg picture:
  - States time-independent;
  - Fields are $\phi(x)$, etc, with all t, x^+ dependence;
  - Lagrangian density specifies theory;
  - Equations of motion for fields in space-time;
  - CRs specified on quantization surface (fixed x^+ or fixed t or . . .).

  Then can treat equal-x^+ and equal-t quantization in same framework.
- Light-front analysis and annihilation and creation operators:

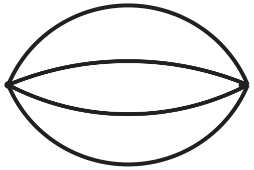
$$\phi(x) = \int \frac{\mathrm{d}k^+\,\mathrm{d}\boldsymbol{k}_T}{(2\pi)^3} e^{-ik^+x^- + i\boldsymbol{k}_T\cdot\boldsymbol{x}_T}\ \widetilde{\phi}(x^+; k^+, \boldsymbol{k}_T)$$

$$= \int_0^\infty \frac{\mathrm{d}k^+}{2k^+} \int \frac{\mathrm{d}\boldsymbol{k}_T}{(2\pi)^3} \left[e^{-ik^+x^- + i\boldsymbol{k}_T\cdot\boldsymbol{x}_T} a_k(x^+) + e^{ik^+x^- - i\boldsymbol{k}_T\cdot\boldsymbol{x}_T} a_k(x^+)^\dagger \right].$$

- x^+-ordered perturbation theory: intermediate states with on-shell particles and "energy" (P^-) denominators.

Paradox I

- Standard statement: Vacuum bubble:



  is zero in LF quantization: External $P^+ = 0$, any possible intermediate states have $k^+ > 0$.

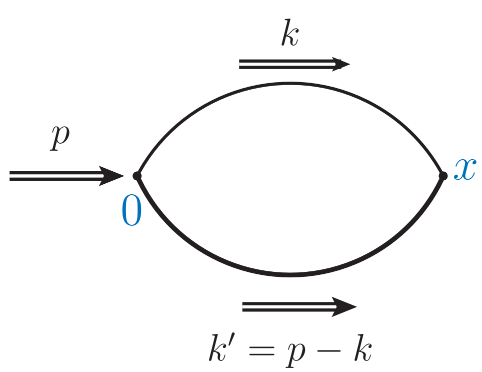
- Same rationale applies to *any* loop graph with imposed external $P^+ = 0$.

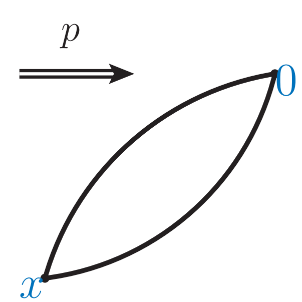
- Example: Green function

$$\Pi(p^2) \stackrel{\text{def}}{=} \int d^2x\ e^{ip\cdot x} \langle 0| T \tfrac{1}{2}\phi^2(x) \tfrac{1}{2}\phi^2(0) |0\rangle_{\text{connected}}$$

  in free scalar QFT in $1+1$ dimensions.

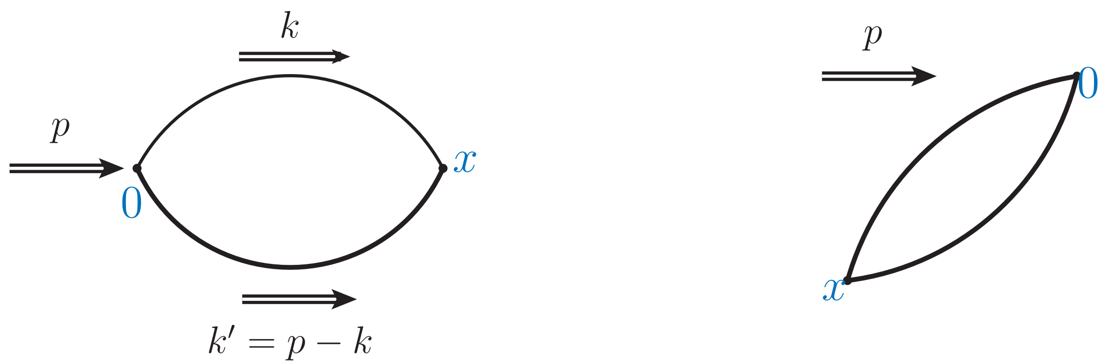
- x^+-ordered graphs:





