COMPARING LIGHT-FRONT QUANTIZATION WITH INSTANT-TIME QUANTIZATION

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Seminar in Virtual ILCAC

November 2020

P. D. Mannheim, P. Lowdon, S. J. Brodsky: Structure of light front vacuum sector diagrams, Phys. Lett. B 797, 134916 (2019). (arXiv:1904.05253)

P. D. Mannheim: Equivalence of light-front quantization and instant-time quantization. Phys. Rev. D 102, 025020 (2020). (arXiv:1909.03548)

P. D. Mannheim: Light-front quantization is the same as instant-time quantization, Proc. Sci. (LC2019) 062 (2019). (arXiv:2001.04603)

P. D. Mannheim, P. Lowdon, S. J. Brodsky: Comparing light-front quantization with instant-time quantization, Phys. Rep. in press. (arXiv:2005.00109)

1 THREE QUESTIONS

(1) Is light-front quantization at equal $x^+ = x^0 + x^3$ the same theory as instant-time quantization at equal x^0, or is it a different theory?

(2) Are the Hamiltonians the same or different. If different, which one describes nature? Are the Hamiltonians even related?

(3) Is there anything in quantum field theory that is not accounted for by the on-shell Light-Front Hamiltonian description of physics?

THREE ANSWERS

(1) **YES**, even though equal x^0 and equal x^+ commutators and anticommutators look to be so different.

(2)**YES**, because of general coordinate invariance. $x^0 \to x^0 + x^3$ is a spacetime dependent translation not a Lorentz transformation. The momentum generators and thus the Hamiltonians are unitarily equivalent.

(3) **YES**, but only in the vacuum sector, because of zero modes with $p_- = 0$.

Mass-shell conditions

$$(p_0)^2 - (p_3)^2 - (p_1)^2 - (p_2)^2 = 4p_+p_- - (p_1)^2 - (p_2)^2 = m^2, 2p_+ = p_0 + p_3, 2p_- = p_0 - p_3,$$
$$\mathrm{Instant}: p_0 = \pm[(p_1)^2 + (p_2)^2 + (p_3)^2 + m^2]^{1/2}, \quad \mathrm{well-behaved\ at\ p_3 = 0.}$$
$$\mathrm{Front}: p_+ = \frac{(p_1)^2 + (p_2)^2 + m^2}{4p_-}, \quad \mathrm{singular\ at\ p_- = 0.} \tag{1.1}$$

Key Features

Light-Front Fock space is equivalent to Light-Front Hamiltonian and both correspond to pole terms in Feynman diagrams. Also correspond to instant-time graphs in infinite momentum frame. However in vacuum graphs also get **circle at infinity contributions**, and with them instant-time and light-front vacuum graphs prove to be equal. But instant-time graphs have no $p_- = 0$ zero mode problem. Therefore zero-mode problem must have a solution.

2 INSTANT-TIME AND LIGHT-FRONT FOCK SPACE EXPANSIONS

Instant-Time Scalar Field Fock Space Expansion with $E_p^2 = p_1^2 + p_2^2 + p_3^2 + m^2$

$$\phi(x^0, x^1, x^2, x^3) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E_p)^{1/2}} [a(\vec{p})e^{-iE_pt+i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{+iE_pt-i\vec{p}\cdot\vec{x}}]. \tag{2.1}$$

Contains $-\infty \leq p_3 \leq \infty$, well-behaved at $p_3 = 0$.

Light-Front Scalar Field Fock Space Expansion with $F_p^2 = (p_1)^2 + (p_2)^2 + m^2$

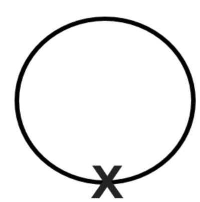
$$\phi(x^+, x^1, x^2, x^-) = \frac{2}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{(4p_-)^{1/2}}$$
$$\times \left[e^{-i(F_p^2 x^+/4p_- + p_-x^- + p_1x^1 + p_2x^2)} a(p_1, p_2, p_-) + e^{i(F_p^2 x^+/4p_- + p_-x^- + p_1x^1 + p_2x^2)} a_p^\dagger(p_1, p_2, p_-)\right]. \tag{2.2}$$

Singular at $p_- = 0$, undefined at $x^+ = 0$, $p_- = 0$. ($p_- = p^+/2$, $p_+ = p^-/2$).

Contains $0 \leq p_- \leq \infty$ only, Light-Front Hamiltonian approach restricts to $p_- > 0$, $p_+ < \infty$.

Thus go beyond Light-Front-Hamiltonian if have processes with $p_- = 0$.

This happens in vacuum sector where tadpole is $-i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle$ with $x^+ = 0$.
If bring zero four-momentum into cross in vacuum tadpole then only allowed momentum in loop has $p_- = 0$. If exclude $p_- = 0$ then tadpole is zero. Potential solution to cosmological constant problem. Fails since have to deal with indeterminacy of x^+/p_- at $x^+ = 0$, $p_- = 0$.



3 INSTANT-TIME AND LIGHT-FRONT COMMUTATORS

$$T^{\mu\nu} = \frac{2}{(-g)^{1/2}} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x(-g)^{1/2} L, \quad P_\mu = \int d^3x T^0{}_\mu, \quad [P_\mu, \phi(x)] = -i\partial_\mu \phi(x). \tag{3.1}$$

Scalar instant-time commutators, $KE = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$**,** $\quad \Pi = \partial^0 \phi = \partial_0 \phi$

$$\begin{aligned}[\phi(x^0, x^1, x^2, x^3), \partial_0 \phi(x^0, y^1, y^2, y^3)] &= i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3), \\ [\phi(x^0, x^1, x^2, x^3), \phi(x^0, y^1, y^2, y^3)] &= 0.\end{aligned} \tag{3.2}$$

Scalar light-front commutators, $KE = \frac{1}{2}[\partial_+ \phi \partial_- \phi - \partial_1 \phi \partial^1 \phi - \partial_2 \phi \partial^2 \phi]$**,** $\quad \partial_+ = \partial_0 + \partial_3$**,** $\partial_- = \partial_0 - \partial_3$**,** $\Pi = \partial^+ \phi = 2\partial_- \phi$

$$\begin{aligned}[\phi(x^+, x^1, x^2, x^-), 2\partial_- \phi(x^+, y^1, y^2, y^-)] &= i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^- - y^-), \\ [\phi(x^+, x^1, x^2, x^-), \phi(x^+, y^1, y^2, y^-)] &= -\frac{i}{4}\epsilon(x^- - y^-)\delta(x^1 - y^1)\delta(x^2 - y^2).\end{aligned} \tag{3.3}$$