Paradox II

- x^+-ordered graphs:



- When $p^+ > 0$:

$$\Pi(p^2) = \frac{1}{4\pi} \int_0^{p^+} \frac{dk^+}{4k^+(p^+ - k^+)} \frac{i}{p^- - \frac{m^2}{2k^+} - \frac{m^2}{2(p^+ - k^+)} + i\epsilon}$$

$$= \frac{i}{8\pi} \int_0^1 d\xi \, \frac{1}{p^2 \xi(1-\xi) - m^2 + i\epsilon}$$

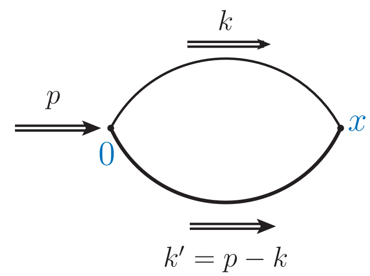
Agrees with Feynman-graph calculation.

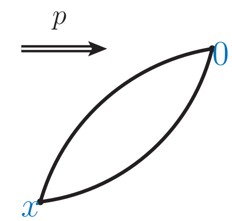
- At $p^+ = 0$, we apparently have $\Pi(0) = 0$, because there are no possible intermediate states.
- But $\Pi(p^2)$ is analytic at all p^2 (except $p^2 = 4m^2$), and $p^2 \to 0$ limit is non-zero.

Feynman graph view

- Chang & Ma (1969), and especially Yan (1973) derive x^+-ordered perturbation theory from Feynman graphs.
- That gives standard rules at $P^+ \neq 0$.
- But a correction is needed at $P^+ = 0$.
- But after Brodsky & Shrock (2011), we have to worry that LF quantization might give different results than equal-time quantization and standard Feynman graphs.
- Therefore we must address issue within derivation of x^+-ordered perturbation theory from LFQ.

Light-front calculation: its derivation





Integrate over x^+ to get energy denominators

$$\Pi(p^2) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\frac{dk^+}{2k^+}\int_{-\infty}^{\infty}\frac{dk'^+}{2k'^+}\int_{-\infty}^{\infty}dx^- e^{ix^-(p^+-k^+-k'^+)}$$

$$\left[\theta(k^+)\theta(k'^+)\frac{i}{p^- - \frac{m^2}{2k^+} - \frac{m^2}{2k'^+} + i\epsilon} + \theta(-k^+)\theta(-k'^+)\frac{i}{-p^- + \frac{m^2}{2k^+} + \frac{m^2}{2k'^+} + i\epsilon}\right]$$

Then use

$$\int dx^- e^{ix^-(p^+-k^+-k'^+)} = 2\pi\delta(p^+ - k^+ - k'^+),$$

BUT only if integrated with function continuous at $p^+ = k^+ + k'^+$.

Hence standard derivation fails at $p^+ = 0$.

The primary problem

Fundamentally, the problem is a failure to properly treat the mathematics of distributions correctly.

Quantum fields are operator-valued distributions, not operator valued functions of space-time coordinates. I.e., primary object is $\phi[f]$ interpreted as

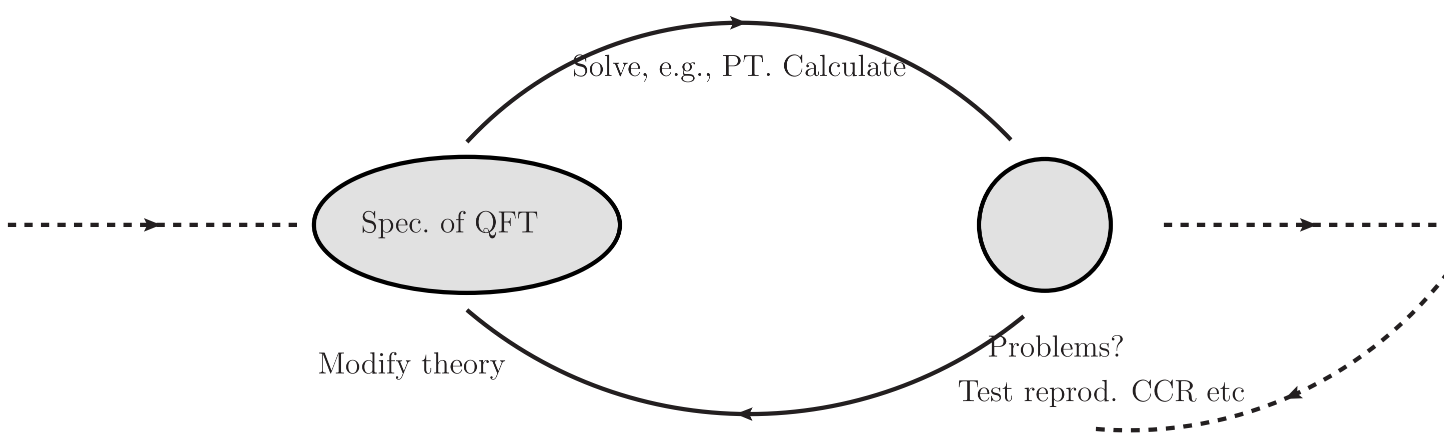
$$\phi[f] \text{ `='} \int f(x)\phi(x)\,\mathrm{d}^4x,$$

with f a test function, i.e., a (real or complex) function of position which is smooth and of fast decrease at infinity.

A proper solution of the problems requires reconsidering the treatment of quantum fields and quantization methods from the start. That includes ET quantization, in fact. (The difficulties are more hidden, but well-known to mathematical physicists.)

Project (with Peter Lowdon)

- Redo quantization formulation, both equal-time and light-front, from foundations, using distribution-theory formulation of fields.
- *But* aim at getting formulas etc as near as possible to the textbook ones, but no nearer, with suitable definitions/constructions.
- Make extensive use of perturbative examples to show what the difficulties are, and *where*, and to motivate and inspire next steps.
- With ETQ, use OPE to delimit where problems arise. Then work out modified methods to use instead of textbook ones.
- Repeat for LFQ, but now using asymptotic results beyond OPE.
- Do all this strictly within the final theory.



Rest of seminar

- Specification of ϕ^4 theory, for examples.
- Examples of difficulty of localizing fields (ET).
- Some methods.
- Examine some problems on LF.

Specification of ϕ^4 theory, integer space-time dimension n

Lagrangian density (with omission of any UV counterterms needed):

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

Euler-Lagrange equations of motion:

$$\partial^2\phi + m^2\phi + \frac{\lambda}{3!}\phi^3 = 0$$

Evolution operator and CCR in ET quantization, with $\pi = \dot{\phi}$:

$$H = \int \mathrm{d}^{n-1}\mathbf{x} \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right]$$

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta^{(n-1)}(\mathbf{x}-\mathbf{y})$$

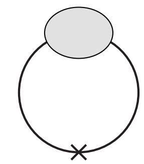
LF quantization:

$$P^- = \int \mathrm{d}^{n-2}\boldsymbol{x}_T \, \mathrm{d}x^- \left[\frac{1}{2}(\boldsymbol{\nabla}_T\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right]$$

$$[\phi(x^+, x^-, \mathbf{x}_T), \partial_-\phi(x^+, y^-, \boldsymbol{y}_T)] = \frac{i}{2}\delta(x^- - y^-)\delta^{(n-2)}(\boldsymbol{x}_T - \boldsymbol{y}_T)$$

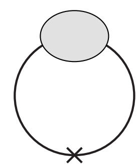
CCR determined by consistency between Euler-Lagrange and Heisenberg evolution

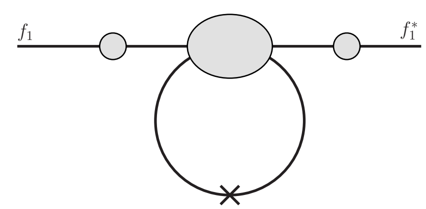
Is $\phi(x)$ an operator on state space? No!

$$\|\phi(x)\,|0\rangle\|^2 = \langle 0|\,\phi(x)^2\,|0\rangle =$$  $$= \infty$$

in all cases, even free field theory in 2 dimensions.