Gauge field instant-time commutators, $\Pi^\mu = -\partial^0 A^\mu = -\partial_0 A^\mu$

$$\begin{aligned}[A_\nu(x^0, x^1, x^2, x^3), \partial_0 A_\mu(x^0, y^1, y^2, y^3)] &= -ig_{\mu\nu}\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3), \\ [A_\nu(x^0, x^1, x^2, x^3), A_\mu(x^0, y^1, y^2, y^3)] &= 0.\end{aligned} \tag{3.4}$$

Gauge field light-front commutators, $\Pi^\mu = -\partial^+ A^\mu = -2\partial_- A^\mu$

$$\begin{aligned}[A_\nu(x^+, x^1, x^2, x^-), 2\partial_- A_\mu(x^+, y^1, y^2, y^-)] &= -ig_{\mu\nu}\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^- - y^-), \\ [A_\nu(x^+, x^1, x^2, x^-), A_\mu(x^+, y^1, y^2, y^-)] &= \frac{i}{4}g_{\mu\nu}\epsilon(x^- - y^-)\delta(x^1 - y^1)\delta(x^2 - y^2).\end{aligned} \tag{3.5}$$

4 INSTANT-TIME AND LIGHT-FRONT ANTICOMMUTATORS

Fermion instant-time anticommutators, $\Pi = i\psi^\dagger$

$$\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(x^0,y^1,y^2,y^3)\right\} = \delta_{\alpha\beta}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3). \tag{4.1}$$

Fermion light-front anticommutators, $\Pi = i\psi^\dagger\gamma^0(\gamma^0+\gamma^3) = 2i\psi^\dagger_{(+)}$

$$\left\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\} = \Lambda^+_{\alpha\beta}\delta(x^- -y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{4.2}$$

Projectors

$$\Lambda^\pm = \tfrac{1}{2}(1\pm\gamma^0\gamma^3),\quad \Lambda^+ + \Lambda^- = I,\quad (\Lambda^+)^2=\Lambda^+,\quad (\Lambda^-)^2=\Lambda^-,\quad \Lambda^+\Lambda^-=0,\quad \gamma^\pm=\gamma^0\pm\gamma^3,\quad (\gamma^\pm)^2=0,$$
$$\psi_{(\pm)}=\Lambda_\pm\psi,\quad \textbf{non-invertible projectors.} \tag{4.3}$$

$$\psi_{(-)}(x^+,x^1,x^2,x^-) = -\frac{i}{4}\int du^-\epsilon(x^- -u^-)[-i\gamma^0(\gamma^1\partial_1+\gamma^2\partial_2)+m\gamma^0]\psi_{(+)}(x^+,x^1,x^2,u^-).$$
$$\textbf{constrained variable.} \tag{4.4}$$

$$\left\{[\psi_{(+)}]_\nu(x),[\psi^\dagger_{(-)}]_\sigma(y)\right\} = \tfrac{i}{8}\epsilon(x^- -y^-)[i(\gamma^-\gamma^1\partial^x_1+\gamma^-\gamma^2\partial^x_2)-m\gamma^-]_{\nu\sigma}\delta(x^1-y^1)\delta(x^2-y^2), \tag{4.5}$$

$$\begin{aligned}&\left\{\psi^{(-)}_\mu(x^+,x^1,x^2,x^-),[\psi^\dagger_{(-)}]_\nu(x^+,y^1,y^2,y^-)\right\}\\ &=\frac{1}{16}\Lambda^-_{\mu\nu}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\int du^-\epsilon(x^- -u^-)\epsilon(y^- -u^-)\delta(x^1-y^1)\delta(x^2-y^2).\end{aligned} \tag{4.6}$$

Sure look different, but..(Mannheim 2020)....

5 UNEQUAL TIME COMMUTATORS AND ANTICOMMUTATORS – MASSLESS FIELDS

UNEQUAL TIME Scalar instant-time commutator

$$\begin{aligned}
i\Delta(x-y) &= [\phi(x^0,x^1,x^2,x^3),\phi(y^0,y^1,y^2,y^3)] \\
&= \int\frac{d^3pd^3q}{(2\pi)^3(2p)^{1/2}(2q)^{1/2}}\Big([a(\vec{p}),a^\dagger(\vec{q})]e^{-ip\cdot x+iq\cdot y}+[a^\dagger(\vec{p}),a(\vec{q})]e^{ip\cdot x-iq\cdot y}\Big) \\
&= \int\frac{d^3p}{(2\pi)^3 2p}\big(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\big) \\
&= -\frac{i}{2\pi}\frac{\delta(x^0-y^0-|\vec{x}-\vec{y}|)-\delta(x^0-y^0+|\vec{x}-\vec{y}|)}{2|\vec{x}-\vec{y}|} \\
&= -\frac{i}{2\pi}\epsilon(x^0-y^0)\delta[(x^0-y^0)^2-(x^1-y^1)^2-(x^2-y^2)^2-(x^3-y^3)^2]. \qquad (5.1)
\end{aligned}$$

Since holds ALL times, also holds at EQUAL light front time.

Substitute $x^0=(x^++x^-)/2$**,** $x^3=(x^+-x^-)/2$**,** $y^0=(y^++y^-)/2$**,** $y^3=(y^+-y^-)/2$**:**

$$i\Delta(x-y)=-\frac{i}{2\pi}\epsilon[\tfrac{1}{2}(x^++x^--y^+-y^-)]\delta[(x^+-y^+)(x^--y^-)-(x^1-y^1)^2-(x^2-y^2)^2]. \qquad (5.2)$$

$$i\Delta(x-y)\big|_{x^+=y^+} = [\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)]=-\frac{i}{4}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \qquad (5.3)$$

At $x^+=y^+$ **UNEQUAL instant-time commutator is EQUAL light-front time commutator**

Light-front quantization is instant-time quantization, and does not need to be independently postulated.

UNEQUAL TIME Abelian gauge field instant-time commutator

$$[A_\nu(x^0,x^1,x^2,x^3),A_\mu(y^0,y^1,y^2,y^3)] = ig_{\mu\nu}\Delta(x-y)$$
$$= -\frac{i}{2\pi}g_{\mu\nu}\epsilon(x^0-y^0)\delta[(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2]. \tag{5.4}$$

Leads to

$$[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)] = \frac{i}{4}g_{\mu\nu}\epsilon(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{5.5}$$

At $x^+ = y^+$ UNEQUAL instant-time commutator is EQUAL light-front time commutator Similar result holds for non-Abelian gauge field.

6 FERMION UNEQUAL INSTANT-TIME ANTICOMMUTATOR

$$\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi_\beta^\dagger(y^0,y^1,y^2,y^3)\right\}=\left[(i\gamma^\mu\gamma^0\partial_\mu\right]_{\alpha\beta}i\Delta(x-y). \tag{6.1}$$

Apply projector and set $x^+=y^+$

$$\begin{aligned}&\Lambda^+_{\alpha\gamma}\left\{\psi_\gamma(x^+,x^1,x^2,x^-),\psi_\delta(x^+,y^1,y^2,y^-)\right\}\Lambda^+_{\delta\beta}\\ &=\left\{[\psi_{(+)}(x^+,x^1,x^2,x^-)]_\alpha,[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\}=\Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2).\end{aligned} \tag{6.2}$$

At $x^+=y^+$ UNEQUAL instant-time anticommutator is EQUAL light-front time anticommutator. Can also derive anticommutators involving bad fermions in the same way.