But on normalized one-particle state (in- or out-):

$$\|\phi(x)\,|f_1\rangle\|^2 = \langle f_1|\,\phi(x)^2\,|f_1\rangle =$$  $$\times \langle f_1|f_1\rangle$$

$$+$$ 

$$+$$ 

Copy of vacuum case, plus log divergence for 1 particle matrix element of ϕ^2 in renormalizable interacting theory ($n = 4$).

ETC

Hence: Either $\phi(x)$ doesn't exist, or it takes physical state to state far outside Hilbert space of states.

But distribution $\phi[f]$ is safe

E.g.,

$$\left\| \phi[f] \left|0\right\rangle \right\|^2 \text{`='} \int \mathrm{d}^n x \, \mathrm{d}^n y \, f(x)^* f(y) \left\langle 0\right| \phi(x)\phi(y) \left|0\right\rangle$$

This is fine: $\langle 0|\, \phi(x)\phi(y)\, |0\rangle$ is known to correspond to an ordinary (analytic) function with an integrable light-cone singularity. High-momentum components have oscillations that give suppression.

But $\phi(y)\,|0\rangle$ and $\phi(x)\phi(y)\,|0\rangle$ don't exist (in state space). so that standard linear algebra to get $\langle 0|\, \phi(x)\phi(y)\, |0\rangle$ fails.

Localization or not on surface of constant time

Define time-averaged field by function h with unit integral:

$$\phi(t; \Delta t, h.g) \text{ `='} \int \mathrm{d}t' \, \mathrm{d}^{n-1}\mathbf{x} \, \phi(t', \mathbf{x}) \frac{h((t'-t)/\Delta t)}{\Delta t} g(\mathbf{x})$$

i.e., $\phi[f]$ with $f(t', \mathbf{x}) = \frac{h((t'-t)/\Delta t)}{\Delta t} g(\mathbf{x})$. Define time-localized field by

$$\phi(t.g) = \lim_{\Delta t \to 0} \phi(t; \Delta t, h.g) \text{ `='} \int \mathrm{d}^{n-1}\mathbf{x} \, \phi(t, \mathbf{x}) g(\mathbf{x})$$

if the limit exists.

Then standard ETCCR is strictly given as

$$[\phi(t, f), \pi(t, g)] = i \int \mathrm{d}^{n-1}\mathbf{x} \, f(\mathbf{x}) g(\mathbf{x})$$

These time-localized fields do exist in free theory.

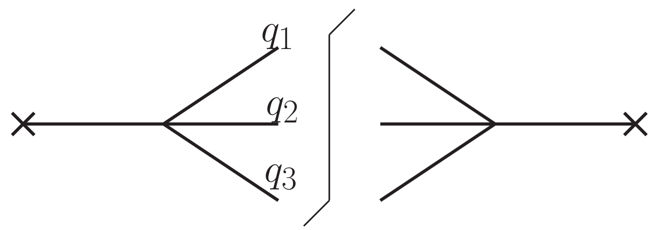
Time localization with interactions is dynamics-dependent

In momentum space:

$$\phi(t,g) = \int \frac{\mathrm{d}q^0}{2\pi} e^{iq^0 t} \int \frac{\mathrm{d}^{n-1}\mathbf{q}}{(2\pi)^{n-1}} \tilde{g}(\mathbf{q}) \tilde{\phi}(q^0, \mathbf{q})$$

Hence

$$\|\phi(t,g)\,|0\rangle\|^2 =$$



$+$ finite 1-body term $+ O(\lambda^3)$

$$= \int \prod_j \frac{\mathrm{d}^{n-1}\mathbf{q}_j}{(2\pi)^{n-1} 2E_j} \left|\tilde{g}(\textstyle\sum \mathbf{q}_j)\right|^2 \frac{1}{(q^2 - m^2)^2} \frac{\lambda^2}{6} \;+\; \ldots$$

UV degree $= \Lambda^{2n-2-3-4} = \Lambda^{2n-9}$. Hence finite at $n = 4$ and lower.