Light-front quantization is instant-time quantization, and does not need to be independently postulated. The seemingly different structure between EQUAL instant-time and EQUAL light-front time commutators is actually a consequence of the structure of UNEQUAL instant-time time commutators and anticommutators.

GENERAL RULE: ANY TWO DIRECTIONS OF QUANTIZATION THAT CAN BE CONNECTED BY A GENERAL COORDINATE TRANSFORMATION DESCRIBE THE SAME THEORY.

7 MASSIVE FIELDS – SCALAR INSTANT-TIME CASE

$$
\begin{aligned}
i\Delta(IT; x-y) &= [\phi(x^0,x^1,x^2,x^3),\phi(y^0,y^1,y^2,y^3)] \\
&= \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3 \frac{1}{(2\pi)^3 2E_p}\left(e^{-iE_p(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{iE_p(x^0-y^0)-i\vec{p}\cdot(\vec{x}-\vec{y})}\right).
\end{aligned} \tag{7.1}
$$

Here p_3 ranges from $-\infty$ to ∞ and integrand is well-behaved at $p_3 = 0$.

$$
\begin{aligned}
i\Delta(IT;(x-y)^2 > 0) &= \frac{im}{4\pi}\epsilon(x^0-y^0)\frac{J_1(m[(x-y)^2]^{1/2})}{[(x-y)^2]^{1/2}}, \\
i\Delta(IT;(x-y)^2 = 0) &= -\frac{i}{2\pi}\epsilon(x^0-y^0)\delta[(x-y)^2], \\
i\Delta(IT;(x-y)^2 < 0) &= 0.
\end{aligned} \tag{7.2}
$$

Discontinuous at $m = 0$, go off shell and write a contour integral in p_0 since $\epsilon(t) = \theta(t) - \theta(-t)$ and $\delta(t)$ are distributions with

$$
\theta(t) = -\frac{1}{2\pi i}\oint d\omega \frac{e^{-i\omega t}}{\omega + i\epsilon} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega + i\epsilon}, \tag{7.3}
$$

with $t \neq 0$ suppressing circle at infinity. As we will see, in $t = 0$ vacuum case, no suppression. Get $\theta(0) = 1/2$.

$$
\begin{aligned}
i\Delta(IT; x-y) = -\frac{1}{2\pi i}\frac{1}{8\pi^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3 \oint dp_0 \\
\times \left[\frac{\theta(x^0-y^0)e^{-ip\cdot(x-y)} - \theta(-x^0+y^0)e^{ip\cdot(x-y)}}{(p_0)^2-(p_3)^2-(p_1)^2-(p_2)^2-m^2+i\epsilon} + \frac{\theta(x^0-y^0)e^{ip\cdot(x-y)} - \theta(-x^0+y^0)e^{-ip\cdot(x-y)}}{(p_0)^2-(p_3)^2-(p_1)^2-(p_2)^2-m^2-i\epsilon}\right].
\end{aligned} \tag{7.4}
$$

Introduce exponential regulator, with the $i\epsilon$ term suppressing the $\alpha = \infty$ contribution when A is real

$$
\int_0^{\infty} d\alpha \exp[i\alpha(A + i\epsilon)] = -\frac{1}{iA}, \tag{7.5}
$$

Obtain

$$
i\Delta(IT; x-y) = -\frac{i}{4\pi^2}\epsilon(x^0-y^0)\int_0^{\infty}\frac{d\alpha}{4\alpha^2}\left[e^{-i(x-y)^2/4\alpha - i\alpha m^2 - \alpha\epsilon} + e^{i(x-y)^2/4\alpha + i\alpha m^2 - \alpha\epsilon}\right]. \tag{7.6}
$$

8 MASSIVE FIELDS – SCALAR LIGHT-FRONT CASE

$$
\begin{aligned}
&i\Delta(LF;x-y) = [\phi(x^+,x^1,x^2,x^-),\phi(y^+,y^1,y^2,y^-)] \\
&= \frac{1}{4\pi^3}\int_{-\infty}^{\infty} dp_1 dp_2 \int_0^{\infty}\frac{dp_-}{4p_-}\left[e^{-i[F_p^2(x^+-y^+)/4p_-+p_-(x^--y^-)+p_1(x^1-y^1)+p_2(x^2-y^2)]} - e^{i[F_p^2(x^+-y^+)/4p_-+p_-(x^--y^-)+p_1(x^1-y^1)+p_2(x^2-y^2)]}\right].
\end{aligned}
\tag{8.1}
$$

Here p_- only ranges from 0 to ∞ and integrand is singular at $p_-=0$. So put p_- into the exponential.

$$
\begin{aligned}
i\Delta(LF;x-y) &= -\frac{1}{2\pi i}\frac{1}{4\pi^3}\int_{-\infty}^{\infty} dp_1\int_{-\infty}^{\infty} dp_2\int_0^{\infty} dp_- \oint dp_+ \\
&\times\left[\frac{\theta(x^+-y^+)e^{-ip\cdot(x-y)}-\theta(-x^++y^+)e^{ip\cdot(x-y)}}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon}+\frac{\theta(x^+-y^+)e^{ip\cdot(x-y)}-\theta(-x^++y^+)e^{-ip\cdot(x-y)}}{4p_+p_- -(p_1)^2-(p_2)^2-m^2-i\epsilon}\right] \\
&= -\frac{i}{4\pi^2}\epsilon(x^+-y^+)\int_0^{\infty}\frac{d\alpha}{4\alpha^2}\left[e^{-i(x-y)^2/4\alpha-i\alpha m^2-\alpha\epsilon}+e^{i(x-y)^2/4\alpha+i\alpha m^2-\alpha\epsilon}\right].
\end{aligned}
\tag{8.2}
$$

$$
\begin{aligned}
&i\Delta(LF;(x-y)^2>0) = \frac{im}{4\pi}\epsilon(x^+-y^+)\frac{J_1(m[(x-y)^2]^{1/2})}{[(x-y)^2]^{1/2}} = \frac{im}{4\pi}\epsilon(x^--y^-)\frac{J_1(m[(x-y)^2]^{1/2})}{[(x-y)^2]^{1/2}}, \\
&i\Delta(LF;(x-y)^2=0) = -\frac{i}{2\pi}\epsilon(x^+-y^+)\delta[(x-y)^2] = -\frac{i}{2\pi}\epsilon(x^--y^-)\delta[(x-y)^2], \\
&i\Delta(LF;(x-y)^2<0) = 0.
\end{aligned}
\tag{8.3}
$$

$$
\begin{aligned}
i\Delta(LF;x-y) &= -\frac{i}{4\pi^2}\epsilon(x^--y^-)\int_0^{\infty}\frac{d\alpha}{4\alpha^2}\left[e^{-i(x-y)^2/4\alpha-i\alpha m^2-\alpha\epsilon}+e^{i(x-y)^2/4\alpha+i\alpha m^2-\alpha\epsilon}\right], \\
i\Delta(IT;x-y) &= -\frac{i}{4\pi^2}\epsilon(x^0-y^0)\int_0^{\infty}\frac{d\alpha}{4\alpha^2}\left[e^{-i(x-y)^2/4\alpha-i\alpha m^2-\alpha\epsilon}+e^{i(x-y)^2/4\alpha+i\alpha m^2-\alpha\epsilon}\right].
\end{aligned}
\tag{8.4}
$$

Substitute $x^0=(x^++x^-)/2$, $x^3=(x^+-x^-)/2$, $y^0=(y^++y^-)/2$, $y^3=(y^+-y^-)/2$, so that $(x-y)^2=(x^0-y^0)^2-(x^3-y^3)^2-(x^1-y^1)^2-(x^2-y^2)^2\to(x^+-y^+)(x^--y^-)-(x^1-y^1)^2-(x^2-y^2)^2$ the instant-time $i\Delta(IT;x-y)$ transforms into the light-front $i\Delta(LF;x-y)$. We have thus achieved our main objective, showing that $i\Delta(IT;x-y)$ and $i\Delta(LF;x-y)$ are related by a coordinate transformation, and are thus **COMPLETELY EQUIVALENT**.

9 MASSIVE FERMION FIELDS

For instant-time case need **FOUR**-component fermion

$$\begin{aligned}&\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(y^0,y^1,y^2,y^3)\right\}\\ &=\left[\left(i\gamma^0\partial^x_0+i\gamma^3\partial^x_3+i\gamma^1\partial^x_1+i\gamma^2\partial^x_2+m\right)\gamma^0\right]_{\alpha\beta}i\Delta(IT;x-y).\end{aligned}\tag{9.1}$$

For light-front case again need **FOUR**-component fermion

$$\begin{aligned}&\left\{\psi_\alpha(x^+,x^1,x^2,x^-),\psi^\dagger_\beta(y^+,y^1,y^2,y^-)\right\}\\ &=\left[\left(i\gamma^+\partial^x_+ +i\gamma^-\partial^x_- +i\gamma^1\partial^x_1+i\gamma^2\partial^x_2+m\right)\gamma^0\right]_{\alpha\beta}i\Delta(LF;x-y).\end{aligned}\tag{9.2}$$

Thus can derive unequal light-front time anticommutators from unequal instant-time anticommutators. **PROVIDED INCLUDE GOOD AND BAD FERMIONS**

But what happened to projected fermion anticommutators. We now derive them by projecting (9.2).

$$\left\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(y^+,y^1,y^2,y^-)\right\}=2\Lambda^+_{\alpha\beta}i\frac{\partial}{\partial x^-}i\Delta(LF;x-y), \quad (9.3)$$

$$\left\{[\psi_{(-)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(-)}]_\beta(y^+,y^1,y^2,y^-)\right\}=2\Lambda^-_{\alpha\beta}i\frac{\partial}{\partial x^+}i\Delta(LF;x-y). \quad (9.4)$$

$$\left\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\}=\Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \quad (9.5)$$

$$\left\{\frac{\partial}{\partial x^-}\psi^{(-)}_\alpha(x^+,x^1,x^2,x^-),\frac{\partial}{\partial y^-}[\psi^\dagger_{(-)}]_\beta(y^+,y^1,y^2,y^-)\right\}$$
$$=2i\Lambda^-_{\alpha\beta}\frac{1}{4}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\frac{\partial}{\partial x^-}i\Delta(LF;x-y). \quad (9.6)$$

$$\left\{\frac{\partial}{\partial x^-}\psi^{(-)}_\mu(x^+,x^1,x^2,x^-),\frac{\partial}{\partial y^-}[\psi^\dagger_{(-)}]_\nu(x^+,y^1,y^2,y^-)\right\}$$
$$=\frac{1}{4}\Lambda^-_{\mu\nu}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \quad (9.7)$$

$$\left\{[\psi_{(+)}]_\nu(x),[\psi^\dagger_{(-)}]_\sigma(y)\right\}$$
$$=\tfrac{i}{8}\epsilon(x^--y^-)[i(\gamma^-\gamma^1\partial^x_1+\gamma^-\gamma^2\partial^x_2)-m\gamma^-]_{\nu\sigma}\delta(x^1-y^1)\delta(x^2-y^2), \quad (9.8)$$

10 UNITARY EQUIVALENCE VIA TRANSLATION INVARIANCE

So far the discussion has only dealt with free theory commutators, and they just happen to be c-numbers. However, for interacting theories we can only discuss matrix elements. With

$$[P_\mu, \phi] = -i\partial_\mu \phi, \quad [P_\mu, P_\nu] = 0 \tag{10.1}$$

to all orders in perturbation theory because of Poincare invariance, we introduce

$$U(P_0, P_3) = \exp(ix^3 P_0) \exp(ix^0 P_3). \tag{10.2}$$

It effects

$$U\phi(IT; x^0, x^1, x^2, -x^3)U^{-1} = \phi(IT; x^0 + x^3, x^1, x^2, x^0 - x^3) = \phi(LF; x^+, x^1, x^2, x^-) \tag{10.3}$$

Then with $|\Omega_F\rangle = U|\Omega_I\rangle$ we obtain

$$\begin{aligned} -i\langle\Omega_I|[\phi(IT; x^0, x^1, x^2, -x^3), \phi(0)]|\Omega_I\rangle &= -i\langle\Omega_I|U^\dagger U[\phi(IT; x^0, x^1, x^2, -x^3), \phi(0)]U^\dagger U|\Omega_I\rangle \\ &= -i\langle\Omega_F|[\phi(LF; x^+, x^1, x^2, x^-), \phi(0)]|\Omega_F\rangle, \end{aligned} \tag{10.4}$$

to all orders in perturbation theory. We thus establish the unitary equivalence of matrix elements of instant-time and light-front commutators to all orders.

The same equivalence holds for the all-order Lehmann representations. For the instant-time case we have

$$\langle\Omega|[\phi(IT;x),\phi(IT;y)]|\Omega\rangle = \frac{1}{(2\pi)^3}\int_0^\infty d\sigma^2\rho(\sigma^2,IT)\int d^4q\epsilon(q_0)\delta(q^2-\sigma^2)e^{-iq\cdot(x-y)}$$
$$= \int_0^\infty d\sigma^2\rho(\sigma^2,IT)i\Delta(IT,FREE;x-y,\sigma^2), \tag{10.5}$$

where

$$\rho(q^2,IT)\theta(q_0) = (2\pi)^3\sum_n \delta^4(p^n_\mu - q_\mu)|\langle\Omega|\phi(0)|p^n_\mu\rangle|^2, \quad P_\mu|p^n_\mu\rangle = p^n_\mu|p^n_\mu\rangle, \tag{10.6}$$

as written in instant-time momentum eigenstates.