For $\dot{\phi}$: Two extra powers of q^0, so UV degree $\Lambda^{2n-2-3-4} = \Lambda^{2n-7}$. Linear UV divergence at $n = 4$. So $\dot{\phi} = \pi$ cannot be localized. *ETCCRs are in trouble.*

For $\ddot{\phi}$ (in equation of motion): Another 2 powers, so get linear UV divergence at $n = 3$ – super-renormalizable theory – and cubic at $n = 4$.

Lessons from these results

- Problems with standard localized formulations are analyzable UV/short-distance problems.
- They are limited in scope (cf. standard UV renormalization).
- To get an adequate treatment, one must start with field as distribution $\phi[f]$.
- Then use calculated short-distance behavior to determine what dynamics-dependent modifications are needed.
- Warning: *Localization limits/asymptotes are highly non-uniform.*
  E.g., in $\lim_{\Delta t \to 0}$ op.$(t; \Delta t, g)\, |\text{state}\rangle$, to be close to limit, Δt needs to be (much) smaller than $1/E$, (Typical energy in state is E.)

Start on reformulation

- Define renormalized composite operators $R[\phi^2]$ etc by point-splitting, and then using OPE, factor out short-distance Wilson coefficient as splitting goes to zero.
- Cf. Valatin (**1954**), Wilson & Zimmermann (1972). But they had a lot of IR dependence in Wilson coefficient.
- Use of $\overline{\mathrm{MS}}$ etc gives pure short-distance contribution to coefficients. Then in asymptotically free theory, low-order calculation + RG allows definition with definite formula for $R[\phi^2]$ etc.
- ETCCR: Use relevant Wilson coefficient to replace $\delta^{(n-1)}(\mathbf{x}-\mathbf{y})$ by something else.
- Applications: . . .

Where now?

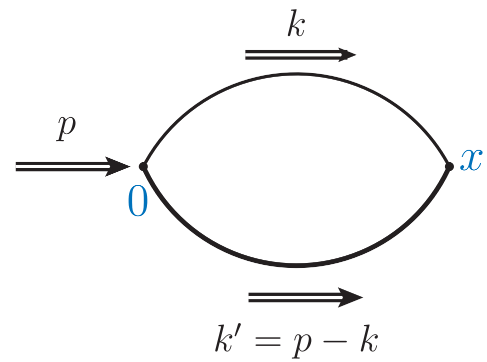
- Finish treatment of ETQ.
- Generalize to LFQ:
- Extra localization complications: light-cone instead of short distance, with issue of‘zero-mode (i.e., $P^+ \to 0$, or $P^- \to \infty$).
- Problems even in free theory (e.g., Nakanishi & Yamawaki, NPB 122, 15 (1977)).
- Much work on “the” $P^+ = 0$ mode. I don’t find them enough . . . . Real problem is not so much *at* $P^+ = 0$, but in asymptotic behavior as $P^+ \to 0$.
- Biggest issue (IMO): status of LF creation and annihilation operators as genuine operators (even after smearing in k^+ and $\boldsymbol{k}_T$). Problems from:
  - effects of UV renormalization.
  - rapidity divergences in gauge theories.

  Symptom of difficulties: Need for important modifications to definitions of pdfs and TMD pdfs, with associated evolution equations.