For the light-front case we have

$$\langle\Omega|[\phi(LF;x),\phi(LF;y)]|\Omega\rangle = \frac{2}{(2\pi)^3}\int_0^\infty d\sigma^2\rho(\sigma^2,LF)\int d^4q\epsilon(q_+)\delta(q^2-\sigma^2)e^{-iq\cdot(x-y)}.$$
$$= \int_0^\infty d\sigma^2\rho(\sigma^2,LF)i\Delta(LF,FREE;x-y,\sigma^2), \tag{10.7}$$

where

$$\rho(q_\mu,LF) = \frac{(2\pi)^3}{2}\sum_n \delta^4(p^n_\mu - q_\mu)|\langle\Omega|\phi(0)|p^n_\mu\rangle|^2 = \rho(q^2,LF)\theta(q_+), \tag{10.8}$$

as written in light-front momentum eigenstates. Specifically, with

$$U|p^n_0\rangle = |p^n_+\rangle, \quad U|p^n_3\rangle = |p^n_-\rangle, \quad U|p^n_1\rangle = |p^n_1\rangle, \quad U|p^n_2\rangle = |p^n_2\rangle \tag{10.9}$$

we obtain the all-order

$$\langle\Omega|[\phi(IT;x),\phi(IT;y)]|\Omega\rangle = \langle\Omega|[\phi(LF;x),\phi(LF;y)]|\Omega\rangle. \tag{10.10}$$

With the all-order momentum operators having real and complete eigenspectra we have the all-order

$$P_\mu(IT)=\sum|p^n(IT)\rangle p^n_\mu(IT)\langle p^n(IT)|,\quad P_\mu(LF)=\sum|p^n(LF)\rangle p^n_\mu(LF)\langle p^n(LF).\tag{10.11}$$

With eigenvalues not changing under a unitary transformation, we obtain

$$\begin{aligned}P_0(IT)=UP_0(IT)U^{-1}&=U\sum|p^n(IT)\rangle p^n_0\langle p^n(IT)|U^\dagger\\&=\sum|p^n(LF)\rangle(p^n_++p^n_-)\langle p^n(LF)|=P_+(LF)+P_-(LF).\end{aligned}\tag{10.12}$$

Given (10.11) and (10.12), there initially appears to be a mismatch between the eigenstates of $P_0(IT)$ and $P_+(LF)$. However, for any timelike set of instant-time momentum eigenvalues we can Lorentz boost p_1, p_2 and p_3 to zero, to yield

$$p_1=0,\quad p_2=0,\quad p_3=0,\quad p_0=m.\tag{10.13}$$

If we impose this same $p_1=0$, $p_2=0$, $p_3=0$ condition on the light-front momentum eigenvalues we would set $p_+=p_-$, $p^2=4p_+^2=m^2$, and thus obtain

$$p_1=0,\quad p_2=0,\quad p_+=p_-,\quad p_0=2p_+=m\tag{10.14}$$

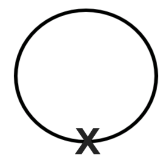
When written in terms of contravariant vectors with $p^\mu=g^{\mu\nu}p_\nu$ this condition takes the form

$$p^0=p^-=m.\tag{10.15}$$

Thus in **the instant-time rest frame** the eigenvalues of $P^0(IT)$ and $P^-(LF)$ coincide. In this sense then instant-time and light-front Hamiltonians are equivalent.

Having now established the equivalence of commutators and the equivalence of Hamiltonian operators, we now proceed to establish the same equivalence for both free and interacting instant-time and light-front Green's functions.

11 EQUIVALENCE OF INSTANT-TIME AND LIGHT-FRONT PROPAGATORS AND TADPOLES



Construct tadpole as $x^\mu \to 0$ limit of propagator (not two-point function), i.e., use x^μ as a regulator.

$$D(x^\mu) = -i\langle\Omega|[\theta(\sigma)\phi(x)\phi(0) + \theta(-\sigma)\phi(0)\phi(x)]|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon}, \quad \sigma = \mathrm{x}^0 \text{ or } \sigma = \mathrm{x}^+ \tag{11.1}$$

$$D(x^\mu = 0) = -i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p \frac{1}{p^2 - m^2 + i\epsilon}. \tag{11.2}$$

$$\begin{aligned}
D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{e^{-i(p_0x^0 + p_1x^1 + p_2x^2 + p_3x^3)}}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon}, \\
D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{e^{-i(p_+x^+ + p_1x^1 + p_2x^2 + p_-x^-)}}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon}, \\
D(x^\mu = 0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon}, \\
D(x^\mu = 0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon}.
\end{aligned} \tag{11.3}$$

For all of these Feynman contours there are only poles, except $D(x^\mu = 0, \text{front})$, for which the circle at infinity in the complex p_+ plane is not suppressed.

12 THE NON-VACUUM INSTANT-TIME CASE

In the instant-time case the Feynman integral is readily performed since it is just pole terms and for the **forward** $D(x^0 > 0, \text{instant}) = -i\langle\Omega_I|\theta(x^0)\phi(x^0, x^1, x^2, x^3)\phi(0)|\Omega_I\rangle$ we obtain

$$\begin{aligned} D(x^0 > 0, \text{instant}) &= D(x^0 > 0, \text{instant}, \text{pole}) \\ &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}} = \frac{1}{8\pi}\left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}). \end{aligned} \tag{12.1}$$

Insertion of the Fock space expansion for $\phi(x^0, x^1, x^2, x^3)$ yields

$$D(x^0 > 0, \text{instant}, \text{ Fock}) = -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}}. \tag{12.2}$$

We recognize (12.2) as (12.1), to thus establish the equivalence of the instant-time Feynman and Fock space prescriptions.

13 THE NON-VACUUM LIGHT-FRONT CASE

In the light-front case poles in the complex p_+ plane occur at

$$p_+ = E'_p - \frac{i\epsilon}{4p_-}, \quad E'_p = \frac{(p_1)^2 + (p_2)^2 + m^2}{4p_-}. \tag{13.1}$$

Poles with $p_- \geq 0^+$ thus all lie below the real p_+ axis and have positive E'_p, while poles with $p_- \leq 0^-$ all lie above the real p_+ axis and have negative E'_p. For $x^+ > 0$, closing the p_+ contour below the real axis (which for $x^+ > 0$ suppresses the circle at infinity contribution) then restricts to poles with $E'_p > 0$, $p_- \geq 0^+$. However, in order to evaluate the pole terms one has to deal with the fact that the pole at $p_- = 0^+$ has $E'_p = \infty$. Momentarily exclude the region around $p_- = 0$, and thus only consider poles below the real p_+ axis that have $p_- \geq \delta$. Evaluating the contour integral in the lower half of the complex p_+ plane thus gives

$$\begin{aligned} D(x^+ > 0, \text{front}, \text{pole}) &= -\frac{2i}{(2\pi)^3} \int_\delta^\infty \frac{dp_-}{4p_-} \int_{-\infty}^\infty dp_1 \int_{-\infty}^\infty dp_2 e^{-i(E'_p x^+ + p_- x^- + p_1 x^1 + p_2 x^2) - \epsilon x^+/4p_-} \\ &= -\frac{1}{4\pi^2 x^+} \int_\delta^\infty dp_- e^{-ip_- x^- + i[(x^1)^2 + (x^2)^2]p_-/x^+ - im^2 x^+/4p_- - \epsilon x^+/4p_-} \\ &= -\frac{1}{4\pi^2 x^+} \int_\delta^\infty dp_- e^{-ip_- x^2/x^+ - im^2 x^+/4p_- - \epsilon x^+/4p_-}. \end{aligned} \tag{13.2}$$

If we now set $\alpha = x^+/4p_-$, we obtain

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{1}{16\pi^2} \int_0^{x^+/4\delta} \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon}. \tag{13.3}$$

In (13.3) we can now take the limit $\delta \to 0$, $x^+/4\delta \to \infty$ without encountering any ambiguity **AS LONG AS x^+ IS NONZERO**, and with $x^+ > 0$ thus obtain

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon} = \frac{1}{8\pi} \left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}). \tag{13.4}$$

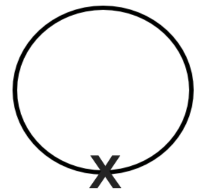
Comparing with (12.1) we see that $D(x^+ > 0, \mathrm{instant})$ and $D(x^+ > 0, \mathrm{front})$ are equal.

Inserting the Fock space expansion for $\phi(x^+, x^1, x^2, x^-)$ gives precisely the same result, and thus we obtain

$$\begin{aligned} &D(x^0 > 0, \mathrm{instant}) = D(x^0 > 0, \mathrm{instant}, \mathrm{pole}) = D(x^0 > 0, \mathrm{instant}, \mathrm{Fock}) \\ &= D(x^+ > 0, \mathrm{front}) = D(x^+ > 0, \mathrm{front}, \mathrm{pole}) = D(x^+ > 0, \mathrm{front}, \mathrm{Fock}). \end{aligned} \tag{13.5}$$

General rule: the Feynman and Fock space prescriptions will coincide whenever the only contribution to Feynman contours is poles. Thus for $x^+ > 0$ the Feynman and Light-Front Hamiltonian approaches coincide. But what about $x^+ = 0$?

14 THE INSTANT-TIME VACUUM CASE



In the instant-time case one can readily set x^μ to zero, and obtain

$$\begin{aligned} D(x^\mu=0,\mathrm{instant}) &= \frac{1}{(2\pi)^4}\int dp_0dp_1dp_2dp_3\frac{1}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon} \\ &= D(x^\mu=0,\mathrm{instant},\mathrm{pole}) = D(x^\mu=0,\mathrm{instant},\mathrm{Fock}) \\ &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p} = -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}. \end{aligned} \tag{14.1}$$

15 THE LIGHT-FRONT VACUUM CASE - POLE AND FOCK SPACE CONTRIBUTIONS

In the light-front case we set x^μ to zero and evaluate

$$D(x^\mu=0,\mathrm{front})=\frac{2}{(2\pi)^4}\int dp_+dp_1dp_2dp_-\frac{1}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon}. \tag{15.1}$$

Again we need to take care of the $p_-=0$ region, so we again introduce the δ cutoff at small p_-. On closing below the real p_+ axis the only poles are those with $p_->0$, and for them we obtain a pole contribution of the form

$$D(x^\mu=0,\mathrm{front,pole})=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{\delta}^{\infty}\frac{dp_-}{4p_-}. \tag{15.2}$$

Then on setting $p_-=1/\alpha$, we are able to let p_- go to zero, to obtain

$$D(x^\mu=0,\mathrm{front,pole})=-\frac{i}{16\pi^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{0}^{1/\delta}\frac{d\alpha}{\alpha}=-\frac{i}{16\pi^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{0}^{\infty}\frac{d\alpha}{\alpha}. \tag{15.3}$$

For the Fock space prescription we set $x^\mu=0$ in (2.2), viz.

$$\phi(0)=\frac{2}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{0}^{\infty}\frac{dp_-}{(4p_-)^{1/2}}[a_p+a_p^\dagger], \tag{15.4}$$

and on inserting $\phi(0)$ into $-i\langle\Omega|\phi(0)\phi(0)]|\Omega\rangle$ obtain

$$D(x^\mu=0,\mathrm{front,Fock})=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{0}^{\infty}\frac{dp_-}{4p_-}=D(x^\mu=0,\mathrm{front,pole}). \tag{15.5}$$

Comparing with (15.2) we again see the equivalence of the pole and Fock space prescriptions.

However, something is wrong. We are evaluating the m-dependent $D(x^\mu=0,\mathrm{front})$ as given in (15.1), and yet we obtain an answer that does not depend on m at all. **What went wrong is that we left out the circle at infinity.**

16 THE LIGHT-FRONT VACUUM CASE - CIRCLE AT INFINITY CONTRIBUTION

To evaluate the circle at infinity contribution we introduce the regulator

$$\frac{1}{(A+i\epsilon)} = -i\int_0^\infty d\alpha e^{i\alpha(A+i\epsilon)}. \tag{16.1}$$

For $p_- > 0$ the regulator converges on the **UPPER** half circle, and there are no poles at all. We obtain

$$\begin{aligned}
&D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) \\
&= \frac{2i}{(2\pi)^4}\int_0^\infty dp_- \int_{-\infty}^\infty dp_1 \int_{-\infty}^\infty dp_2 \int_0^\pi iRe^{i\theta}d\theta \int_0^\infty d\alpha e^{i\alpha(4p_-Re^{i\theta}-(p_1)^2-(p_2)^2-m^2+i\epsilon)} \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\int_0^\pi iRe^{i\theta}d\theta e^{4i\alpha p_-Re^{i\theta}} \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{(e^{4i\alpha p_-Re^{i\theta}} - e^{-4i\alpha p_-Re^{i\theta}})}{4i\alpha p_-}\Bigg|_0^\pi \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{(e^{-4i\alpha p_-R} - e^{4i\alpha p_-R})}{4i\alpha p_-} \\
&= -\frac{1}{4\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{\sin(4\alpha p_-R)}{4\alpha p_-}.
\end{aligned} \tag{16.2}$$

Then, on letting R go to infinity we obtain

$$\begin{aligned}
&D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) = -\frac{1}{4\pi^2}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-) \\
&= -\frac{1}{8\pi^2}\int_{-\infty}^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-) = -\frac{1}{32\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}.
\end{aligned} \tag{16.3}$$

We thus establish the centrality of $p_- = 0$ modes.