—EXTRAS—

—FEYNMAN GRAPH VIEW—

Feynman graph I



$$\Pi(p^2) = -\frac{1}{8\pi^2} \int dk^+ dk^- \frac{1}{[2k^+k^- - m^2 + i\epsilon][2(k^+ - p^+)(k^- - p^-) - m^2 + i\epsilon]}$$

At $p = 0$

$$\Pi(0) = -\frac{1}{8\pi^2} \int dk^+ dk^- \frac{1}{[2k^+k^- - m^2 + i\epsilon]^2}$$

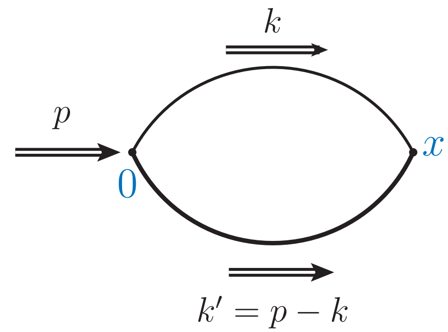
Then

– When $k^+ > 0$ close in upper half plane and get zero.
– When $k^+ < 0$ close in lower half plane and get zero.

But convergence at large $|k^-|$ is non-uniform.

Non-zero contribution from $|k^-| \to \infty$, $|k^+| = O(1/k^-)$, but only if $p^+ = 0$

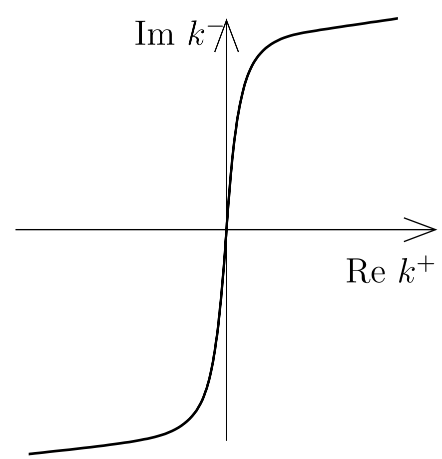
Feynman graph II: Problem of break in contour



$$\Pi(0) = -\frac{1}{8\pi^2}\int dk^+ dk^- \frac{1}{[2k^+k^- - m^2 + i\epsilon]^2}$$

When $k^+ \gtrless 0$ close in upper/lower half plane.

Need to fill in break in contour:



—LIGHT-FRONT QUANTIZATION ISSUES—

An observation

- In LFQ, we use $a^\dagger_{\mathrm{LF}}(k^+, \boldsymbol{k}_T)$ to create states of partons from vacuum.
- In renormalizable theory, construct $a^\dagger$ from *renormalized* field, so
$$a^\dagger = \frac{1}{\sqrt{Z}} \times \text{Bare } a^\dagger_0 \text{ with number interpretation}$$
- But Z has UV divergences in perturbation theory, and resummed true Z may be zero.
- So $a^\dagger_{\mathrm{LF}}(k^+, \boldsymbol{k}_T)|0\rangle$ has infinitely wrong normalization (even after averaging with a wave function to try to get normalizable state).

Problem with light-front normalization even in free-field theory

From LF mode expansion, compute

$$\langle 0|\, \phi(x^+, x^-, \boldsymbol{x}_T)\, \phi(x^+, y^-, \boldsymbol{y}_T)\, |0\rangle \propto \int_0^\infty \frac{\mathrm{d}k^+}{k^+} \int \mathrm{d}^{n-2}\boldsymbol{k}_T\, e^{ik^+(x^- - y^-) - i\boldsymbol{k}_T\cdot(\boldsymbol{x}_T - \boldsymbol{y}_T)}$$
$$\propto \delta^{(n-2)}(\boldsymbol{x}_T - \boldsymbol{y}_T) \int_0^\infty \frac{\mathrm{d}k^+}{k^+} e^{ik^+(x^- - y^-)}$$

Divergence at $k^+ = 0$.

But it is known that the correlator is a function $G_2((x-y)^2) = G_2((\boldsymbol{x}_T - \boldsymbol{y}_T)^2$, singular at $\boldsymbol{x}_T = \boldsymbol{y}_T$.

Derivative with respect to x^- OK.

Off-light-front, factor

$$e^{ik^-(x^+ - y^+)}$$

gives oscillations that remove divergence, since $k^- \propto 1/k^+$.

[E.g., Nakanishi & Yamawaki, NPB 122, 15 (1977)]