Similarly, for $p_- < 0$ close on the **LOWER** half circle, and again there are no poles. We obtain

$$D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) = D(x^\mu = 0, p_- < 0, \text{front}, \text{lower circle}), \tag{16.4}$$

and thus

$$
\begin{aligned}
D(x^\mu = 0, \text{front}) &= D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) + D(x^\mu = 0, p_- < 0, \text{front}, \text{lower circle}) \\
&= -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \qquad (16.5)
\end{aligned}
$$

Now not only is there now an m dependence, we obtain

$$
D(x^\mu = 0, \text{front}) = D(x^\mu = 0, \text{instant}). \qquad (16.6)
$$

So again, light-front quantization is instant-time quantization. And even though there is only a circle at infinity contribution in the light front case, it is this circle at infinity that enables the light-front and instant-time vacuum graphs to be the same.

17 RECONCILING THE FOCK SPACE AND FEYNMAN CALCULATIONS

To avoid $p_- = 0$ difficulties we use the regulator on the real p_+ axis, and set

$$
\begin{aligned}
&D(x^\mu, \text{front}, \text{regulator}) \\
&= -\frac{2i}{(2\pi)^4}\int_{-\infty}^{\infty} dp_+ \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_- e^{-i(p_+x^+ + p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon)} \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} dp_- e^{-i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(-(p_1)^2 - (p_2)^2 - m^2 + i\epsilon)}\delta(4\alpha p_- - x^+) \\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{0} dp_- e^{-i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{i\alpha(-(p_1)^2 - (p_2)^2 - m^2 + i\epsilon)}\delta(4\alpha p_- - x^+).
\end{aligned} \tag{17.1}
$$

On changing the signs of p_-, p_1 and p_2 in the last integral and setting F_p^2 equal to the positive $(p_1)^2 + (p_2)^2 + m^2$ we obtain

$$
\begin{aligned}
&D(x^\mu, \text{front}, \text{regulator}) \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} e^{-i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{ix^+(-F_p^2 + i\epsilon)/4p_-}\delta(\alpha - x^+/4p_-) \\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} e^{i(p_-x^- + p_1x^1 + p_2x^2)} \int_0^\infty d\alpha e^{ix^+(F_p^2 - i\epsilon)/4p_-}\delta(\alpha + x^+/4p_-) \\
&= -\frac{2i\theta(x^+)}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} e^{-i(F_p^2x^+/4p_- + p_-x^- + p_1x^1 + p_2x^2 + ix^+\epsilon/4p_-)} \\
&-\frac{2i\theta(-x^+)}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{4p_-} e^{i(F_p^2x^+/4p_- + p_-x^- + p_1x^1 + p_2x^2 - ix^+\epsilon/4p_-)},
\end{aligned} \tag{17.2}
$$

and note that the structure of (17.2) is such that for $x^+ > 0$ (forward in time) one only has positive energy propagation, while for $x^+ < 0$ (backward in time) one only has negative energy propagation. With the insertion into $D(x^\mu) = -i\langle\Omega|[\theta(x^+)\phi(x)\phi(0) + \theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle$ of the Fock space expansion for $\phi(x^\mu)$ given in (2.2) precisely leading to (17.2), we recognize (17.2) as the $x^\mu \neq 0$ $D(x^\mu, \text{front}, \text{Fock})$.

Now if we set $x^\mu = 0$ in (17.2) we would appear to obtain the m-independent $D(x^\mu = 0, \text{front}, \text{Fock})$ given in (15.5). However, we cannot take the $x^+ \to 0$ limit since the quantity $x^+/4p_-$ is undefined if p_- is zero, and $p_- = 0$ is included in the integration range. Hence, just as discussed in regard to (13.3), the limit is singular.

To obtain a limit that is not singular we note that we can set x^μ to zero in (17.1) as there the limit is well-defined, and this leads to

$$\begin{aligned}
&D(x^\mu = 0, \text{front}, \text{regulator}) \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} dp_- \int_0^{\infty} d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-) \\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{0} dp_- \int_0^{\infty} d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-) \\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_- \int_0^{\infty} \frac{d\alpha}{4\alpha} e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(p_-),
\end{aligned} \tag{17.3}$$

and again see the **centrality of $p_- = 0$ modes**. If we do the momentum integrations we obtain the m-dependent

$$D(x^\mu = 0, \text{front}, \text{regulator}) = -\frac{1}{16\pi^2}\int_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \tag{17.4}$$

We recognize (17.4) as being of the same form as the m-dependent $D(x^\mu = 0, \text{front})$ given in (16.5). We thus have to conclude that the limit $x^\mu \to 0$ of (17.2) is not (15.5) but is (17.4) instead, and that

$$D(x^\mu = 0, \text{front}) = D(x^\mu = 0, \text{instant}) = -\frac{1}{16\pi^2}\int_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \tag{17.5}$$

Setting $p_- = 0$ and then $x^+ = 0$ is not the same as setting $x^+ = 0$ and then $p_- = 0$.

Thus because of singularities we first have to point split, and when we do so we find that it is the m-dependent (17.4) that is the correct value for the light-front vacuum graph. And it is equal to the instant-time vacuum graph.

18 INFINITE MOMENTUM FRAME CONSIDERATIONS

Under a Lorentz boost with velocity u in the 3-direction the contravariant and covariant components of a general four-vector A^μ transform as

$$A^0 \to \frac{A^0 + uA^3}{(1-u^2)^{1/2}}, \quad A^3 \to \frac{A^3 + uA^0}{(1-u^2)^{1/2}}, \quad A_0 \to \frac{A_0 - uA_3}{(1-u^2)^{1/2}}, \quad A_3 \to \frac{A_3 - uA_0}{(1-u^2)^{1/2}}. \tag{18.1}$$

If we set $(1-u) = \epsilon^2/2$, then with ϵ small, to leading order we obtain

$$\begin{aligned}
&A^0 \to \frac{A^0 + A^3}{\epsilon} + O(\epsilon), \quad A^3 \to \frac{A^3 + A^0}{\epsilon} + O(\epsilon), \quad A_0 \to \frac{A_0 - A_3}{\epsilon} + O(\epsilon), \quad A_3 \to \frac{A_3 - A_0}{\epsilon} + O(\epsilon), \\
&(A^0)^2 - (A^3)^2 = A^+A^- \to A^+A^-,
\end{aligned} \tag{18.2}$$

where $A^\pm = A^0 \pm A^3$. This leads to

$$p^3 \to \frac{p^+}{\epsilon} = \frac{2p_-}{\epsilon}, \quad E_p \to \frac{2p_-}{\epsilon}, \quad \frac{dp^3}{E_p} \to \frac{dp_-}{p_-}, \tag{18.3}$$

where $E_p = [(p_3)^2 + (p_1)^2 + (p_2)^2 + m^])^{1/2}$.

On transforming to the infinite momentum frame we obtain

$$\begin{aligned}
&D(x^\mu = 0, \text{instant}, \text{Fock}) = D(x^\mu = 0, \text{instant}, \text{pole}) = -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty} \frac{d^3p}{2E_p} \\
&\to -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{2p_-} = D(x^\mu = 0, \text{front}, \text{Fock}) = D(x^\mu = 0, \text{front}, \text{pole}).
\end{aligned} \tag{18.4}$$

$$
\begin{aligned}
D(x^\mu=0,\text{instant},\text{pole}) &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p} \\
&\to -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{2p_-}=D(x^\mu=0,\text{front},\text{pole})
\end{aligned}
\tag{18.5}
$$

and as such, the infinite momentum frame is doing what it is supposed to do, namely it is transforming an instant-time on-shell graph into a light-front on-shell graph. However, this is not the correct answer as it does not depend on m. As we showed in (17.5) the correct answer is the m-dependent

$$
D(x^\mu=0,\text{front})=D(x^\mu=0,\text{instant})=-\frac{1}{16\pi^2}\int_0^{\infty}\frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}. \tag{18.6}
$$

Thus in this respect not only is the on-shell prescription failing for light-front vacuum graphs, so is the infinite momentum frame prescription.

We thus have two puzzles: How could the limit in (18.5) lose its m dependence to begin with if it is a Lorentz transformation. And second how do we recover the m dependence anyway.

For the first puzzle we note that since the mass-dependent quantity $dp_3/2E_p$ is Lorentz invariant, under a Lorentz transformation with a velocity less than the velocity of light it must transform into itself and thus must remain mass dependent. However, in the infinite momentum frame it transforms into a quantity $dp_-/2p_-$ that is mass independent. This is because velocity less than the velocity of light and velocity equal to the velocity of light are inequivalent, since an observer that is able to travel at less than the velocity of light is not able to travel at the velocity of light. Lorentz transformations at the velocity of light are different than those at less than the velocity of light, and at the velocity of light observers (viz. observers on the light cone) can lose any trace of mass.

The resolution to the second puzzle lies in the contribution of the circle at infinity to the Feynman contour. In the instant-time case the integral

$$\int \frac{dp_0 dp_3}{(p_0)^2-(p_3)^2-(p_1)^2-(p_2)^2-m^2+i\epsilon} \tag{18.7}$$

is suppressed on the circle at infinity in the complex p_0 plane (p_3 being finite), and only poles contribute. However, when one goes to the infinite momentum frame in the instant-time case dp_3 also becomes infinite ($p^3 = mv/(1-v^2)^{1/2}$) and the circle contribution is no longer suppressed. Specifically, on the instant-time circle at infinity, the term that is of relevance behaves as

$$\int \frac{Rie^{i\theta} d\theta dp_3}{R^2 e^{2i\theta}-(p_3)^2}, \tag{18.8}$$

and on setting $\epsilon = 1/R$ in the infinite momentum frame limit, as per (18.3) the circle term behaves as the unsuppressed

$$\int \frac{Rie^{i\theta} d\theta R dp_-}{R^2 e^{2i\theta}-R^2 p_-^2} = \int \frac{ie^{i\theta} d\theta dp_-}{e^{2i\theta}-p_-^2}. \tag{18.9}$$

Thus in the instant-time case one cannot ignore the circle at infinity in the infinite momentum frame even though one can ignore it for observers moving with finite momentum. Consequently, the initial reduction from the instant-time Feynman diagram to the on-shell instant-time Hamiltonian prescription is not valid in the infinite momentum frame, and one has to do the full four-dimensional Feynman contour integral instead.

19 INTERACTIONS

Two c-number approaches: path integrals and Feynman diagrams. Path integrals involve integrals of classical variables in coordinate space. Feynman diagrams involve integrals of classical variables in momentum space. For both we can transform from instant-time to light-front coordinate and momentum variables using general coordinate transformations. Thus if underlying theory and its renormalization procedure are general coordinate invariant the equivalence of instant-time and light-front Green's functions is established.

However, there is a caveat. For Feynman diagrams we need to start out with fully covariant four-dimensional contour integrals if we want to establish the equivalence. We can obscure the equivalence if we do the pole integrations in the complex frequency plane first, as then we would have on-shell three-dimensional integrals. Also we would then have a zero momentum mode problem. We can avoid this by not doing the frequency integrations until after we have introduced the exponential regulators.

That the zero mode problem must be avoidable is apparent from the path integral approach as it is purely in coordinate space and involves no zero momentum modes at all.

20 THE MORAL OF THE STORY

When we let $p_- \to 0$ we are letting $p_+ = [(p_1)^2 + (p_2)^2 + m^2]/4p_- \to \infty$.
However x^+ is the conjugate of p_+, and thus as $p_+ \to \infty$, $x^+ \to 0$.
The $p_- \to 0$ and the $x^+ \to 0$ limits are thus intertwined.
If we stay away from $x^+ = 0$ and restrict to $x^+ > 0$ and thus $p_- > 0$ as in the Light-Front Hamiltonian approach, there is no difficulty as there are only poles and nothing is singular, with the forward scattering on-shell Light-Front Hamiltonian approach thus being validated.
However this does become a concern for tadpole graphs as they have $x^+ = 0$, since we need both $\theta(x^+)$ and $\theta(-x^+)$ time orderings in the limit, with $\langle\Omega|[\theta(x^+)\phi(x)\phi(0) + \theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle \to \langle\Omega|[\theta(0^+)\phi(0)\phi(0) + \theta(0^-)\phi(0)\phi(0)]|\Omega\rangle = \langle\Omega|\phi(0)\phi(0)|\Omega\rangle$.
If we compare

$$\begin{aligned} D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{e^{-i(p_0x^0+p_1x^1+p_2x^2+p_3x^3)}}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon}, \\ D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{e^{-i(p_+x^++p_1x^1+p_2x^2+p_-x^-)}}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon}, \end{aligned} \tag{20.1}$$

$$\begin{aligned} D(x^\mu=0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon}, \\ D(x^\mu=0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon}, \end{aligned} \tag{20.2}$$

we can transform each instant-time graph into each corresponding light-front graph by a change of variable. Thus they must be equal. However, that does not mean that pole equals pole or that circle equals circle, only that pole plus circle equals pole plus circle, as it is only on the full closed contour that the integrals are equal.

The transformation $x^0 \to x^0 + x^3$, $x^3 \to x^0 - x^3$ is a spacetime-dependent general coordinate transformation (not a Lorentz transformation), and thus by the general coordinate invariance of the fundamental interactions it must be the case that

LIGHT-FRONT QUANTIZATION IS INSTANT-TIME QUANTIZATION, JUST ONE THEORY.